Journal of Computational Applied Mechanics 2024, 55 (2): 235-241

DOI: 10.22059/JCAMECH.2024.374235.1012

RESEARCH PAPER

Variational principle for Schrödinger-KdV system with the M-fractional derivatives

Man-Li Jiao [a], Ji-Huan He [b*], Chun-Hui He [c], Abdulrahman Ali Alsolami [d]

[a] *School of Science, Xi'an University of Architecture and Technology, Xi'an, China*
[b] *National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University,199 Ren-Ai Road, Suzhou, China*
[c] *School of Mathematics, China University of Mining and Technology, Xuzhou 221116, Jiangsu, P. R. China*
[d] *Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia*

Abstract

The variational theory is an inextricable part of both continuum mechanics and physics, and plays an important role in mathematics and nonlinear science, however it is difficult to find a variational formulation for a nonlinear system, and it is more difficult for a fractional differential system. This paper is to search for a variational formulation for the Schrödinger-KdV system with M-fractional derivatives. The fractional complex transformation is used to convert the system into a traditional differential system, and the semi-inverse method is further applied to establish a needed variational principle.

Keywords: Hamilton principle, fractional calculus, chain rule

1. Introduction

The variational principle is of paramount importance in engineering applications [1], it is an inextricable part of both continuum mechanics and physics. The most famous one is the Hamilton principle [2], which can be used to establish governing equations for a complex problem [3-5]. The variational principle is also used to search for an approximate solution [6, 7] and numerical solution [8] of a nonlinear differential equation, the most famous ones are Hamiltonian-Based frequency-amplitude formulation for nonlinear vibration systems [6], and the variational iteration method for nonlinear differential equations [9, 10], the variational-based finite element method [11].

Recently the variational theory in a fractal space [12, 13] became a hot topic in both mathematics and physics, because the basic assumptions in continuum mechanics become totally invalid, however the variational theory still holds, that means the energy conservation and mass conservation and Hamilton principle still work in a fractal space though the physical laws in a fractal space cannot be modelled by the differential equations. Variational principles for various solitary waves [14-19] nano/microelectromechanical systems [12], singular waves [13] and microgravity systems [20] in a fractal space were established. In this paper we will search for a variational formulation for Schrödinger-KdV system with M-fractional derivative [21].

2. Schrödinger-KdV system with M-fractional derivatives

The Schrödinger-KdV system is a hot topic in both physics and mathematics [21-23], this paper considers the following Schrödinger-KdV system with M-fractional derivative [21].

E-mail address: hejihuan@suda.edu.cn

$$iw_t + \lambda_1 w_x + \lambda_2 |w|^2 w + \lambda_3 wu = 0 \quad (1)$$

$$u_t^\alpha + \sigma_1 u u_x^\beta + \sigma_2 u_x^{3\beta} + \sigma_3 (|w|^2)_x^\beta = 0 \quad (2)$$

where the coefficients involved in Eq. (1) are real constants, w is a complex function, while u is a real-valued function. The M-fractional derivative is defined as [21, 24, 25].

$$w_t^\alpha = \lim_{\varepsilon \to 0} \frac{w(tE_\gamma(\varepsilon t^{1-\alpha})) - w(t)}{\varepsilon}, \quad (3)$$

$\gamma > 0, 0 < \alpha \le 1.$ where the truncated Mittag-Leffler function is defined as

$$E_\gamma(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k + 1)}, \quad (4)$$

When $\alpha = \beta = 1$, Eqs.(1) and (2) turn to be the traditional Schrödinger-KdV system [22, 23]. When $\lambda_3 = 0$, Eq. (1) is the well-known Schrödinger equation for quantum waves [26, 27]. When the quantum wave travels through a moving dispersive medium with velocity of u, a dispersive term ($\lambda_3 wu$) has to be considered in Eq. (1), while Eq. (2) describes a KdV-like solitary wave [28]. The dispersive medium can be considered as a fractal space, so the system can be modelled by a fractional differential model, which can also be used to model neural networks [29] and Benjamin-Bona-Mahony equation [30].

3. Fractional complex transformation

The fractional complex transformation was first proposed in 2010 [31], it can be explained as an approximate transformation from a fractal space to a smooth space, in literature it was also called as the two-scale fractal transformation [32], and its geometrical explanation is available in Ref. [33].

According to the fractional complex transformation [31, 33], we introduce two new variables (W and U) defined as

$$w = W(\lambda) e^{i\mu}, u = U(\lambda). \quad (5)$$

where the complex variables λ and μ are defined, respectively, as

$$\lambda = \frac{\Gamma(\gamma_1 + 1)}{\beta} k_1 x^\beta + \frac{\Gamma(\gamma_2 + 1)}{\alpha} \rho_1 t^\alpha \quad (6)$$

$$\mu = \frac{\Gamma(\gamma_1 + 1)}{\beta} k_2 x^\beta + \frac{\Gamma(\gamma_2 + 1)}{\alpha} \rho_2 t^\alpha \quad (7)$$

where constants k_1, k_2 and ρ_1, ρ_2 are constants. Substituting Eqs. (5)-(7) into Eqs. (1) and (2), using the chain role given in Ref. [33], and separating the real part and the imaginary part, we have [20]

$$\lambda_1 k_1^2 W'' - (\rho_2 + k_2^2 \lambda_1) W + \lambda_2 W^3 + \lambda_3 WU = 0 \quad (8)$$

$$\sigma_2 k_1^3 U''' + \sigma_1 k_1 U U' + \rho_1 U' + 2\sigma_3 k_1 W W' = 0 \tag{9}$$

$$(\rho_1 + 2k_1 k_2 \lambda_1) W' = 0 \tag{10}$$

where the prime is the derivative with respect to λ. Combining Eq. (9) and Eq. (10) together, we have

$$\sigma_2 k_1^2 U'' + \frac{1}{2}\sigma_1 U^2 - 2k_2 \lambda_1 U + \sigma_3 W^2 = H \tag{11}$$

where H is an integration constant. By the fractional complex transformation [31, 33], the fractional Schrödinger-KdV system given in Eqs.(1) and (2) turns out to be the traditional differential system given in Eqs.(8) and (11), so the problem becomes extremely simple.

4. Variational formulation

This paper is to establish a variational formulation for Eqs. (8) and (11). To this end, we first consider a special case when $\sigma_3 = 0$ and $\lambda_3 = 0$, Eqs. (8) and (11) become, respectively, as

$$\lambda_1 k_1^2 W'' - (\rho_2 + k_2^2 \lambda_1) W + \lambda_2 W^3 = 0 \tag{12}$$

$$\sigma_2 k_1^2 U'' + \frac{1}{2}\sigma_1 U^2 - 2k_2 \lambda_1 U = H \tag{13}$$

Eq. (11) is the well-known Duffing oscillator [34-37], its variational formulation is

$$J_1(W) = \int \left\{ -\frac{1}{2}\lambda_1 k_1^2 W'^2 - \frac{1}{2}(\rho_2 + k_2^2 \lambda_1) W^2 + \frac{1}{4}\lambda_2 W^4 \right\} d\lambda \tag{14}$$

Eq. (12) adopts the following variational formulation

$$J_2(U) = \int \left\{ -\frac{1}{2}\sigma_2 k_1^2 U'^2 + \frac{1}{6}\sigma_1 U^3 - k_2 \lambda_1 U^2 - HU \right\} d\lambda \tag{15}$$

The semi-inverse method [38] has to be adopted to search for a variational formulation for cases when $\lambda_3 \neq 0$ and $\sigma_3 \neq 0$.

The semi-inverse method [38], we can construct a trial-functional in the form

$$J(W,U) = kJ_1(W) + J_2(U) + \int F(W,U,W',U') d\lambda \tag{16}$$

where J_1 and J_2 are defined respectively in Eqs.(14) and (15), k is an unknown constant, and F is an unknown function of W and/or U and/or their derivatives.

The trial-functional given in Eq. (16) turns out to be J_1 or J_2 under special conditions. There are many alternative candidates for the trial-functions, see examples in Refs. [39-44].

The Lagrange function in Eq. (16) can be expressed as

$$L(W,U,W',U') = k\left[-\frac{1}{2}\lambda_1 k_1^2 W'^2 - \frac{1}{2}(\rho_2 + k_2^2\lambda_1)W^2 + \frac{1}{4}\lambda_2 W^4\right] + \left[-\frac{1}{2}\sigma_2 k_1^2 U'^2 + \frac{1}{6}\sigma_1 U^3 - k_2\lambda_1 U^2 - HU\right] + F \tag{17}$$

The Euler-Lagrange equations can be written as

$$\frac{\partial L}{\partial W} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial W'}\right) = 0 \tag{18}$$

$$\frac{\partial L}{\partial U} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial U'}\right) = 0 \tag{19}$$

Eqs. (18) and (19) imply that

$$k\left[\lambda_1 k_1^2 W'' - (\rho_2 + k_2^2\lambda_1)W + \lambda_2 W^3\right] + \frac{\delta F}{\delta W} = 0 \tag{20}$$

$$\sigma_2 k_1^2 U'' + \frac{1}{2}\sigma_1 U^2 - 2k_2\lambda_1 U - H + \frac{\delta F}{\delta U} = 0 \tag{21}$$

where the variational derivative is defined as

$$\frac{\delta F}{\delta W} = \frac{\partial F}{\partial W} - \frac{d}{d\lambda}\left(\frac{\partial F}{\partial W'}\right) \tag{22}$$

Eqs.(20) and (21) should be equivalent to Eqs.(8) and (11), to this end, we set

$$\frac{\delta F}{\delta W} = -k\left[\lambda_1 k_1^2 W'' - (\rho_2 + k_2^2\lambda_1)W + \lambda_2 W^3\right] = k\lambda_3 WU \tag{23}$$

$$\frac{\delta F}{\delta U} = -\left[\sigma_2 k_1^2 U'' + \frac{1}{2}\sigma_1 U^2 - 2k_2\lambda_1 U - H\right] = \sigma_3 W^2 \tag{24}$$

According to the consistency of F, it requires

$$\frac{\partial}{\partial U}\left(\frac{\delta F}{\delta W}\right) = \frac{\partial}{\partial W}\left(\frac{\delta F}{\delta U}\right) \tag{25}$$

Eq. (25) implies that

$$k\lambda_3 = 2\sigma_3 \tag{26}$$

From Eq. (26), the unknown parameter (k) can be determined:

$$k = \frac{2\sigma_3}{\lambda_3} \tag{27}$$

From Eqs. (23) and (24), and in view of Eq. (27), F can be identified as

$$F = \frac{1}{2} k\lambda_3 W^2 U = \sigma_3 W^2 U \tag{28}$$

Finally, we obtain the following variational formulation

$$J(W,U) = \int L(W,U,W',U')d\lambda = \frac{2\sigma_3}{\lambda_3} J_1(W) + J_2(U) + \int \sigma_3 W^2 U d\lambda \tag{29}$$

where the Lagrange function reads

$$\begin{aligned} L(W,U,W',U') = \frac{2\sigma_3}{\lambda_3}\left[-\frac{1}{2}\lambda_1 k_1^2 W'^2 - \frac{1}{2}(\rho_2 + k_2^2\lambda_1)W + \frac{1}{4}\lambda_2 W^4\right] \\ + \left[-\frac{1}{2}\sigma_2 k_1^2 U'^2 + \frac{1}{6}\sigma_1 U^3 - k_2\lambda_1 U^2 + \sigma_3 W^2 U - HU\right] \end{aligned} \tag{28}$$

5. Conclusion

This paper applies the semi-inverse method to finding a variational formulation for the Schrödinger-KdV system with the M-fractional derivatives.

References

[1] T. Belytschko, Y. Y. Lu, L. Gu, Element-free Galerkin methods, *International Journal for Numerical Methods in Engineering,* Vol. 37, No. 2, pp. 229-256, 1994.

[2] E. L. Hill, Hamilton's Principle and the Conservation Theorems of Mathematical Physics, *Reviews of Modern Physics,* Vol. 23, No. 3, pp. 253-260, 07/01/, 1951.

[3] S. Limkatanyu, W. Sae-Long, J. Rungamornrat, C. Buachart, P. Sukontasukkul, S. Keawsawasvong, P. Chindaprasirt, BENDING, BUCKLING AND FREE VIBRATION ANALYSES OF NANOBEAM-SUBSTRATE MEDIUM SYSTEMS, *Facta Universitatis Series Mechanical Engineering,* Vol. 20, pp. 561-587, 12/01, 2022.

[4] S. A. Faghidian, A. Tounsi, Dynamic characteristics of mixture unified gradient elastic nanobeams, *Facta Universitatis, Series: Mechanical Engineering,* Vol. 20, No. 3, pp. 539-552, 2022.

[5] P. Kooloth, L. M. Smith, S. N. Stechmann, Hamilton's principle with phase changes and conservation principles for moist potential vorticity, *Quarterly Journal of the Royal Meteorological Society,* Vol. 149, No. 752, pp. 1056-1072, 2023.

[6] H. Ma, Simplified Hamiltonian-based frequency-amplitude formulation for nonlinear vibration systems, *Facta Universitatis, Series: Mechanical Engineering,* Vol. 20, No. 2, pp. 445-455, 2022.

[7] S.-Q. Wang, A variational approach to nonlinear two-point boundary value problems, *Computers & Mathematics with Applications,* Vol. 58, No. 11-12, pp. 2452-2455, 2009.

[8] X. Li, D. Wang, T. Saeed, Multi-scale numerical approach to the polymer filling process in the weld line region, *Facta Universitatis, Series: Mechanical Engineering,* Vol. 20, No. 2, pp. 363-380, 2022.

[9] S. Deng, X. Ge, The variational iteration method for Whitham-Broer-Kaup system with local fractional derivatives, *Thermal Science,* Vol. 26, No. 3 Part B, pp. 2419-2426, 2022.

[10] Y. Zhang, D. Tian, J. Pang, A fast estimation of the frequency property of the microelectromechanical system oscillator, *Journal of Low Frequency Noise, Vibration and Active Control*, Vol. 41, No. 1, pp. 160-166, 2022/03/01, 2021.
[11] G. Breitbach, J. Altes, M. Sczimarowsky, Solution of radiative problems using variational based finite element method, *International Journal for Numerical Methods in Engineering*, Vol. 29, No. 8, pp. 1701-1714, 1990.
[12] C.-H. He, A variational principle for a fractal nano/microelectromechanical (N/MEMS) system, *International Journal of Numerical Methods for Heat & Fluid Flow*, Vol. 33, No. 1, pp. 351-359, 2022.
[13] C.-H. He, C. Liu, Variational principle for singular waves, *Chaos, Solitons & Fractals*, Vol. 172, pp. 113566, 07/01, 2023.
[14] Y. Wang, Q. Deng, Fractal derivative model for tsunami travelling, *Fractals*, Vol. 27, 09/24, 2018.
[15] Y. WANG, J. AN, X. WANG, A VARIATIONAL FORMULATION FOR ANISOTROPIC WAVE TRAVELING IN A POROUS MEDIUM, *Fractals*, Vol. 27, No. 04, pp. 1950047, 2019.
[16] K.-L. WANG, C.-H. HE, A REMARK ON WANG'S FRACTAL VARIATIONAL PRINCIPLE, *Fractals*, Vol. 27, No. 08, pp. 1950134, 2019.
[17] W.-W. Ling, P.-X. Wu, A fractal variational theory of the Broer-Kaup system in shallow water waves, *Thermal Science*, Vol. 25, pp. 87-87, 01/01, 2021.
[18] K.-J. WANG, A FRACTAL MODIFICATION OF THE UNSTEADY KORTEWEG–DE VRIES MODEL AND ITS GENERALIZED FRACTAL VARIATIONAL PRINCIPLE AND DIVERSE EXACT SOLUTIONS, *Fractals*, Vol. 30, No. 09, pp. 2250192, 2022.
[19] K.-J. WANG, G.-D. WANG, F. SHI, H.-W. ZHU, GENERALIZED VARIATIONAL PRINCIPLES AND NEW ABUNDANT WAVE STRUCTURES OF THE FRACTAL COUPLED BOUSSINESQ EQUATION, *Fractals*, Vol. 30, No. 07, pp. 2250152, 2022.
[20] S. W. Yao, Variational Perspective To Fractal Kawahara Model In Microgravity Space, *Fractals*, Vol. 31, No. 01, pp. 2350012, 2023.
[21] B. Hong, Abundant explicit solutions for the M-fractional coupled nonlinear Schrödinger–KdV equations, *Journal of Low Frequency Noise, Vibration and Active Control*, Vol. 42, No. 3, pp. 1222-1241, 2023.
[22] F.-F. Liang, X.-P. Wu, C.-L. Tang, Ground State Solution for Schrödinger–KdV System with Periodic Potential, *Qualitative Theory of Dynamical Systems*, Vol. 22, 01/23, 2023.
[23] F.-F. Liang, X.-P. Wu, C.-L. Tang, Normalized Ground-State Solution for the Schrödinger–KdV System, *Mediterranean Journal of Mathematics*, Vol. 19, 10/18, 2022.
[24] S.-W. Yao, R. Manzoor, A. Zafar, M. Inc, S. Abbagari, A. Houwe, Exact soliton solutions to the Cahn–Allen equation and Predator–Prey model with truncated M-fractional derivative, *Results in Physics*, Vol. 37, pp. 105455, 2022.
[25] S. Salahshour, A. Ahmadian, S. Abbasbandy, D. Baleanu, M-fractional derivative under interval uncertainty: Theory, properties and applications, *Chaos, Solitons & Fractals*, Vol. 117, pp. 84-93, 12/01, 2018.
[26] B. Hong, Exact solutions for the conformable fractional coupled nonlinear Schrödinger equations with variable coefficients, *Journal of Low Frequency Noise, Vibration and Active Control*, Vol. 42, No. 2, pp. 628-641, 2023.
[27] M. Suleman, D. Lu, C. Yue, J. Ul Rahman, N. Anjum, He–Laplace method for general nonlinear periodic solitary solution of vibration equations, *Journal of Low Frequency Noise, Vibration and Active Control*, Vol. 38, No. 3-4, pp. 1297-1304, 2019.
[28] D. Zhao, D. Lu, S. A. Salama, P. Yongphet, M. M. Khater, Soliton wave solutions of ion-acoustic waves a cold plasma with negative ions, *Journal of Low Frequency Noise, Vibration and Active Control*, Vol. 41, No. 3, pp. 852-895, 2022.
[29] P.-H. Kuo, Y.-R. Tseng, P.-C. Luan, H.-T. Yau, Novel fractional-order convolutional neural network based chatter diagnosis approach in turning process with chaos error mapping, *Nonlinear Dynamics*, Vol. 111, 01/21, 2023.
[30] J. Lu, L. Ma, Numerical analysis of space-time fractional Benjamin-Bona-Mahony equation, *Thermal Science*, Vol. 27, pp. 1755-1762, 01/01, 2023.
[31] Z.-B. Li, J.-H. He, Fractional Complex Transform for Fractional Differential Equations, *Mathematical and Computational Applications*, Vol. 15, No. 5, pp. 970-973, 2010.
[32] J. Lu, L. Chen, Numerical analysis of a fractal modification of Yao-Cheng oscillator, *Results in Physics*, Vol. 38, pp. 105602, 05/01, 2022.

[33] J.-H. He, S. K. Elagan, Z. B. Li, Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus, *Physics Letters A*, Vol. 376, pp. 257–259, 01/09, 2012.
[34] C.-H. HE, C. LIU, A MODIFIED FREQUENCY–AMPLITUDE FORMULATION FOR FRACTAL VIBRATION SYSTEMS, *Fractals*, Vol. 30, No. 03, pp. 2250046, 2022.
[35] C.-H. He, T. S. Amer, D. Tian, A. F. Abolila, A. A. Galal, Controlling the kinematics of a spring-pendulum system using an energy harvesting device, *Journal of Low Frequency Noise, Vibration and Active Control*, Vol. 41, No. 3, pp. 1234-1257, 2022.
[36] C.-H. HE, C. LIU, J.-H. HE, K. A. GEPREEL, LOW FREQUENCY PROPERTY OF A FRACTAL VIBRATION MODEL FOR A CONCRETE BEAM, *Fractals*, Vol. 29, No. 05, pp. 2150117, 2021.
[37] H. Ma, Fractal variational principle for an optimal control problem, *Journal of Low Frequency Noise, Vibration and Active Control*, Vol. 41, No. 4, pp. 1523-1531, 2022.
[38] J.-H. He, Variational principles for some nonlinear partial differential equations with variable coefficients, *Chaos, Solitons & Fractals*, Vol. 19, pp. 847-851, 03/01, 2004.
[39] X.-Y. Liu, Y.-P. Liu, Z.-W. Wu, Variational principle for one-dimensional inviscid flow, *Thermal Science*, Vol. 26, No. 3 Part B, pp. 2465-2469, 2022.
[40] X.-Q. Cao, B.-N. Liu, M.-Z. Liu, K.-C. Peng, W.-L. Tian, Variational principles for two kinds of non-linear geophysical KdV equation with fractal derivatives, *Thermal Science*, Vol. 26, No. 3 Part B, pp. 2505-2515, 2022.
[41] M.-Z. Liu, X.-Q. Zhu, X.-Q. Cao, B.-N. Liu, K.-C. Peng, Internal solitary waves in the ocean by semi-inverse variational principle, *Thermal Science*, Vol. 26, No. 3 Part B, pp. 2517-2525, 2022.
[42] X.-Y. Liu, Y.-P. Liu, Z.-W. Wu, Optimization of a fractal electrode-level charge transport model, *Thermal Science*, Vol. 25, No. 3 Part B, pp. 2213-2220, 2021.
[43] W.-W. Ling, P.-X. Wu, Variational theory for a kind of non-linear model for water waves, *Thermal Science*, Vol. 25, No. 2 Part B, pp. 1249-1254, 2021.
[44] Y. Shen, C.-H. He, A. A. Alsolami, D. Tian, Nonlinear Vibration With Discontinuities In A Fractal Space: Its Variational Formulation And Periodic Property, *FRACTALS (fractals)*, Vol. 31, No. 07, pp. 1-9, 2023.