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RESEARCH PAPER

From Data to Stability: A Novel Approach for Controlling Unknown Linear Time-Invariant Systems with Performance Enhancement

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Abstract

A novel data-driven control methodology is introduced in this paper, specifically designed for unknown linear time-invariant systems. Schur stability is established through the application of Linear Matrix Inequality (LMI) conditions, and system performance is improved by leveraging the concept of D-stability. Stability and performance are ensured by incorporating LMI features, with reliance solely on a finite set of collected data, eliminating the necessity for system model identification. Hence, the original performance mapping problem undergoes a transformation into a stability issue, incorporating modified system matrices. Then, the stability condition is formulated within the framework of LMI. The effectiveness of our approach is exemplified through two specific examples, highlighting the significant and impactful results obtained. These examples serve to showcase the practical application and outcomes of our methodology within the defined scope, providing a clear demonstration of its performance and efficacy in addressing relevant scenarios.

Keywords: Data driven control; control design; unknown system; linear matrix inequalities.

1. Introduction

Learning from data is crucial in various scientific fields, serving as the foundation for statistics and artificial intelligence. Its significance is increasingly recognized in engineering, particularly in control engineering.

Data-driven control is a contemporary approach in control engineering that leverages empirical data to design and optimize control systems. Unlike traditional methods that rely on precise mathematical models of a system, data-driven control harnesses the power of collected data to make informed decisions about control strategies. This approach is particularly valuable in scenarios where accurate models of the underlying system dynamics are challenging to obtain. By utilizing techniques such as machine learning and statistical analysis, data-driven control seeks to understand and adapt to the behavior of a system based on real-world observations. This methodology not only offers flexibility in handling complex and nonlinear systems but also allows for adaptive adjustments over time, making it a versatile and promising avenue in the field of control systems design [1].

The central challenge in data-driven control lies in substituting process models with actual data. An initial resolution to this matter, specifically for linear systems, was proposed by Willems et al. [2]. They posit that a linear system can be dynamically characterized through a finite set of system trajectories, granted these trajectories stem from sufficiently excited dynamics. This lemma has been employed, either overtly or implicitly, in the development of data-driven control strategies [3]. As an example, the works [4,5] presented the design of optimal data-driven control tailored to optimize the well-known Linear-Quadratic Regulation (LQR) problem. Additionally, [6] explored the stabilization of switched systems through the application of data-driven control. Also, in [7,8], a robust data-driven state-feedback design has been designed for linear time-invariant systems in the presence of noise. Furthermore, in [9,10], different control design strategies have been presented for discrete-time linear time-varying systems with unknown dynamics, emphasizing identification-free methods. Furthermore, various data-driven model predictive control approaches have been suggested to stabilize unknown systems in [11-13]. In [14], different data-driven model predictive control methods have been proposed for unknown linear time-invariant systems. In [15], design of data driven controllers has been discussed for nonlinear systems.

D-stability, a fundamental concept in control theory, is defined by the placement of eigenvalues within a specific domain of the unit circle in the context of closed-loop control systems. This criterion not only ensures stability but also holds the promise of enhancing the system's performance. The key lies in the strategic positioning of eigenvalues, reflecting the system's robustness against disturbances and uncertainties. By adhering to the principles of **D**-stability, control strategies can be devised to not only maintain stability but also optimize the system's dynamic response, making it a valuable tool in achieving superior performance in control applications [14-18].

Linear Matrix Inequalities (LMIs) play a pivotal role in control theory, offering a versatile framework for addressing stability and performance criteria in control system design. LMIs provide a set of algebraic inequalities involving matrices, and their application allows for the formulation of robust conditions to ensure system stability and desired performance. The appeal of LMIs lies in their ability to encompass a broad range of system dynamics and uncertainties, making them a valuable tool for both linear and nonlinear systems. Their applicability extends to various control strategies, including state feedback, observer design, and optimization problems, making LMIs a foundation in the development of robust and efficient control systems [19-22].

Building upon the advantages of Linear Matrix Inequalities (LMIs) and D-stability in augmenting system performance, this paper is primarily motivated to introduce an innovative approach for stabilizing unknown linear time-invariant systems. This approach capitalizes on the merits of data-driven control, D-stability, and LMIs. Building upon the advantages offered by these methodologies, the proposed stabilization technique contributes to enhance performance for unknown linear time invariant systems. The main contribution of this paper is to present a data driven control ensuring the **D**-stability of the closed-loop system. In summary, the paper introduces several key innovations, outlined as follows:

1. Achieving stability in the closed-loop control system is made possible through the implementation of a novel data-driven approach.
2. Enhancing system performance is accomplished by employing D-stability analysis.

The manuscript is structured as follows: Section 2 offers foundational insights into data-driven control. Moving to Section 3, a data-driven mechanism is introduced, accompanied by an analysis of stability and the improvement of the ensuing closed-loop system using **D**-stability concept. Section 4 is dedicated to the presentation of numerical simulations. The paper is brought to a close with concluding remarks in Section 5.

2. Preliminaries

To devise a data-driven control system, it is imperative to possess an ample amount of data. This necessity underscores the significance of the concept known as the persistence of excitation, which asserts that that we can collect sufficiently rich data. The subsequent definition articulates the concept of the persistence of excitation.

Definition 1: Given a signal $\mathbb{R}^r$, the Hankel matrix can be given as follows:

$$Z_{i,h,j} = \begin{bmatrix} z(i) & z(i+1) & \cdots & z(i+j-1) \\ z(i+1) & z(i+2) & \cdots & z(i+j) \\ \vdots & \vdots & \vdots & \vdots \\ z(i+h-1) & z(i+h) & \cdots & z(i+h+j-2) \end{bmatrix}, i \in \mathbb{Z}, h, j \in \mathbb{N}. \quad (1)$$

For $h=1$, the Hankel matrix is given by (2).

$$Z_{i,j} = \begin{bmatrix} z(i) & z(i+1) & \cdots & z(i+j-1) \end{bmatrix}. \quad (2)$$

Definition 2: The signal $z_{[0,T-1]} : [0, T-1] \cap \mathbb{Z} \to \mathbb{R}^r$ is considered persistently exciting of order L if the matrix $z_{0,L,T-L+1}$ has full rank.

Now, assume that the LTI system is in the following form:

$$x(k+1) = Ax(k) + bu(k), \quad (3)$$

Where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Moreover, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$ are two unknown matrices. Suppose we conducted an experiment of duration $T \in \mathbb{N}$, collecting input and state data $u_{d,[0,T]}$ and $x_{d,[0,T]}$. Consider the corresponding Hankel matrices as follows:

$$\begin{cases} U_{0,T} = [u_d(0) \quad u_d(1) \quad \cdots \quad u_d(T-1)], \\ X_{0,T} = [x_d(0) \quad x_d(1) \quad \cdots \quad x_d(T-1)], \\ X_{1,T} = [x_d(0) \quad x_d(1) \quad \cdots \quad x_d(T-1)]. \end{cases} \tag{4}$$

Lemma 1 [۲]: Assuming the controllability of system (1). If the input signal $u_{d,[0,T-1]}$ is persistently exciting of order $n+1$, then

$$rank \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} = n + m. \tag{5}$$

Condition (5) conveys the idea that the data content is rich enough, facilitating the design of data-driven control. A direct implication of the aforementioned result is that every input-state sequence of the system can be represented as a linear combination of the collected input-state data. As stated in [1], condition (5) can also be employed to parameterize any feedback interconnection, providing additional versatility in its application. Therefore, consider an arbitrary $m \times n$ matrix K. By the Rouche-Capelli theorem, there must exist $T \times n$ matrix F such that

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} F, \tag{6}$$

Where I_n is an identity matrix with dimension $n \times n$. Therefore, one has

$$A + BK = [B \quad A] \begin{bmatrix} K \\ I_n \end{bmatrix} = [B \quad A] \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} F = X_{1,T} F, \tag{7}$$

In which $X_{1,T} = AX_{0,T} + BU_{0,T}$.

In next section, we will introduce the concept of **D**-stability which, which extends the idea of Schur stability.

Definition 3 [۱۴,15]: Let **D** be a domain on the complex plan which is symmetric about real axis. Then, a matrix $\Omega \in \mathbb{R}^{n \times n}$ is considered **D**-stability if

$$\lambda_i(\Omega) \in D, \qquad i = 1, \cdots, n, \tag{8}$$

Where $\lambda_i(\Omega)$ indicates the eigenvalues of the matrix Ω.

Based on **Definition 3**, the **D**-stability concept offers the potential to enhance the system's performance. This is evident as the eigenvalues of the closed-loop control system are situated within a specific domain of the unit circle.

Lemma 2 (Schur complement Lemma) [21]: For any affine matrices $Q(x), S(x)$ and $R(x)$, the following inequalities are equivalent:

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0 \Leftrightarrow \begin{cases} R(x) > 0 \quad Q(x) - S(x)R^{-1}(x)S^T(x) > 0 \\ \qquad\qquad or \\ Q(x) > 0 \quad R(x) - S^T(x)R^{-1}(x)S(x) > 0 \end{cases}$$

3. Main Result

This section, our objective is to formulate a data-driven control utilizing the collected data $U_{0,T}, X_{0,T}$ and $X_{1,T}$.

Furthermore, throughout the paper, the **D**-stability concept is applied to enhance the performance of the closed-loop control system. Moreover, to derive the LMI conditions satisfying the **D**-stability of a matrix, the Lyapunov theorem is used. Based on this theorem, the states $x(k)$ converges to zero if and only if:

1. For all state $x(k)$, the function $V(k)$ is positive definite,
2. For all state $x(k)$, $\Delta V(k) = V(k+1) - V(k) < 0$.

Therefore, in the next lemma at first a **D**-stability condition is derived. To better illustrate the concept of D-stability, **Fig.1** indicates the disc $D_{r_d} = \{\mathrm{Re} + j\,\mathrm{Im} \mid (\mathrm{Re})^2 + (\mathrm{Im})^2 < {r_d}^2\}$ (the grey disc). In this scenario, the D-stability condition necessitates that all eigenvalues of the closed-loop control system reside within the disc region (depicted in gray in **Fig. 1**).

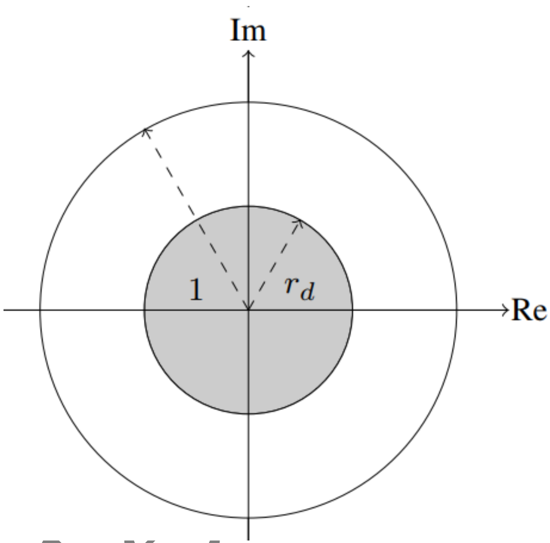

Fig. 1. Disc region D_{r_d} (grey disc).

Lemma 3: Suppose the discrete linear system $x(k+1) = H(k)x(k)$ consist of one eigenvalue situated on the circular path around the origin with a radius of r_d. In such a scenario, it follows that system $x(k+1) = \frac{H(k)}{r_d}x(k)$ possesses at least one eigenvalue positioned on the circular path around the origin with a radius of 1.

Proof:

Let v_i be an eigenvector with the corresponding eigenvalue $H(k)$ with the eigenvalue λ_i. In this context, one can express it as follows:

$$\frac{H(k)}{r_d}v_i = \frac{\lambda_i}{r_d}v_i. \tag{9}$$

Owing to the absolute homogeneity of norms, it is established that if $|\lambda_i| = r_d$ then one has

$$\left|\frac{\lambda_i}{h_d}\right| = \frac{|\lambda_i|}{|h_d|} \to \frac{|r_d|}{|r_d|} = 1. \tag{10}$$

□

In the upcoming theorem, our objective is to formulate a data-driven control strategy utilizing the collected data $U_{0,T}$, $X_{0,T}$ and $X_{1,T}$ in (4). This will be achieved by leveraging the Lyapunov theorem and features of LMIs.

Theorem 1: Let the condition (5) hold. Then, the state feedback controller $K = U_{0,T} X (X_{0,T} X)^{-1}$ can stabilize the unknown system (1), where X is solution of the following LMI:

$$\begin{bmatrix} r_d X_{0,T} X & X_{1,T} X \\ (X_{1,T} X)^T & r_d X_{0,T} X \end{bmatrix} > 0 \quad , 0 < r_d \leq 1. \tag{11}$$

Proof: By the Lyapunov theorem, the first step is to define a positive Lyapunov function. By considering the positive Lyapunov function $V(k) = x(k)^T P x(k), P > 0$, we have

$$\Delta V = x(k)^T \left((A+BK)^T P (A+BK) - P \right) x(k) < 0 \rightarrow (A+BK)^T P (A+BK) - P < 0, \tag{12}$$

Or consequently

$$(A+BK) P (A+BK)^T - P < 0. \tag{13}$$

From (7) and **Lemma 3**, (13) can be rewritten as follows:

$$X_{1,T} F P F^T X_{1,T}^T - r_d{}^2 P < 0. \tag{14}$$

Consider the change of variable $F = XP^{-1}$ and substitute it into (14). This yields the subsequent inequality:

$$X_{1,T} F P P^{-1} P^T F^T X_{1,T}^T - r_d{}^2 P < 0 \rightarrow X_{1,T} X P^{-1} X^T X_{1,T}^T - r_d{}^2 P < 0. \tag{15}$$

Using the **Schur complement Lemma (Lemma 2)** and the equation (7), the LMI (11) can be readily derived from (15).□

Remark 1: According to Theorem 1, the state feedback controller $K = U_{0,T} X (X_{0,T} X)^{-1}$ can stabilize the unknown system. Notably, the methodology advocated in this paper is data-driven control, as it involves utilizing the collected data $U_{0,T}$, $X_{0,T}$ and $X_{1,T}$ within the controller to stabilize the closed-loop control system. Moreover, the parameter $0 < r_d \leq 1$ is employed to enhance the system's performance. If $r_d = 1$, Schur stability is assured. However, this paper presents a generalization of Schur stability.

Remark 2. The methodologies offer advantages in ensuring the stability of the closed-loop control system and enhancing overall system performance. However, a notable drawback is their inapplicability to nonlinear systems. The primary limitation of the methodology lies in the feasibility constraints of the LMI presented in equation (11).

4. Illustrative Examples

Example 1: To demonstrate the effectiveness of the proposed approach, we analyze the following discrete-time dynamical system.

$$\left[A \,\middle|\, B \right] = \left[\begin{array}{cccc|cc} 1 & -0.2077 & 3.1 & -5 & 0.2 & 0.1 \\ -0.05 & 4 & 0 & 1 & 0.3 & 0.1 \\ 1 & 0.2 & -2.1 & 0.4 & 1 & -0.3 \\ 0.1 & 2 & 0.4 & -2 & 0.1 & 0 \end{array} \right]. \tag{16}$$

The system demonstrates open-loop instability. Because the eigenvalues $\lambda_1 = 1.7666, \lambda_2 = -4.3302, \lambda_3 = -3.4882, \lambda_4 = -1.0482$ are not located in the unit circle. The control design process is implemented using MATLAB. We generate datasets with random initial conditions by applying a sequence of length $T = 15$ via the MATLAB command *rand*. We use CVX [23] to solve the optimization problems.

Figs 2 to 5 depict the resulting outputs using the proposed method from Theorem 1 for $r_d = 0.3, 1.$ As evident in the figures, the proposed method for $r_d = 0.3$ demonstrate superior performance in the sense of faster responses and smaller overshoots when compared to $r_d = 1$ [6]. Furthermore, to validate the results presented in Theorem 1, the locations of the closed-loop control system's eigenvalues have been depicted in Fig 6. As observed in this figure, all eigenvalues are situated within a circle with the radius $r_d = 0.3$ (highlighted by red color). This means that the conclusions drawn from Lemma 1 and Theorem 1 hold true.

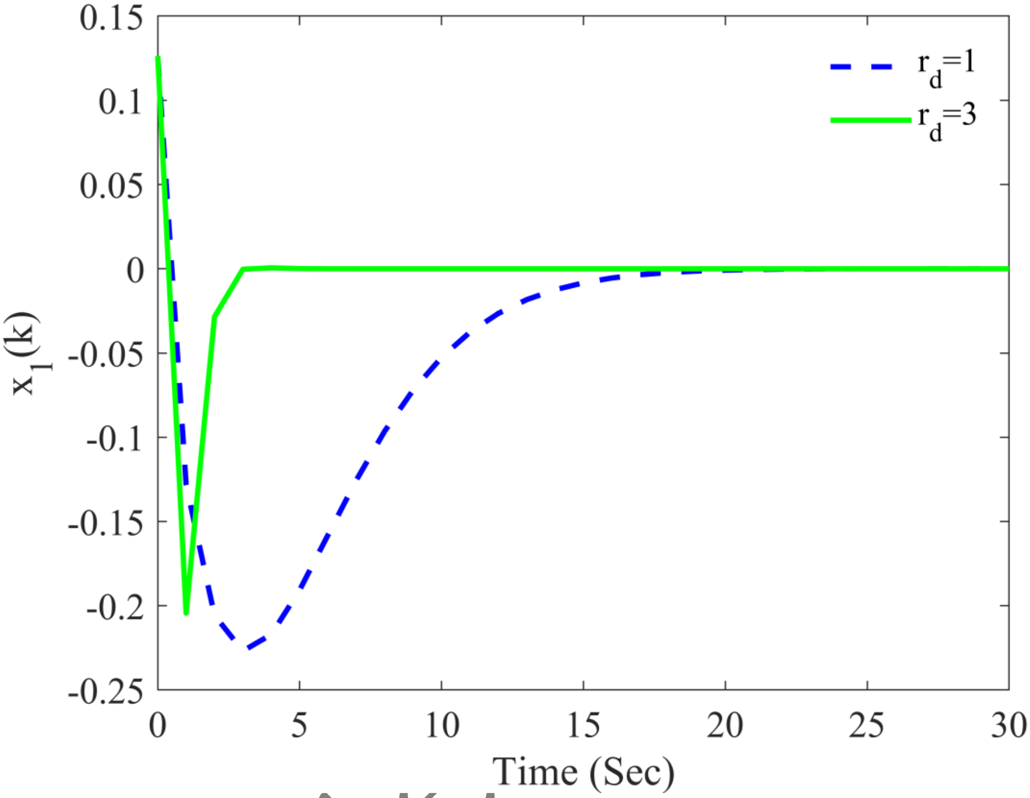

Fig. 2. Trajectory of $x_1(k)$.

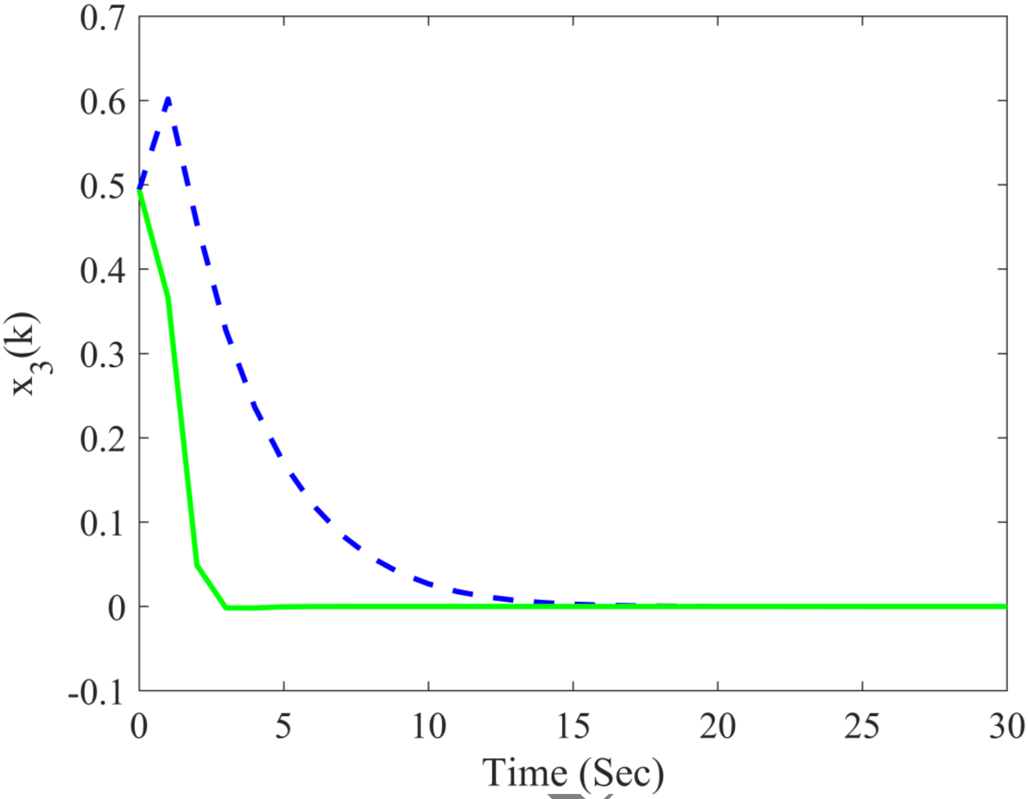

Fig. 3. Trajectory of $x_2(k)$ **.**

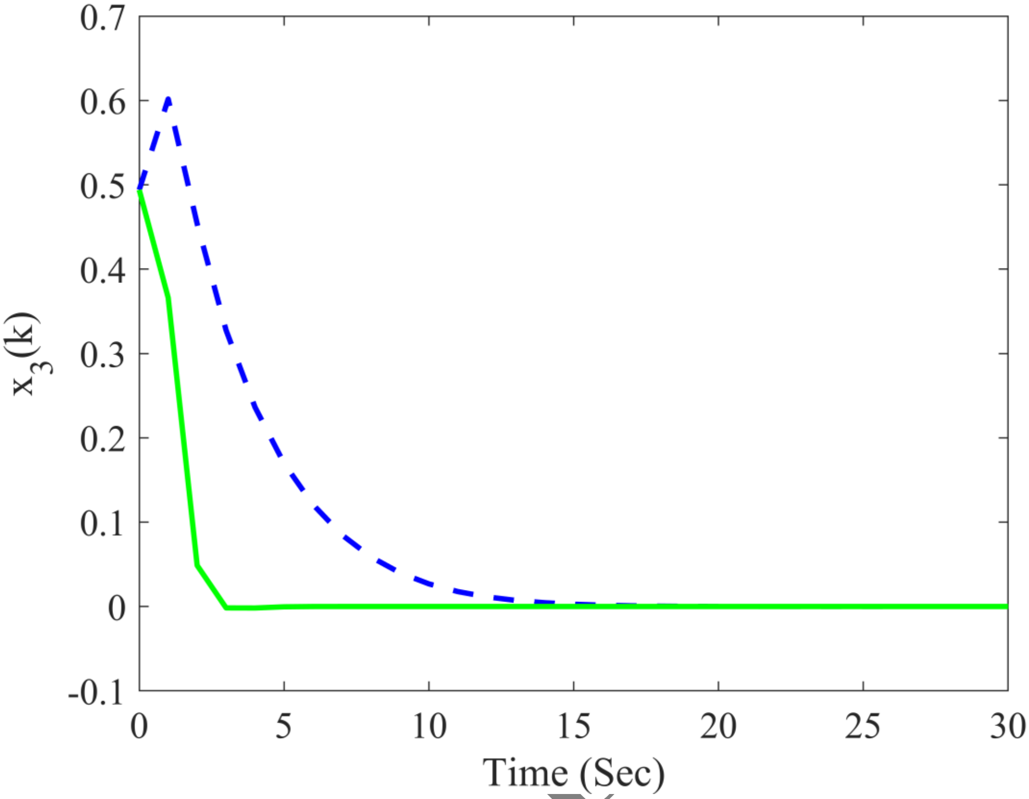

Fig. 4. Trajectory of $x_3(k)$ **.**

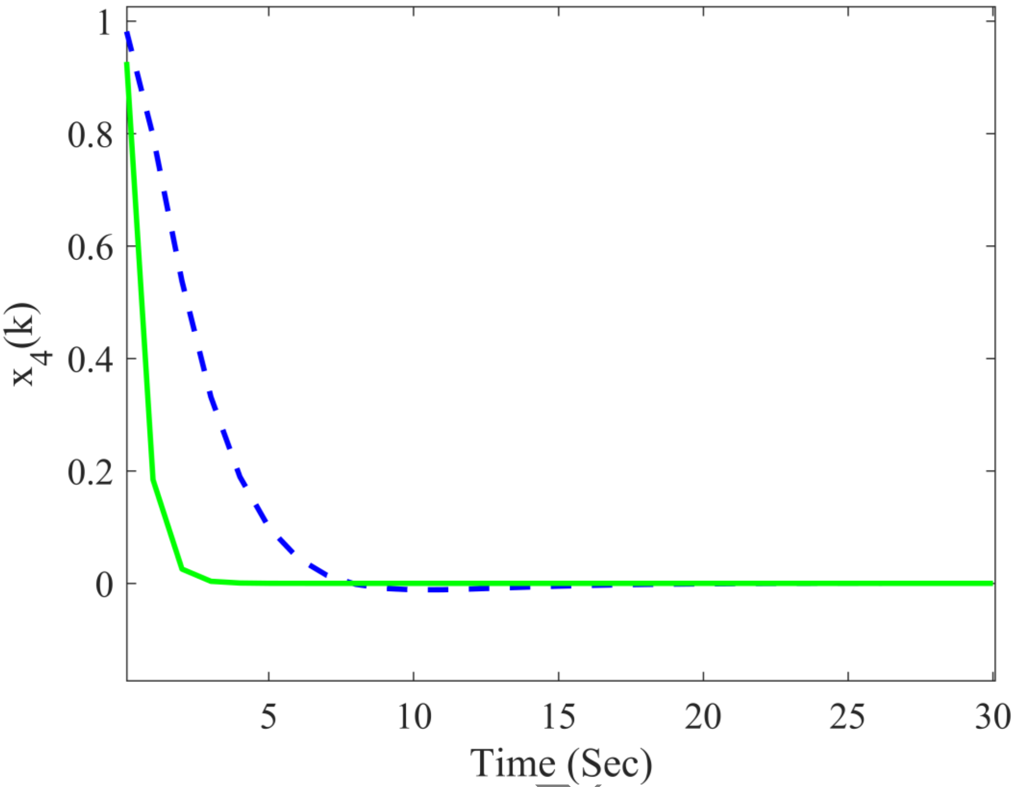

Fig. 5. Trajectory of $x_4(k)$ **.**

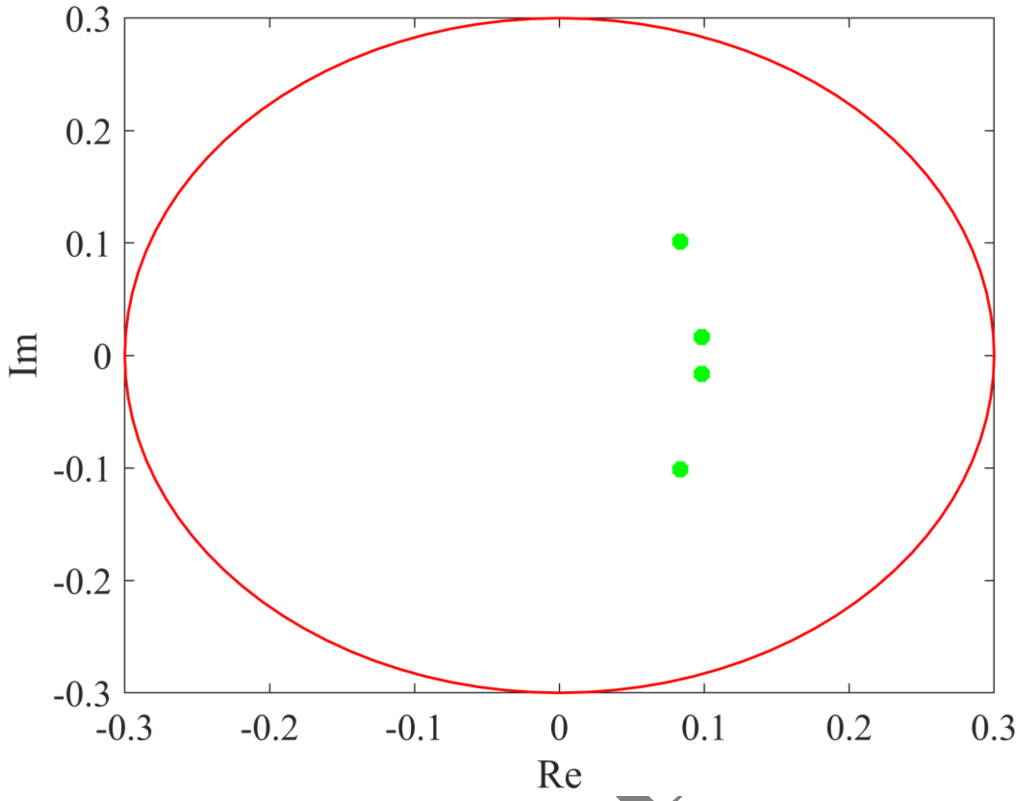

Fig. 6: Location of the eigenvalues in Example 1 (green color) and a circle with the radius $r_d = 0.3$ (red color).

Example 2: Consider a continuous stirred tank reactor system in [26] with a linearization point $[0.9831, 0.3918]^T$ and a sampling time of 0.5 seconds. Subsequently, the following system matrices are derived:

$$\begin{bmatrix} A & \vert B \end{bmatrix} = \left[\begin{matrix} 0.9749 & 0.0135 \\ 0.0004 & 0.9888 \end{matrix} \middle| \begin{matrix} 0.041\times 10^{-4} \\ 5.934\times 10^{-4} \end{matrix}\right]. \quad (17)$$

It can be readily shown that the open-loop system is unstable.

Fig. 7 to Fig. 8 show the trajectories of states $x(k)$ for $r_d = 0.6$. As observed from these figures, all states converge to zero. This implies that the closed-loop control system in stable by applying the data-driven control. In this scenario, the stabilizing controller K is acquired through the application of Theorem 1, as outlined below:

$$K = 10^4 \begin{bmatrix} \text{-5.0006} & \text{-0.1726} \end{bmatrix}. \quad (18)$$

This implies that even though the system is not known, it can be stabilized by implementing the controller (18). Additionally, to corroborate the outcomes stated in Theorem 1, the positions of the system's eigenvalues are illustrated in Fig 9. As depicted, all eigenvalues are enclosed within a circle with a radius $r_d = 0.6$. This verifies the validity of the conclusions derived from Theorem 1.

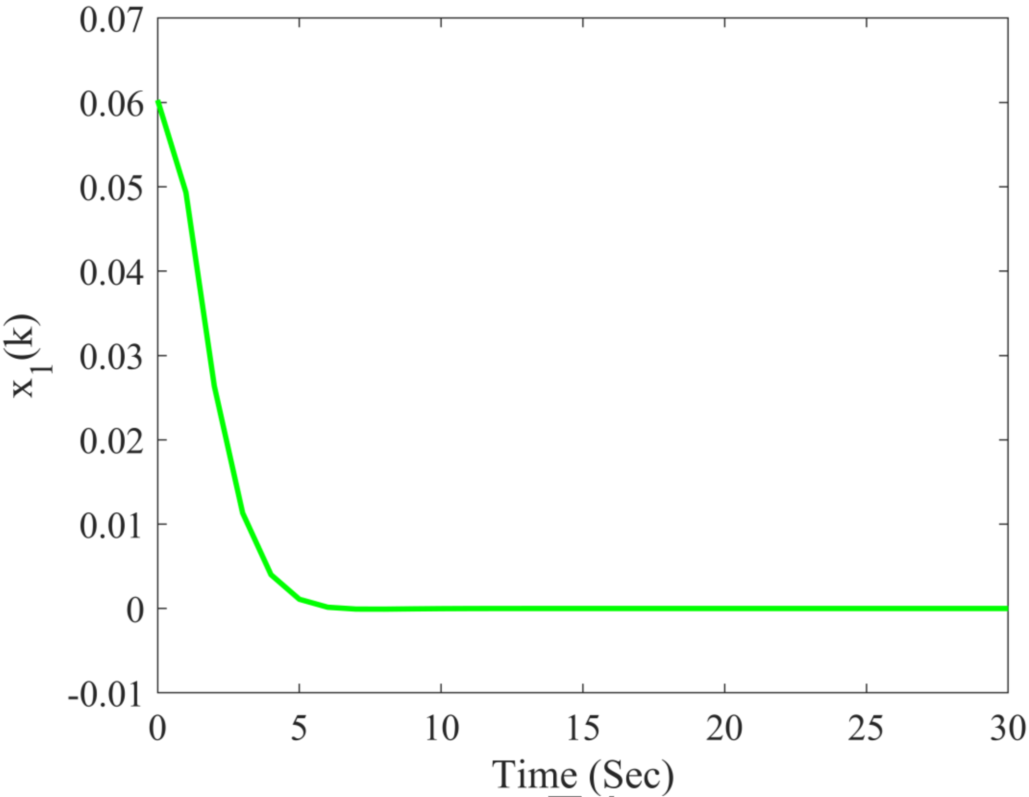

Fig. 7. Trajectory of $x_1(k)$ **.**

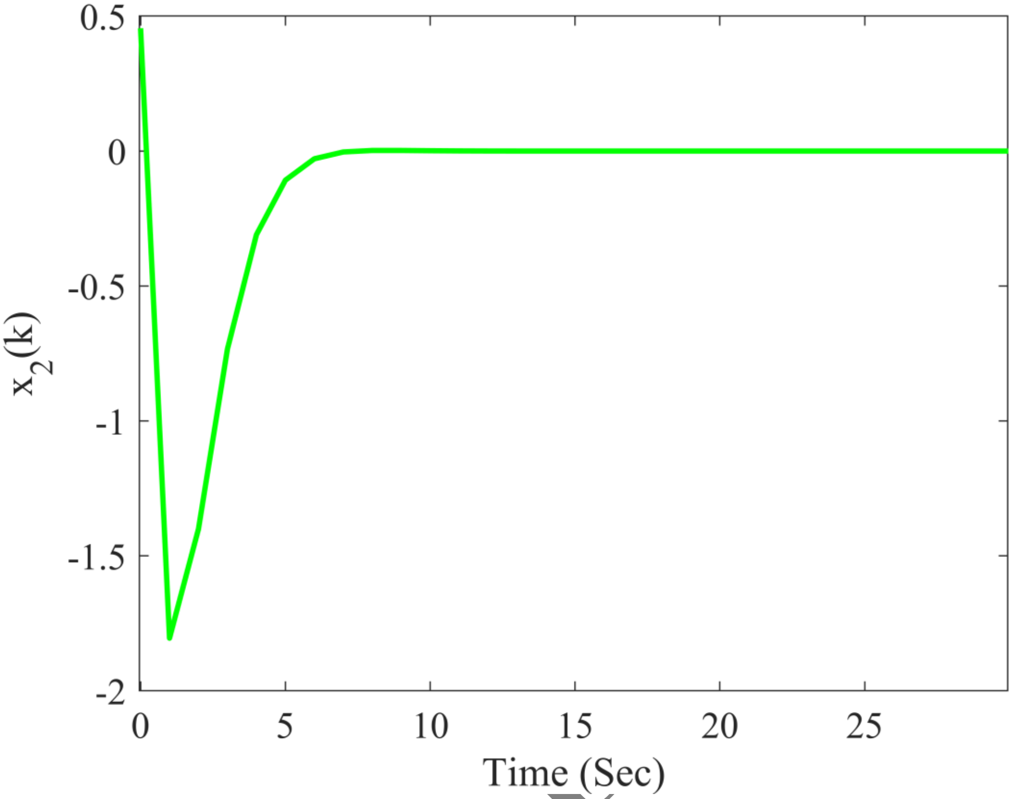

Fig. 8. Trajectory of $x_2(k)$ **.**

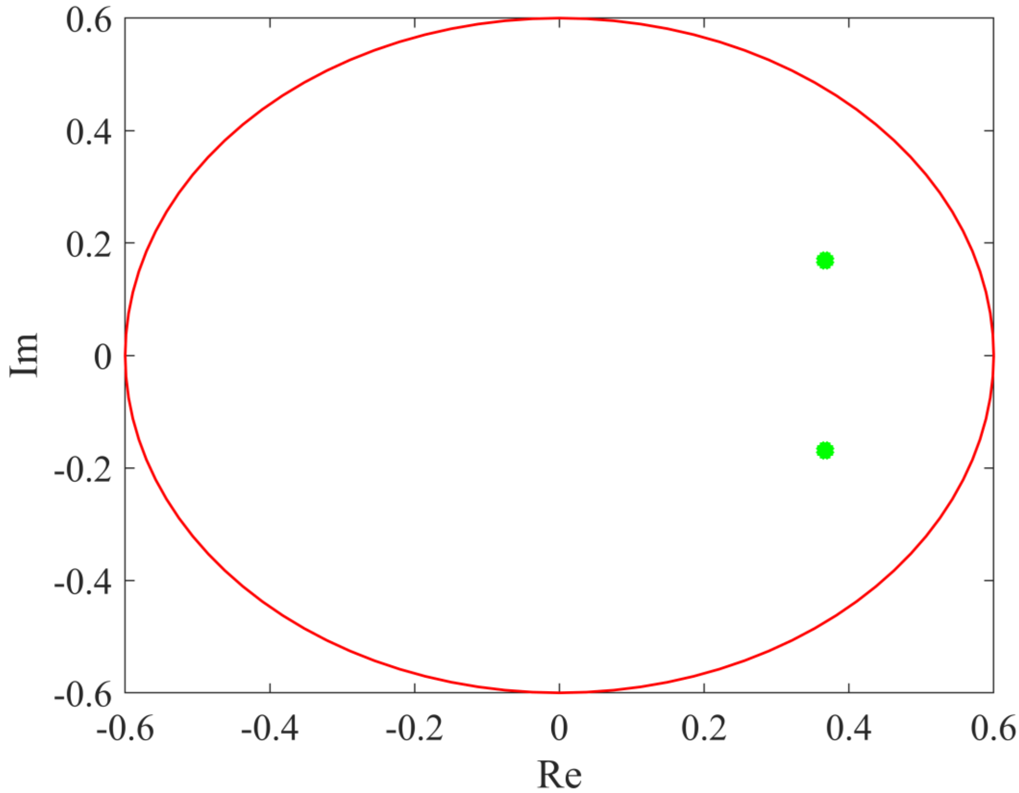

Fig. 9. Location of the eigenvalues in Example 2 (green color) and a circle with the radius $r_d = 0.6$.

5. Conclusion

In this paper, a data-driven control approach was introduced for addressing unknown linear time-invariant systems, operating under the assumption that the system matrices were unknown, with only a set of collected data available. Lemma 1 was formulated to establish a D-stability condition. Additionally, Theorem 1 was presented, wherein a stabilizing state feedback for the unknown system was designed using the Lyapunov theorem and LMIs. Notably, Theorem 1 highlighted the utilization of collected data in the controller design process. Moreover, to improve system performance, integration of D-stability into the derived LMI framework was pursued. The effectiveness of the proposed methodology was illustrated through two examples, showcasing its applicability and impact in practical scenarios. Future work could involve exploring the implementation and adaptation of our proposed findings in the development of data-driven controllers for time-delay systems. Another area for future exploration could involve determining an approach for integrating our proposed outcomes into robust control strategies and real-world practical applications.

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