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RESEARCH PAPER

Variational principle for Schrödinger-KdV system with the M-fractional derivatives

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Abstract

The variational theory is an inextricable part of both continuum mechanics and physics, and plays an important role in mathematics and nonlinear science, however it is difficult to find a variational formulation for a nonlinear system, and it is more difficult for a fractional differential system. This paper is to search for a variational formulation for the Schrödinger-KdV system with M-fractional derivatives. The fractional complex transformation is used to convert the system into a traditional differential system, and the semi-inverse method is further applied to establish a needed variational principle.

Keywords: Hamilton principle, fractional calculus, chain rule

1. Introduction

The variational principle is of paramount importance in engineering applications[1], it is an inextricable part of both continuum mechanics and physics. The most famous one is the Hamilton principle [2], which can be used to establish governing equations for a complex problem [3-5]. The variational principle is also used to search for an approximate solution [6,7] and numerical solution[8] of a nonlinear differential equation, the most famous ones are Hamiltonian-Based frequency-amplitude formulation for nonlinear vibration systems[6], and the variational iteration method for nonlinear differential equations[9,10], the variational-based finite element method[11].

Recently the variational theory in a fractal space[12,13] became a hot topic in both mathematics and physics, because the basic assumptions in continuum mechanics become totally invalid, however the variational theory still holds, that means the energy conservation and mass conservation and Hamilton principle still work in a fractal space though the physical laws in a fractal space can not be modelled by the differential equations. Variational principles for various solitary waves[14-19] nano/microelectromechanical systems[12], singular waves[13] and microgravity systems[20] in a fractal space were established. In this paper we will search for a variational formulation for Schrödinger-KdV system with M-fractional derivative[21].

2. Schrödinger-KdV system with M-fractional derivatives

The Schrödinger-KdV system is a hot topic in both physics and mathematics[21-23], this paper considers the following Schrödinger-KdV system with M-fractional derivative[21]

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$$iw_t + \lambda_1 w_x + \lambda_2 \left|w\right|^2 w + \lambda_3 wu = 0 \tag{1}$$

$$u_t^\alpha + \sigma_1 u u_x^\beta + \sigma_2 u_x^{3\beta} + \sigma_3 \left(\left|w\right|^2\right)_x^\beta = 0 \tag{2}$$

where the coefficients involved in Eq.(1) are real constants, w is a complex function, while u is a real-valued function. The M-fractional derivative is defined as[21, 24,25]

$$w_t^\alpha = \lim_{\varepsilon \to 0} \frac{w(tE_\gamma(\varepsilon t^{1-\alpha})) - w(t)}{\varepsilon}, \tag{3}$$

$\gamma > 0, 0 < \alpha \le 1.$ where the truncated Mittag-Leffler function is defined as

$$E_\gamma(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k + 1)}, \tag{4}$$

When $\alpha = \beta = 1$, Eqs.(1) and (2) turn to be the traditional Schrödinger-KdV system[22,23]. When $\lambda_3 = 0$, Eq.(1) is the well-known Schrödinger equation for quantum waves[26,27]. When the quantum wave travels through a moving dispersive medium with velocity of u, a dispersive term ($\lambda_3 wu$) has to be considered in Eq.(1), while Eq.(2) describes a KdV-like solitary wave[28]. The dispersive medium can be considered as a fractal space, so the system can be modelled by a fractional differential model, which can also be used to model neural networks[29] and Benjamin-Bona-Mahony equation[30].

3. Fractional complex transformation

The fractional complex transformation was first proposed in 2010 [31], it can be explained as an approximate transformation from a fractal space to a smooth space, in literature it was also called as the two-scale fractal transformation[32], and its geometrical explanation is available in Ref.[33].

According to the fractional complex transformation[31,33], we introduce two new variables (W and U) defined as

$$w = W(\lambda)e^{i\mu}, u = U(\lambda). \tag{5}$$

where the complex variables λ and μ are defined, respectively, as

$$\lambda = \frac{\Gamma(\gamma_1 + 1)}{\beta} k_1 x^\beta + \frac{\Gamma(\gamma_2 + 1)}{\alpha} \rho_1 t^\alpha \tag{6}$$

$$\mu = \frac{\Gamma(\gamma_1 + 1)}{\beta} k_2 x^\beta + \frac{\Gamma(\gamma_2 + 1)}{\alpha} \rho_2 t^\alpha \tag{7}$$

where constants k_1, k_2 and ρ_1, ρ_2 are constants. Substituting Eqs.(5)-(7) into Eqs.(1) and (2) , using the chain role given in Ref.[33], and separating the real part and the imaginary part, we have[20]

$$\lambda_1 k_1^2 W'' - (\rho_2 + k_2^2 \lambda_1) W + \lambda_2 W^3 + \lambda_3 WU = 0 \tag{8}$$

$$\sigma_2 k_1^3 U''' + \sigma_1 k_1 U U' + \rho_1 U' + 2\sigma_3 k_1 W W' = 0 \tag{9}$$

$$(\rho_1 + 2k_1 k_2 \lambda_1) W' = 0 \tag{10}$$

where the prime is the derivative with respect to λ. Combining Eq.(9) and Eq.(10) together, we have

$$\sigma_2 k_1^2 U'' + \frac{1}{2}\sigma_1 U^2 - 2k_2\lambda_1 U + \sigma_3 W^2 = H \tag{11}$$

where H is an integration constant. By the fractional complex transformation[31,33], the fractional Schrödinger-KdV system given in Eqs.(1) and (2) turns out to be the traditional differential system given in Eqs.(8) and (11), so the problem becomes extremely simple.

4. Variational formulation

This paper is to establish a variational formulation for Eqs.(8) and (11). To this end, we first consider a special case when $\sigma_3 = 0$ and $\lambda_3 = 0$, Eqs.(8) and (11) become, respectively, as

$$\lambda_1 k_1^2 W'' - (\rho_2 + k_2^2 \lambda_1) W + \lambda_2 W^3 = 0 \tag{12}$$

$$\sigma_2 k_1^2 U'' + \frac{1}{2}\sigma_1 U^2 - 2k_2\lambda_1 U = H \tag{13}$$

Eq.(11) is the well-known Duffing oscillator[34-37], its variational formulation is

$$J_1(W) = \int \left\{ -\frac{1}{2}\lambda_1 k_1^2 W'^2 - \frac{1}{2}(\rho_2 + k_2^2\lambda_1) W^2 + \frac{1}{4}\lambda_2 W^4 \right\} d\lambda \tag{14}$$

Eq.(12) adopts the following variational formulation

$$J_2(U) = \int \left\{ -\frac{1}{2}\sigma_2 k_1^2 U'^2 + \frac{1}{6}\sigma_1 U^3 - k_2\lambda_1 U^2 - HU \right\} d\lambda \tag{15}$$

The semi-inverse method[38] has to be adopted to search for a variational formulation for cases when $\lambda_3 \neq 0$ and $\sigma_3 \neq 0$.

The semi-inverse method[38], we can construct a trial-functional in the form

$$J(W,U) = kJ_1(W) + J_2(U) + \int F(W,U,W',U') d\lambda \tag{16}$$

where J_1 and J_2 are defined respectively in Eqs.(14) and (15), k is an unknown constant, and F is an unknown function of W and/or U and/or their derivatives.

The trial-functional given in Eq.(16) turns out to be J_1 or J_2 under special conditions. There are many alternative candidates for the trial-functions, see examples in Refs.[39-44].

The Lagrange function in Eq.(16) can be expressed as

$$L(W,U,W',U') = k\left[-\frac{1}{2}\lambda_1 k_1^2 W'^2 - \frac{1}{2}(\rho_2 + k_2^2\lambda_1)W^2 + \frac{1}{4}\lambda_2 W^4\right] + \left[-\frac{1}{2}\sigma_2 k_1^2 U'^2 + \frac{1}{6}\sigma_1 U^3 - k_2\lambda_1 U^2 - HU\right] + F \tag{17}$$

The Euler-Lagrange equations can be written as

$$\frac{\partial L}{\partial W} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial W'}\right) = 0 \tag{18}$$

$$\frac{\partial L}{\partial U} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial U'}\right) = 0 \tag{19}$$

Eqs.(18) and (19) imply that

$$k\left[\lambda_1 k_1^2 W'' - (\rho_2 + k_2^2\lambda_1)W + \lambda_2 W^3\right] + \frac{\delta F}{\delta W} = 0 \tag{20}$$

$$\sigma_2 k_1^2 U'' + \frac{1}{2}\sigma_1 U^2 - 2k_2\lambda_1 U - H + \frac{\delta F}{\delta U} = 0 \tag{21}$$

where the variational derivative is defined as

$$\frac{\delta F}{\delta W} = \frac{\partial F}{\partial W} - \frac{d}{d\lambda}\left(\frac{\partial F}{\partial W'}\right) \tag{22}$$

Eqs.(20) and (21) should be equivalent to Eqs.(8) and (11), to this end, we set

$$\frac{\delta F}{\delta W} = -k\left[\lambda_1 k_1^2 W'' - (\rho_2 + k_2^2\lambda_1)W + \lambda_2 W^3\right] = k\lambda_3 WU \tag{23}$$

$$\frac{\delta F}{\delta U} = -\left[\sigma_2 k_1^2 U'' + \frac{1}{2}\sigma_1 U^2 - 2k_2\lambda_1 U - H\right] = \sigma_3 W^2 \tag{24}$$

According to the consistency of F , it requires

$$\frac{\partial}{\partial U}\left(\frac{\delta F}{\delta W}\right) = \frac{\partial}{\partial W}\left(\frac{\delta F}{\delta U}\right) \tag{25}$$

Eq.(25) implies that

$$k\lambda_3 = 2\sigma_3 \tag{26}$$

From Eq.(26), the unknown parameter (k) can be determined:

$$k = \frac{2\sigma_3}{\lambda_3} \tag{27}$$

From Eqs.(23) and (24), and in view of Eq.(27), F can be identified as

$$F = \frac{1}{2} k\lambda_3 W^2 U = \sigma_3 W^2 U \tag{28}$$

Finally we obtain the following variational formulation

$$J(W,U) = \int L(W,U,W',U') d\lambda = \frac{2\sigma_3}{\lambda_3} J_1(W) + J_2(U) + \int \sigma_3 W^2 U d\lambda \tag{29}$$

where the Lagrange function reads

$$\begin{aligned} L(W,U,W',U') &= \frac{2\sigma_3}{\lambda_3}\left[-\frac{1}{2}\lambda_1 k_1^2 W'^2 - \frac{1}{2}(\rho_2 + k_2^2\lambda_1)W + \frac{1}{4}\lambda_2 W^4\right] \\ &+ \left[-\frac{1}{2}\sigma_2 k_1^2 U'^2 + \frac{1}{6}\sigma_1 U^3 - k_2\lambda_1 U^2 + \sigma_3 W^2 U - HU\right] \end{aligned} \tag{28}$$

5. Conclusion

This paper applies the semi-inverse method to finding a variational formulation for the Schrödinger-KdV system with the M-fractional derivatives.

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