Journal of Computational Applied Mechanics 2023, 54 (2): 186-203

DOI: 10.22059/JCAMECH.2023.351563.776

RESEARCH PAPER

Exact Analytical Solutions to Bending Problems of SFrSFr Thin Plates Using Variational Kantorovich-Vlasov Method

Charles Chinwuba Ike*

Department of Civil Engineering Enugu State University of Science and Technology, Agbani 402004 Enugu State, Nigeria.

Abstract

This article applies the variational Kantorovich-Vlasov method to obtain exact mathematical solutions to the bending problem of uniformly loaded thin plate with two opposite simply supported edges and two free edges. The studied problem is a common theme in the analysis and design of structures: Vlasov method was adopted simultaneously in the variational Kantorovich method, and the deflection function $w(x, y)$ is expressed in variable-separable form as single infinite series in terms of the unknown function $g(y)$ and known sinusoidal functions of x coordinate variable $f(x)$ where $f(x)$ satisfies Dirichlet boundary conditions at the simple supports. The total potential energy functional $\prod$, expressed in terms of $g(y)$ and the derivatives $g'(y)$, $g''(y)$ is then minimized with respect to $g(y)$ using the Euler-Lagrange differential equations. The resulting equation of equilibrium is a set of fourth order inhomogeneous ordinary differential equations (ODEs) in $g(y)$. The general solution is found as a single series of infinite terms and boundary conditions are enforced to find the integration constants. The single infinite series expression found for $w(x, y)$ satisfies the governing equations at every point on the domain and the boundaries and is thus exact within the scope of thin plate theory adopted to idealize the plate. Moment-deflection equations are used to obtain exact analytical expressions for the bending moments M_{xx}, M_{yy}. Deflection and bending moments are computed at the plate center; as well as at the middle of the free edges. Comparison of the plate center deflections and bending moments for various aspect ratios illustrate that the exact solutions by the present work are in agreement with Levy solutions presented by Timoshenko and Woinowsky-Krieger and symplectic elasticity solutions presented by Cui Shuang. The present results for bending moments at the free edges for various aspect ratios agree with the Levy results presented by Timoshenko and Woinowsky-Krieger and symplectic elasticity results presented by Cui Shuang. The novelty of the work is that to the author's knowledge this is the first application of the variational Kantorovich-Vlasov method to the formulation of exact analytical solution to bending analysis of thin rectangular plates with SFrSFr boundaries.

Keywords: Variational Kantorovich-Vlasov method, total potential energy functional, deflection, bending moment, Euler-Lagrange differential equation, ordinary differential equation, Kirchhoff plate theory.

* Corresponding author. Tel.: +2348033101883. *E-mail address:* charles.ike@esut.edu.ng Orcid ID: 0000-0001-6952-0993

1. Introduction

Plates are three-dimensional (3D) elements that are usually characterized by transverse dimensions that are smaller than the other in-plane dimensions. They are found as component parts of aircrafts, spacecrafts, machines, structures, and foundations; and may be subjected to transverse static, dynamic and/or in-plane compressive/tensile loads. They can be homogeneous or inhomogeneous; isotropic or non-isotropic, elastic or inelastic. They may be composite, sandwich, laminated, functionally graded (FG), flat, curved, Nasihatgozar and Mohammed Reza Khalili [1]; Zargaripoor et al [2]; Cuba et al [3]; Pourabdy et al [4]; Hadiji and Avcar [5]; Daikh and Zenkour [6]; Ton-That [7]; Ramirez et al [8]; Solanki et al [9-11]; Soltani [12]; Soltani and Soltani [13]; Soltani and Asgarin [14].

They can be found in various geometrical shapes such as rectangular, circular, skew, quadrilateral, elliptical, oval, Varghese [15]; Shishesaz et al [16]; Teo and Liew [17].

Depending on the loading type, the plates behaviour can be classified into three broad types: static elasticity, dynamic or stability Hadi et al [18-21]; Hosseini et al [19], Nejad et al [21].

Plates are categorized using the thickness to least inplane dimension ratio as: thin, moderately thick and thick plates. They have been vastly investigated due to their wide and extensive use in engineering Javidi et al [22]; Chan-Dinh and Le-Tran [23], [24-26].

Studies of Mindlin plates have been done by Ike [27-29]; Ike et al [30] and Nwoji et al [31, 32]. Onah et al [33] derived, using theory of three-dimensional (3D) elasticity, stress functions for solving 3D elastostatic problems and used the obtained stress function for the flexural analysis of thick circular plates. The authors in [33] obtained closed form solutions for the internal stresses, and applied the formulation to obtain satisfactory solutions for stresses.

Sayyad and Ghumare [34] have formulated a new quasi 3D model based on fifth order shear and normal shear deformation theory for FG plates. Sayyad and Shinde [35] derived a novel fifth order shear and normal deformation theory for the static and dynamic behaviours of sandwich functionally graded (FG) plates. They solved the resulting equations using Navier's method and obtained accurate solutions for displacements and stresses in the FG plates.

Bathini and Reddy [36] have derived a new higher order shear deformation theory for FG plates which satisfies the transverse shear stress conditions at the top and bottom boundaries and avoids shear modification factors. They validated their formulation with numerical results for perfect and porous FG plates obtained using Navier's double trigonometric series method. Bathini and Reddy [36] derived a refined inverse hyperbolic shear deformation theory for flexure of FG plates, and applied the formulation to obtain satisfactory solutions for stresses.

Khoram et al [37] presented a review of nano-plates and particularly used various plates theories to explain the linear and nonlinear nano-plates.

Rodrigues et al [38] used the Radial Point Interpolation Method (RPIM), a meshless discretization technique to obtain accurate bending solutions of symmetric cross-ply laminated plates using higher order shear deformation theories (HDSTs). They demonstrated the accuracy of the RPIM and proposed new numerical solutions for the flexure of symmetric laminated plates.

Sayyad and Ghugal [39] used the exponential shear deformation theory to study the flexural and natural vibration of thick isotropic, homogeneous, rectangular plates.

Ghaznavi and Shariyat [40] studied the bending of sandwich plates. Reddy et al [41] worked on the flexure of thick plates using HSDT.

Sayyad and Ghugal [42] studied the flexural analysis of thick plates resting on Winkler foundations using the trigonometric shear deformation theory (TSDT) which considers transverse shear and normal strain effects. The TSDT presents transverse shear stress variation across the thickness which occurs with elasticity theory and satisfies the transverse stress-free conditions at the top and bottom surfaces. The principle of virtual work was used to obtain the equilibrium equations and boundary conditions and Navier's series techniques used to procure closed form solutions for simply supported plate boundaries.

Ghugal and Gaj'bhiye [43] formulated a fifth order shear deformation theory which was applied to the bending analysis of thick plates.

Other seminal publications on plates and elasticity include: Ike et al [44].

Shetty et al [45] presented closed-form bending deflection solution for transversely loaded thick beams whose field equations of equilibrium were formulated using third-order simple single variable theory. Their solutions were for simply supported, cantilever and clamped ends. They considered rectangular cross-sections and isotropic materials. Their derived deflection expression illustrated clearly the contributions of flexural and transverse shear deformations to the overall deflection especially as the beam thickness/span ratio increased.

The Galerkin-Vlasov method has been used to solve various thin plate bending, buckling and vibration problems for different boundary and loading conditions by Osadebe et al [45], Nwoji et al [46], Mama et al [47] and Ike [48]. Kantorovich, Kantorovich-Vlasov and other variants of the Kantorovich methods have been applied for the exact solution of thin plate problems under different cases of boundary and loading conditions by Ike and Mama [49], Nwoji et al [50], Onah et al [51, 52], Ike [53-55] and Ike and Nwoji [56].

The finite Fourier integral transform methods have been applied for satisfactorily accurate solutions of thin plate problems by Mama et al [57-60], Ike [61], Ike et al [62, 63], Bidgoli et al [64] and Zhang et al [65].

Ritz variational methods relying on the total potential energy minimization principle has been used for accurate solutions of thin plate problems by Nwoji et al [51]. Ülker and Civalek [66] used the Harmonic Differential Quadrature (HDQ) method for the static, dynamic and stability analysis of plates. Civalek et al [67] used the method of polynomial based differential quadrature (PDQ) for accurate solutions to static, dynamic and buckling problems of rectangular plate.

Direct variational method (DVM) have been applied to plate problems by Aginam et al [68], Onyeka and Mama [69] and Onyeka et al [70-72].

1.1 Review of Integral Transformation Methods for Plate Problems

Integral transformation methods of Laplace, Elzaki, Sumudu, Fourier, Mellin, Bessel, Hankel have been used to solve problems of continua, beams and plates. The methods rely on use of integral kernel functions to convert the governing Boundary Value Problem (BVP) over the solution domains to integral equations (IEs). The choice of the kernel function used defines the transformation.

Ike [61] applied the exponential Fourier integral transformation method to solve the stress analysis problems of boundary loads on semi-infinite elastic continua. Ike [72] applied the Fourier integral transformation method to two-dimensional (2D) plane strain elastiostatic half-plane problem formulated via Love stress functions. Ike [73] has used the Fourier cosine transform to solve 2D elastostatic problems. Ike [74] used the Elzaki transform method to solve the 2D elasticity problems in polar coordinates formulated using Airy stress functions. Ike [74] applied the Mellin transform to solve 2D half-plane elasticity problems in polar coordinates. Ike [75] used the Hankel transformation method to derive stresses in Westergaard half space due to boundary point, line and distributed loadings. Ike [76] used the cosine integral transformation to solve the Westergaard half-space problem.

Ike [77] has used the Fourier-Bessel transformation to solve axisymmetric elastic half space problems. Ike [78] applied the Sumudu transform method to obtain exact solutions to the eigenvalue problem of sinusoidally vibrating Euler-Bernoulli beams.

Ike et al [62] applied the Generalized Integral Transform Method (GITM) to obtain exact flexural and stability solutions for rectangular thin plate with opposite edges clamped, while the other edges are simply supported.

Mama et al [60] studied the single finite Fourier integral transformation for bending solutions of thin rectangular plates under triangular load distribution. Onyia et al [79] have used the single finite Fourier integral transformation to obtain exact buckling solutions to SSSS and SSCFr plates.

Oguaghamba and Ike [80] also used the single finite Fourier integral transformation to obtain exact eigenfrequencies for transversely vibrating thin plates. Oguaghamba et al [30] have used the single finite Fourier integral transformation to solve the eigenvalue eigenvector problems of bi-symmetric thin-walled beams with Dirichlet boundary conditions.

Among several theories of plates the classical Kirchhoff plate theory (KPT) is adopted in this work for proved good results when the plate is thin and transverse shear deformations are neglected. Thin plates are common structural elements in building constructions. KPT is the basis of the structural analysis of thin plates. It is an approximate zeroth-order shear deformation plate theory. The discrepancy between the fourth order of the governing domain equation and the number of boundary conditions is the main demerit of the KPT Delyavskyy and Rosinski, [81]. The theory disregards transverse shear and normal stresses. However for thin plates where the ratio of thickness to the least in-plane dimension, r_t is such that $r_t \leq 0.05$, the KPT gives satisfactory results and sufficiently accurate predictions of interval forces for practical purposes (Delyavskyy and Rosinski, [81].

Timoshenko and Woinowsky-Krieger [82] have presented benchmark solution for the studied problem using Levy method. Cui [83] have used symplectic elasticity method to solve the buckling problem of SFrSFr plates.

This paper focuses on the application of the variational Kantorovich-Vlasov method for exact solutions to the bending of thin plates with two opposite simply supported edges and two free edges. Kantorovich-Vlasov method is adopted in this work because previous use of the method in thin plates done for other support conditions yielded exact solutions. The novelty of the study is that this is the first time the variational Kantorovich-Vlasov method is applied in a fundamental manner to develop exact mathematical solutions for the bending analysis of uniformly loaded thin rectangular plates with SFrSFr boundaries.

1.2 Advantages of the Variational Kantorovich-Vlasov Method (VK-VM)

The VK-VM is adopted in this study due to the following disadvantages:

(i) The method in general gives a rapidly converging single series solution and sis obviously an efficient computational method.

(ii) The method significantly reduces the computational work involved due to the orthogonality properties of the beam eigenfucntions used in the Vlasov modification of the Kantorovich method.

(iii) The VK-VM is closely related to the variational formulation of the finite element method (FEM) due to the total potential energy minimization principle, which is one of its fundamental principles.

(iv) The method is useful in manual solutions of plate problems since few series terms yield satisfactory solution.

(v) The method does not require the level of computer software use as is needed in more complex methods such as the DSC, HQ, HDQ.

(vi) The method is easily extended to thick plates by incorporating transverse stresses and strains in the formulation of total potential energy.

1.3 Disadvantages of the VK-VM

The disadvantages are:

(i) The amount of computational rigour increases significantly for non-homogeneous, plates and plates with variable thickness.

(ii) It is difficult to obtain closed form solutions for plates with complex loadings and boundaries.

2. Methodology

This paper considers a thin rectangular plate with in-plane dimensions *a* and *b* along the *x* and *y* coordinates respectively. The considered SFrSFr thin plate bending problem as shown in Figure 1 is simply supported at $x = 0$, and $x = a$ and free at $y = \pm b/2$.

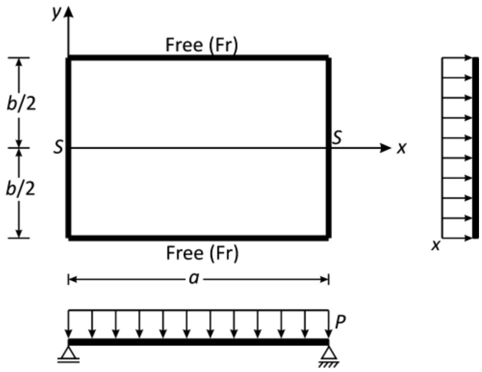


Figure 1. SFrSFr thin plate under uniform loading

The total potential energy functional $\prod$ is expressed by [49]:

$$\prod = U + V \quad (1)$$

where *U* is the strain energy in bending, *V* is the potential energy of loading.

$$\Pi = \frac{D}{2}\int_{(-b/2)}^{b/2}\int_{0}^{a}\left[(\nabla^2 w)^2 + 2(1-\mu)(w_{xy}^2 - w_{xx}w_{yy})\right]dxdy - \int_{(-b/2)}^{b/2}\int_{0}^{a} p(x,y)\,w(x,y)\,dxdy \quad (2)$$

where $w(x, y)$ is the deflection,

$p(x, y)$ is the transverse load distribution,

a and *b* are the in-plane dimensions of the plate in the *x* and *y* coordinate directions respectively,

μ is the Poisson's ratio of the plate,

D is the flexural rigidity of the plate,

∇^2 is the Laplacian,

w_{xy} is the mixed partial derivative of $w(x, y)$ with respect to *x* and *y*,

w_{xx} is the second partial derivative of $w(x, y)$ with respect to *x*,

w_{yy} is the second partial derivative of $w(x, y)$ with respect to *y*.

Let the unknown deflection $w(x, y)$ be assumed in variable-separable form as:

$$w(x,y) = f(x)g(y) \tag{3}$$

where $f(x)$ is a function of x only, $g(y)$ is a function of y only.

2.1 Boundary Conditions

For the considered SFrSFr thin plate, the boundary conditions are:

$$w(x=0,y) = w(x=a,y) = 0 \tag{4a}$$

$$w_{xx}(x=0,y) = w_{xx}(x=a,y) = 0 \tag{4b}$$

$$M_{yy}\left(x=\frac{a}{2}, y=\pm\frac{b}{2}\right) = 0 \tag{4c}$$

$$V_y\left(x=\frac{a}{2}, y=\pm\frac{b}{2}\right) = 0 \tag{4d}$$

where M_y is the bending moment
V_y is the effective shear force

$$M_{yy} = -D(w_{yy} + \mu w_{xx}) \tag{5}$$

$$V_y = -D(w_{yyy} + (2-\mu)w_{xxy}) \tag{6}$$

$$w_{yyy} = \frac{\partial^3 w}{\partial y^3}, \quad w_{xxy} = \frac{\partial^3 w}{\partial x^2 \partial y}.$$

By Kantorovich-Vlasov method a suitable shape function $f(x)$ that satisfies the Dirichlet boundary conditions along the simply supported edges is:

$$f(x) = \sin\frac{n\pi x}{a} \tag{7}$$

$n = 1, 2, 3, 4, \ldots$

n is an integer.

Hence $w(x, y)$ becomes expressed in the single infinite series form:

$$w(x,y) = \sum_{n=1}^{\infty} g(y)\sin\frac{n\pi x}{a} = \sum_{n=1}^{\infty} g(y)\sin\alpha_n x \tag{8}$$

α_n is a parameter defined as:

$$\alpha_n = \frac{n\pi}{a} \tag{9}$$

Let $p(x,y)$ be expressed as the single series given by

$$p(x,y) = \sum_{n=1}^{\infty} p_n \sin\frac{n\pi x}{a} \tag{10}$$

where p_n is the Fourier coefficient of the series representation of $p(x,y)$.

Then $\prod$ becomes:

$$\prod = \frac{D}{2}\sum_{n=1}^{\infty}\int_{-b/2}^{b/2}\int_{0}^{a}\left[(\nabla^2 g(y)\sin\alpha_n x)^2 + 2(1-\mu)\left(\frac{\partial^2}{\partial x \partial y} g(y)\sin\alpha_n x\right)^2 - \left(\frac{\partial^2}{\partial x^2} g(y)\sin\alpha_n x\right)\right.$$
$$\left.\left(\frac{\partial^2}{\partial y^2} g(y)\sin\alpha_n x\right)\right] dxdy - \sum_{n=1}^{\infty}\int_{-b/2}^{b/2}\int_{0}^{a}\sum_{n=1}^{\infty} p_n \sin\alpha_n x\, g(y)\sin\alpha_n x\, dxdy \tag{11}$$

By algebraic simplifications, a modified total potential energy functional $\prod^*$ is constructed as

$$\prod^* = \sum_{n=1}^{\infty}\int_{-b/2}^{b/2}\int_{0}^{a}\left(\sin^2\alpha_n x (g''(y))^2 + \alpha_n^4 \sin^2\alpha_n x (g(y))^2 - 2\alpha_n^2 \sin^2\alpha_n x\, g(y)g''(y) + \right.$$
$$\left. 2(1-\mu)\alpha_n^2\cos^2\alpha_n x (g'(y))^2 + 2(1-\mu)\alpha_n^2 \sin^2\alpha_n x\, g(y)g''(y) - \frac{2p_n}{D}\sin^2\alpha_n x\, g(y)\right) dxdy \tag{12}$$

Let $I_{n_1} = \int_0^a \sin^2 \alpha_n x\, dx$ (13a)

$$I_{n_2} = \int_0^a \cos^2 \alpha_n x\, dx \tag{13b}$$

The functional $\prod^*$ becomes expressed as a functional in $g(y)$ and its derivatives as follows:

$$\prod^* = \int_{-b/2}^{b/2} \Big(I_{n_1}(g''((y))^2 + \alpha_n^4 I_{n_1}(g(y))^2 - 2\alpha_n^2 I_{n_1} g(y) g''(y) + 2(1-\mu)\alpha_n^2 I_{n_2}(g'(y))^2 + 2(1-\mu)\alpha_n^2 I_{n_1} g(y) g''(y) - \frac{2p_n}{D} I_{n_1} g(y) \Big) dy \tag{14}$$

In general,

$$\prod^* = \int_{-b/2}^{b/2} F(g(y), g'(y), g''(y)) dy \tag{15}$$

where the integrand in Equation (15) is:

$$F(g(y), g'(y), g''(y)) = I_{n_1}(g''(y))^2 + \alpha_n^4 I_{n_1}(g(y))^2 - 2\alpha_n^2 I_{n_1} g(y) g''(y) + 2(1-\mu)\alpha_n^2 I_{n_2}(g'(y))^2 + 2(1-\mu)\alpha_n^2 I_{n_1} g(y) g''(y) - \frac{2p_n}{D} I_{n_1} g(y) \tag{16}$$

By integration,

$$I_{n_1} = I_{n_2} = \frac{a}{2} \tag{17}$$

By the Kantorovich-Vlasov method, the equation of equilibrium is obtained by minimizing $\prod^*$ with respect to the unknown $g(y)$. The condition for minimizing $\prod^*$ is the Euler-Lagrange equation.

2.2 Euler-Lagrange Equation

The Euler-Lagrange equation for the problem is given by the differential equation:

$$\frac{\partial F}{\partial g(y)} - \frac{d}{dx}\frac{\partial F}{\partial g'(y)} + \frac{d^2}{dx^2}\frac{\partial F}{\partial g''(y)} = 0 \tag{18}$$

Finding and substituting $\frac{\partial F}{\partial g(y)}$, $\frac{\partial F}{\partial g'(y)}$ and $\frac{\partial F}{\partial g''(y)}$ in Equation (18) gives:

$$2\alpha_n^4 I_{n_1} g(y) - 2\alpha_n^2 I_{n_1} g''(y) + 2(1-\mu)\alpha_n^2 I_{n_1} g''(y) - \frac{2p_n}{D} I_{n_1} - \frac{d}{dx}(2(1-\mu)\alpha_n^2 I_{n_2} 2g'(y)) + \frac{d^2}{dx^2}(2I_{n_1} g''(y) - 2\alpha_n^2 I_{n_1} g(y) + 2(1-\mu)\alpha_n^2 I_{n_1} g(y)) = 0 \tag{19}$$

Simplifying,

$$g^{iv}(y) - 2\alpha_n^2 g''(y) + \alpha_n^4 g(y) - \frac{p_n}{D} = 0 \tag{20}$$

For uniformly distributed load of intensity p_0 over the plate domain, p_n is given from Fourier series theory as:

$$p_n = \frac{2}{a}\int_0^a p_0 \sin \alpha_n x\, dx \tag{21}$$

Evaluating the integral, p_n is obtained as:

$$p_n = \frac{4p_0}{n\pi} \tag{22a}$$

$n = 1, 3, 5, 7, \ldots$

$$p_n = 0 \tag{22b}$$

$n = 2, 4, 6, 8, \ldots$

Hence, the Euler-Lagrange equation is the fourth order inhomogeneous ODE:

$$g^{iv}(y) - 2\alpha_n^2 g''(y) + \alpha_n^4 g(y) = \frac{4p_0}{n\pi D} \tag{23}$$

3. Results

3.1 General Solution to the Euler-Lagrange Equation

The general solution is obtained using the superposition principle as the sum of the homogeneous solution $g_h(y)$ and the particular solution $g_p(y)$.

3.2 Homogeneous Solution $g_h(y)$

The homogeneous solution $g_h(y)$ is the solution to the homogeneous part of the Euler-Lagrange equation

$$g^{iv}(y) - 2\alpha_n^2 g''(y) + \alpha_n^4 g(y) = 0 \tag{24}$$

Let $g(y)$ be assumed in exponential form as:

$$g(y) = e^{sy} \tag{25}$$

where s is a parameter to be found.

Then, $$(s^4 - 2\alpha_n^2 s^2 + \alpha_n^4)e^{sy} = 0 \tag{26}$$

For nontrivial solutions, $e^{sy} \neq 0$.

Hence the characteristic equation is the quartic polynomial in s given by:

$$s^4 - 2\alpha_n^2 s^2 + \alpha_n^4 = 0 \tag{27}$$

$$(s^2 - \alpha_n^2)^2 = 0 \tag{28}$$

The four roots are:

$$s = +\alpha_n \text{ (twice)} \tag{29a}$$

$$s = -\alpha_n \text{ (twice)} \tag{29b}$$

The homogeneous solution $g_n(y)$ is:

$$g_n(y) = A_n \cosh\alpha_n y + B_n\alpha_n y \sinh\alpha_n y + C_n\alpha_n y\cosh\alpha_n y + D_n \sinh\alpha_n y \tag{30}$$

3.3 Particular Solution $g_p(y)$

The particular solution is found from:

$$g_p^{iv}(y) - 2\alpha_n^2 g_p''(y) + \alpha_n^4 g_p(y) = \frac{4p_0}{n\pi D} \tag{31}$$

Let $$g_p''(y) = g_p^{iv}(y) = 0 \tag{32}$$

then $$g_p(y) = \frac{4p_0}{n\pi D\alpha_n^4} \tag{33a}$$

or, $$g_p(y) = \frac{4p_0 a^4}{(n\pi)^5 D} \tag{33b}$$

3.4 General Solution g(y)

The general solution is obtained in terms of four unknown sets of constants as:

$$g(y) = A_n \cosh\alpha_n y + B_n\alpha_n y\sinh\alpha_n y + C_n\alpha_n y\cosh\alpha_n y + D_n \sinh\alpha_n y + \frac{4}{(n\pi)^5}\frac{p_0 a^4}{D} \tag{34}$$

Thus, the general solution for $w(x, y)$ is found as:

$$w(x,y) = \sum_{n=1}^{\infty}(A_n \cosh\alpha_n y + B_n\alpha_n y\sinh\alpha_n y + C_n\alpha_n y\cosh\alpha_n y + D_n \sinh\alpha_n y) + \left(\frac{4}{(n\pi)^5}\frac{p_0 a^4}{D}\right)\sin\alpha_n x \tag{35}$$

3.5 Determination of the Constants of Integration

The constants of integration are found using the boundary conditions on the free edges and symmetry conditions of the problem.

From symmetry in the y direction:

$$g_h(-y) = g_h(y) \tag{36}$$

Using Equation (36) in Equation (34), it is found that

$$C_n = D_n = 0 \tag{37}$$

Hence, $w(x, y)$ becomes:

$$w(x,y) = \sum_{n=1}^{\infty}\left(A_n \cosh\alpha_n y + B_n\alpha_n y\sinh\alpha_n y + \frac{4}{(n\pi)^5}\frac{p_0 a^4}{D}\right)\sin\alpha_n x \tag{38}$$

$n = 1, 3, 5, 7, 9, \ldots$

From the force boundary condition Equation (4c),

$$M_y\left(x = \frac{a}{2}, y = \frac{b}{2}\right) = 0 =$$

$$M_y\left(x=\frac{a}{2}, y=\frac{b}{2}\right) = -D\left\{\sum_{n=1}^{\infty}\left(\alpha_n^2 A_n \cosh\frac{\alpha_n b}{2} + B_n\left(\frac{\alpha_n^3 b}{2}\sinh\frac{\alpha_n b}{2} + 2\alpha_n^2\cosh\frac{\alpha_n b}{2}\right)\right)\sin\frac{\alpha_n a}{2} - \right.$$

$$\left.\mu\left(A_n\cosh\frac{\alpha_n b}{2} + B_n\frac{\alpha_n b}{2}\sinh\frac{\alpha_n b}{2} + \frac{4}{(n\pi)^5}\frac{p_0 a^4}{D}\right)\alpha_n^2\sin\frac{\alpha_n a}{2}\right\} = 0 \tag{39}$$

Simplifying, Equation (39) gives:

$$A_n\alpha_n^2(1-\mu)\cosh\frac{\alpha_n b}{2} + B_n\left((1-\mu)\frac{\alpha_n^3 b}{2}\sinh\frac{\alpha_n b}{2} + 2\alpha_n^2\cosh\frac{\alpha_n b}{2}\right) = \frac{4\alpha_n^2}{(n\pi)^5}\frac{\mu p_0 a^4}{D} \tag{40}$$

From the force boundary condition Equation (4d),

$$V_y\left(x = \frac{a}{2}, y \pm \frac{b}{2}\right) = 0 =$$

$$V_y\left(x = \frac{a}{2}, y \pm \frac{b}{2}\right) = -D\sum_{n=1}^{\infty}\left(A_n\alpha_n^3\sinh\frac{\alpha_n b}{2} + B_n\left(\frac{\alpha_n^4 b}{2}\cosh\frac{\alpha_n b}{2} + 3\alpha_n^3\sinh\frac{\alpha_n b}{2}\right) + \right.$$

$$\left.(2-\mu)\left(-A_n\alpha_n^3\sinh\frac{\alpha_n b}{2} + B_n\left(\frac{\alpha_n^4 b}{2}\cosh\frac{\alpha_n b}{2} + \alpha_n^3\sinh\frac{\alpha_n b}{2}\right)\right)\right)\sin\frac{\alpha_n a}{2} = 0 \tag{41}$$

Simplifying, Equation (41) gives:

$$-A_n\alpha_n^3(1-\mu)\sinh\frac{\alpha_n b}{2} + B_n\left(\frac{\alpha_n^4 b}{2}(3-\mu)\cosh\frac{\alpha_n b}{2} + 4\alpha_n^3\sinh\frac{\alpha_n b}{2}\right) = 0 \tag{42}$$

In matrix format, the system of Equations (40) and (42) becomes:

$$\begin{pmatrix}\delta_{11} & \delta_{12}\\ \delta_{21} & \delta_{22}\end{pmatrix}\begin{pmatrix}A_n\\ B_n\end{pmatrix} = \begin{pmatrix}\dfrac{4\alpha_n^2\mu}{(n\pi)^5}\dfrac{p_0 a^4}{D}\\ 0\end{pmatrix} \tag{43}$$

where δ_{11}, δ_{12}, δ_{21}, δ_{22} are elements of the coefficient matrix in Equation (43).

Thus, $\delta_{11} = \alpha_n^3(1-\mu)\sinh\dfrac{\alpha_n b}{2}$ (44a)

$$\delta_{12} = \frac{\alpha_n^4 b}{2}(3-\mu)\cosh\frac{\alpha_n b}{2} + 4\alpha_n^3\sinh\frac{\alpha_n b}{2} \tag{44b}$$

$$\delta_{21} = -\alpha_n^2(1-\mu)\cosh\frac{\alpha_n b}{2} \tag{44c}$$

$$\delta_{22} = \frac{\alpha_n^3 b}{2}(1-\mu)\sinh\frac{\alpha_n b}{2} + 2\alpha_n^2\cosh\frac{\alpha_n b}{2} \tag{44d}$$

Using Cramer's rule, A_n and B_n are found from:

$$A_n = \frac{\Delta_1}{\Delta} \tag{45a}$$

$$B_n = \frac{\Delta_2}{\Delta} \tag{45b}$$

where Δ_1, Δ_2 and Δ are determinants expressed by:

$$\Delta = \begin{vmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{vmatrix} = \delta_{11}\delta_{22} - \delta_{12}\delta_{21} \tag{46a}$$

$$\Delta_1 = \begin{vmatrix} \frac{4\alpha_n^2\mu}{(n\pi)^5}\frac{p_0 a^4}{D} & \delta_{12} \\ 0 & \delta_{22} \end{vmatrix} = \frac{4\alpha_n^2\mu\delta_{22}}{(n\pi)^5}\frac{p_0 a^4}{D} \tag{46b}$$

$$\Delta_2 = \begin{vmatrix} \delta_{11} & \frac{4\alpha_n^2\mu}{(n\pi)^5}\frac{p_0 a^4}{D} \\ \delta_{22} & 0 \end{vmatrix} = \frac{-4\alpha_n^2\mu p_0 a^4\delta_{21}}{(n\pi)^5 D} \tag{46c}$$

Hence from Equations (45a) and (45b)

$$A_n = \frac{4\alpha_n^2\mu}{(n\pi)^5}\frac{p_0 a^4}{D}\left(\frac{\delta_{22}}{\delta_{11}\delta_{22} - \delta_{12}\delta_{21}}\right) \tag{47a}$$

$$B_n = \frac{-4\mu\alpha_n^2}{(n\pi)^5}\frac{p_0 a^4}{D}\left(\frac{\delta_{21}}{\delta_{11}\delta_{22} - \delta_{12}\delta_{21}}\right) \tag{47b}$$

3.6 Bending moments M_{yy}, M_{xx}

The bending moment expression for M_{yy} is found using the moment-deflection equations as:

$$M_{yy} = -D\left\{\sum_{n=1}^{\infty} -\mu\left(A_n\cosh\alpha_n y + B_n\alpha_n y\sinh\alpha_n y + \frac{4}{(n\pi)^5}\frac{p_0 a^4}{D}\right)\alpha_n^2\sin\alpha_n x + \right.$$
$$\left.(A_n\alpha_n^2\cosh\alpha_n y + B_n\alpha_n(\alpha_n^2 y\sinh\alpha_n y + 2\alpha_n\cosh\alpha_n y)\sin\alpha_n x\right\} \tag{48a}$$

where A_n and B_n are given by Equation (47a) and (47b).
Similarly,

$$M_{xx} = -D\left\{\sum_{n=1}^{\infty}\left(A_n\cosh\alpha_n y + B_n\alpha_n y\sinh\alpha_n y + \frac{4}{(n\pi)^5}\frac{pa^4}{D}\right)\alpha_n^2\sin\alpha_n x + \right.$$
$$\left.\mu(A_n\alpha_n^2\cosh\alpha_n y + B_n\alpha_n(\alpha_n^2 y\sinh\alpha_n y + 2\alpha_n\cosh\alpha_n y)\sin\alpha_n x\right\} \tag{48b}$$

3.7 Effective shear force V_y

The effective shear force V_y is found as:

$$V_y = -D\left\{\sum_{n=1}^{\infty}(A_n\alpha_n^3\sinh\alpha_n y + B_n\alpha_n(\alpha_n^3 y\cosh\alpha_n y + 3\alpha_n^2\sinh\alpha_n y))\sin\alpha_n x + \right.$$
$$\left.(2-\mu)(A_n\alpha_n\sinh\alpha_n y + B_n\alpha_n(\alpha_n y\cosh\alpha_n y + \sinh\alpha_n y)(-\alpha_n^2\sin\alpha_n x)\right\} \tag{49}$$

where A_n and B_n are given by Equations (47a) and (47b).
Then the deflection is determined by substituting Equations (47a) and (47b) in Equation (38).
The deflection at the centre $(x = a/2, y = 0)$ is found as the single infinite series:

$$w\left(x = \frac{a}{2}, y = 0\right) = \sum_{n=1}^{\infty}\left(A_n + \frac{4}{(n\pi)^5 D}p_0 a^4\right)\sin\frac{n\pi}{2} \tag{50}$$

Similarly the expressions for bending moments at the center $(x = a/2, y = 0)$ are found as:

$$M_{yy}\left(x=\frac{a}{2}, y=0\right) = -D\sum_{n=1}^{\infty}(\alpha_n^2 A_n + 2B_n\alpha_n^2)\sin\frac{n\pi}{2} + \mu\alpha_n^2\left(-\left(A_n + \frac{4}{(n\pi)^5}\frac{p_0a^4}{D}\right)\right)\sin\frac{n\pi}{2} \tag{51}$$

$$M_{yy}\left(x=\frac{a}{2}, y=0\right) = -D\sum_{n=1}^{\infty}\left[A_n\alpha_n^2(1-\mu) + 2B_n\alpha_n^2 - \frac{4\mu\alpha_n^2}{(n\pi)^5}\frac{p_0a^4}{D}\right]\sin\frac{n\pi}{2} \tag{52}$$

$$M_{xx}\left(x=\frac{a}{2}, y=0\right) = -D\sum_{n=1}^{\infty}\left[-\left(A_n + \frac{4}{(n\pi)^5}\frac{p_0a^4}{D}\right)\alpha_n^2 + \mu(A_n\alpha_n^2 + 2B_n\alpha_n^2)\right]\sin\frac{n\pi}{2} \tag{53}$$

Simplification gives:

$$M_{xx}\left(x=\frac{a}{2}, y=0\right) = -D\sum_{n=1}^{\infty}\left[\alpha_n^2 A_n(\mu-1) - \frac{4\alpha_n^2}{(n\pi)^5}\frac{p_0a^4}{D} + 2\mu B_n\alpha_n^2\right]\sin\frac{n\pi}{2} \tag{54}$$

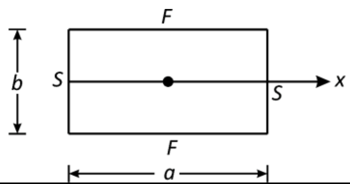
At the center of the free edges $x = a/2$, $y = \pm b/2$,

$$M_{xx}\left(x=\frac{a}{2}, y=\pm\frac{b}{2}\right) = -D\sum_{n=1}^{\infty}\left\{-\left(A_n\alpha_n^2\cosh\frac{\alpha_n b}{2} + B_n\frac{\alpha_n^3 b}{2}\sinh\frac{\alpha_n b}{2} + \frac{4\alpha_n^2}{(n\pi)^5}\frac{p_0a^4}{D}\right) + \mu\left\{\left(A_n\alpha_n^2\cosh\frac{\alpha_n b}{2} + B_n\left(\frac{\alpha_n^3 b}{2}\sinh\frac{\alpha_n b}{2} + 2\alpha_n^2\cosh\frac{\alpha_n b}{2}\right)\right)\right\}\sin\frac{n\pi}{2}\right. \tag{55}$$

Similarly,

$$M_{yy}\left(x=\frac{a}{2}, y=\pm\frac{b}{2}\right) = -D\sum_{n=1}^{\infty}\left\{\left(A_n\alpha_n^2\cosh\frac{\alpha_n b}{2} + B_n\left(\frac{\alpha_n^3 b}{2}\sinh\frac{\alpha_n b}{2} + 2\alpha_n^2\cosh\frac{\alpha_n b}{2}\right)\right) + \mu\left(-\left(A_n\alpha_n^2\cosh\frac{\alpha_n b}{2} + B_n\frac{\alpha_n^3 b}{2}\sinh\frac{\alpha_n b}{2} + \frac{4\alpha_n^2}{(n\pi)^5}\frac{p_0a^4}{D}\right)\right)\right\}\sin\frac{n\pi}{2} \tag{56}$$

Table 1: Deflections and bending moment coefficients at the center ($x = a/2$, $y = 0$) of FrSFrS (or SFrSFr) plate for uniformly distributed load over the plate domain ($a \times b$), for $\mu = 0.30$



$r = a/b$	$W = \alpha p_0 b^4/D$ Timoshenko and W-Krieger [82] α	Present work / Cui Shuang [83]	$M_{xx} = \beta_{xx}qb^2$ Woinowsky-Krieger [82]	Present work / Cui Shuang [83]	$M_{yy} = \beta_{yy}p_0b^2$	Present work / Cui Shuang [83]
2/3	0.0026500	0.0025477	0.0579453	0.0546	0.01952385	0.0151
1.0	0.0132372	0.0130940	0.1253520	0.1225	0.03237516	0.0271
1.5	0.0682493	0.0681020	0.2787449	0.2769	0.04585265	0.0407
2	0.2195266	0.2194097	0.4955701	0.4945	0.05374628	0.0486
3	1.1333532	1.1334448	1.1185097	1.1186	0.06133887	0.0552
4	3.6137824	3.6144728	1.9924895	1.9934	0.06454648	0.0570
5	8.8627411	8.8646689	3.1168416	3.1183	0.06615132	0.0575

Table 2: Deflections and bending moment coefficients for uniformly loaded SFrSFr rectangular thin plate $a \times b$ with $\mu = 0.30$ at the midpoint of the opposite free edges ($x = a/2$, $y = \pm b/2$)

a/b	$W = \alpha p_0 b^4/D$ Timoshenko and W-Krieger [82] α	Cui Shuang [83][114] α	Present work /α	$M_{xx} = \beta_{xx} p_0 b^2$ Present work	Cui Shuang [83]	Timoshenko and Woinowsky-Krieger [82]	$M_{yy} = \beta_{yy} p_0 b^2$ Present work / Cui Shuang [83]
2/3	0.002780343	0.00299418	0.00299418	0.05465043	0.0588431	0.05465	0
1.0	0.014719066	0.01501126	0.01501126	0.12858577	0.1310877	0.1286	0
1.5	0.074561129	0.07489906	0.07489906	0.28949601	0.2905851	0.2895	0
2	0.233901086	0.23431397	0.23431397	0.81083997	0.5162501	0.8108	0
3	1.172559187	1.17335261	1.17335261	1.13816411	1.1378446	1.1382	0
4	3.688649698	3.69022839	3.69022839	2.01400273	2.0132905	2.0140	0
5	8.983764227	8.98672614	8.98672614	3.13928657	3.1384141	3.1393	0

4. Discussion

This article has derived exact mathematical solutions for the flexural analysis of SFrSFr thin plates under uniformly distributed load using Kantorovich-Vlasov method. The problem has been challenging due to difficulties associated with the free edges. Hence, few solutions exist for the considered SFrSFr plate bending problem. The thin rectangular plate solved as shown in Figure 1 is simply supported at the boundaries $x = 0$, $x = a$, and free at $y = \pm b/2$.

The solution is derived using Kantorovich-Vlasov methodology which relies on deriving the deflection basis functions that minimize or extremize the total potential energy functional, $\prod$, for the plate flexure problem and at the same time satisfy the geometric and force boundary conditions identically along all the boundaries of the solution domain. The total potential energy functional $\prod$ expression given by Equation (2) for the general plate flexure problem is a function of the transverse deflection $w(x, y)$.

The Cartesian coordinate system for the problem was chosen to fruitfully exploit the benefits of the symmetry of the plate and the loading; as shown in Figure 1. The spatial view of the studied problem is show in Figure 2.

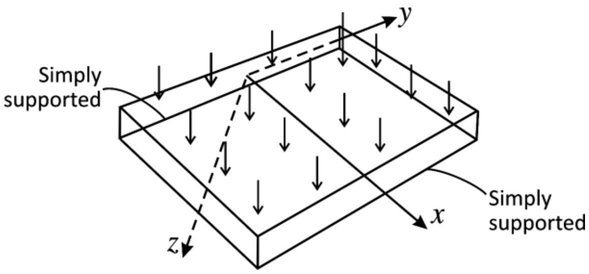


Figure 2: Spatial view of SFrSFr plate bending problem studied

The unknown transverse deflection function $w(x, y)$ is assumed in coordinate variable-separable form as the products of two unknown displacement coordinate (basis) functions $f(x)$ and $g(y)$ given in Equation (3). The geometric and force boundary conditions required to be satisfied are given by Equation (4). Vlasov methodology is adopted in choosing the function $f(x)$ that satisfies the Dirichlet boundary conditions along the edges $x = 0$, $x = a$, as Equation (7). Kantorovich methodology is then adopted in construction of the unknown deflection $w(x, y)$ as the infinite series expression given by Equation (8) where the unknown function $g(y)$ is then sought to minimize $\prod^*$ and simultaneously satisfy the boundary conditions at the free edges ($y = \pm b/2$). The distributed load $p(x, y)$ is similarly expressed in single infinite series for using Fourier series theory as Equation (10) where the Fourier coefficients are determined from the given transverse load distribution. The total potential energy functional $\prod$ is then expressed as a simpler functional given by Equation (14) which is obtained after some algebraic simplifications of Equation (12).

The Euler-Lagrange Equation for minimizing the functional $\prod^*$ with respect to the sought function $g(y)$ is given by Equation (18); which is written explicitly for the considered problem as Equation (19). Simplifications using algebraic process and use of the Fourier series theory results in the Euler-Lagrange differential equation as the fourth order inhomogeneous ODE given by Equation (23); which is expressed in terms of $g(y)$.

The general theory for solving ODEs is used to determine the solution for $g(y)$ as the superposition of the homogeneous solution $g_h(y)$ and the particular solution $g_p(y)$.

The homogeneous solution $g_h(y)$ is found as Equation (30) using the method of trial functions for an exponential trial function solution.

Similarly, the particular solution is obtained as Equation (33b). Hence the general solution $g(y)$ which is the sum of the homogeneous solution and the particular solution is obtained as Equation (34) for $g(y)$. Consequently, the general solution for $w(x, y)$ is found as Equation (35); a single infinite series with unknowns which are the integration constants (A_n, B_n, C_n and D_n).

The integration constants are determined using the boundary conditions on the free edges ($y = \pm b/2$).

The symmetrical nature of the SFrSFr plate and the symmetry of the load demands that $g(y)$ be a symmetrical function. Hence, $g(y)$ is required to satisfy Equation (36).

Enforcement of symmetry of $g(y)$ leads to the vanishing of the two sets of integration constants C_n and D_n as given by Equation (37).

The deflection $w(x, y)$ is then found as Equation (38), a single infinite series with two sets of integration constants (A_n, B_n).

Enforcement of the force boundary conditions at the center of the free edges ($x = a/2$, $y = \pm b/2$) gives the system of equations given after simplifications by Equations (40) and (42).

In matrix form, Equations (40) and (42) are given as Equation (43). Cramer's rule is employed to determine the unknowns A_n, B_n as Equations (47a) and (47b) respectively. The deflection is then fully determined when A_n and B_n are substituted using Equations (47a) and (47b) in Equation (38). The deflection expression is an infinite single series which is constructed/derived to satisfy the equations of equilibrium at all points on the plate domain $(0 \le x \le a;\ \ -b/2 \le y \le b/2)$ as well as the geometric and force boundary conditions along the four edges $(x = 0, x = a,\ \ y = \pm b/2)$. The deflection expression thus solves the SFrSFr plate flexure problem exactly.

Similarly, exact expressions for the bending moment M_{yy} and effective shear force V_y are derived as the single infinite series given by Equations (48) and (49).

The obtained expressions for $w(x, y)$, $M_{xx}(x, y)$, $M_{yy}(x, y)$ are validated by using them to find the deflections and bending moments at the plate center ($x = a/2$, $y = 0$) and the center of the free edges ($x = a/2$, $y = \pm b/2$) for various assumed aspect ratios ($r = a/b$).

The results are compared with results obtained by Timoshenko and Woinowsky-Krieger [114] and Cui Shuang [115]. Table 1 presents the deflections and bending moment coefficients at the center of SFrSFr plate for uniformly distributed load $\mu = 0.30$, and various aspect ratios ($r = a/b$) as well as previous results obtained by Timoshenko and Woinowsky-Krieger [114] and Cui Shuang [115]. Table 1 illustrates that the present results are identical with previously obtained results, even though Cui Shuang used sympletic elasticity method and Timoshenko and Woinowsky-Krieger (1959) used Levy method.

Similarly, the deflections and bending moment coefficients evaluated at the middle of the free edges ($x = a/2$, $y = \pm b/2$) for the studied problem are presented for various aspect ratios ranging from $r = 2/3$ to $r = 5$ (for $\mu = 0.30$) in Table 2. The results as illustrated in Table 2 agree with previous results obtained by Timoshenko and Woinowsky-Krieger [114] and Cui Shuang [115].

5. Conclusion

In conclusion,

(i) Vlasov method was adopted simultaneously with the variational Kantorovich method to express the desired transverse deflection function in coordinate variable separable form as an infinite single series in terms of an unknown function $g(y)$ and known sinusoidal function in the x direction that satisfy the Dirichlet boundary conditions at the simple supports.

(ii) The total potential energy functional $\prod$ is expressed in terms of the function $g(y)$ and the derivatives $g'(y)$, and $g''(y)$.

(iii) The functional $\prod$ is then minimized with respect to $g(y)$ using the Euler-Lagrange differential equations, and fourth order inhomogeneous ODE results expressing the differential equation of equilibrium.

(iv) The general solution for $g(y)$ is obtained using the methods for solving ODEs and the principle of superposition.

(v) Enforcement of boundary conditions leads to the determination of the integration constants and hence the deflection function.

(vi) The moment-deflection equations are used to find the bending moments.

(vii) Bending moment expressions are found for M_{xx}, M_{yy} as infinite single series.
(viii) The expression for $w(x, y)$ is the exact solution for thin plate problem since $w(x, y)$ satisfies the governing equations at all parts on the plate domain and on the boundaries.
(ix) Similarly the expressions obtained for the bending moments are exact within the framework of the Kirchhoff plate theory.
(x) The deflections and bending moments obtained at the plate center and the center of free edges are identical with results presented by Timoshenko and Woinowsky-Krieger [82] and Cui Shuang [83].

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