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RESEARCH PAPER

Study of a Half-Space Via a Generalized Dual Phase-lag Model with Variable Thermal Material Properties and Memory-Dependent Derivative

A. Soleiman [1,*], K.M.Khalil [1,2], M.E.Nasr [1,2], E. Awwad [1,2], F.A. Mohammed [1,3]

[1] *Department of Mathematics, College of Science and Arts, Jouf University, Gurayat, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt*
[3] *Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt*

Abstract

The aim of this work is to investigate a generalized dual phase-lag model with variable thermal material parameters and memory dependent derivatives (VMDPL). In view of this model, the thermoelastic behavior on a half-space under an external body force and subjected to exponentially varying heat is analytically investigated. The governing differential equations are numerically solved using the Laplace transform approach. The effects of the variable thermal material properties and memory dependent derivative on all the physical quantities of a half-space are discussed. The obtained results demonstrate that the physical fields of a half-space depend not only on the distance, but also on the memory time delay and the variable thermal parameter. Furthermore, the variable thermal parameter and the variable thermal parameter has a clear effect on the temperature and the stress but has a negligible effect on the displacement. Finally, the validity of results is acceptable by comparing the displacement, stress and temperature according to the present generalized model (VMDPL) with those due to other thermoelasticity theories

Keywords: Thermoelasticity; Phase-Lags; Variable Thermal Material Properties; Memory Dependent Derivative.

1. Introduction

The classical uncoupled thermoelectricity theory (UCTE) is considered to depend on Fourier 's thermal conduction law and does not address physical structures and materials, such as amorphous media, glassy, human-made porous materials, polymers and colloids. Biot[1] proposed the coupled theory of thermoelasticity model to solve this problem (CTE). Classical models for heat transmission anticipate an infinite speed due to the nature of the parabolic-type heat equation. Cattaneo [2] suggested a generalization of Fourier's law of heat conduction in terms of relaxation time τ_0 to achieve the finite speed of thermal spread waves. Lord and Shulman [3], Green and Lindsay [4], Green and Naghdi [5-7], and Tzou [8] developed generalized models of thermoelasticity to overcome the UCTE and CTE models' absurdity of limitless thermal wave speed. In the temperature gradient and heat flux, Tzou [9] added two separate parameters τ_q and τ_θ, referred to as the phase-lags of the traditional Fourier law, respectively.

For two decades many researchers proved that fractional-order derivatives models have many applications in multi-domain, such as power-law phenomena in fluids, viscoelastic mechanics, ecology, allometric scaling laws in biology, complex networks, colored noise, polarization, electrode-electrolyte and fractional kinetics, dielectric polarization, boundary layer effects, and electromagnetic waves. Using fractional derivatives models for the explanation of viscoelastic materials and proof of the connection between the linear viscoelasticity theory and fractional derivatives, Caputo and Mainardi [10, 11] discovered that there is no conflict with practical results.

* Corresponding Author. Tel.: +966559150980;
Email Address: asoliman@ju.edu.sa , amrsoleiman@yahoo.com

In [12], the author developed a novel fractional generalized thermoelasticity theory based on fractional order heat conduction, which is backed by the corresponding uniqueness theorem. Povstenko [13] looked into the generalized Cattaneo-type model with fractional derivatives of time and came up with a thermal stress model. Some studies relating to generalized local thermoelasticity theories with fractional derivative investigated by [14-16]. Based on fractional computation and Taylor series expansions for time-fractional order, Abouelregal [17] proposes a new model of generalized thermoelasticity as a function of multi-relaxation durations. Similarly, in [18], he provided a modified model of heat conduction that involved a higher order of time derivative and extended Green and Naghdi's theory without energy dissipation. On the other hand, nonlocal elasticity theory is a convenient methodology for considering the small-scale effects that are exhibited by nanoscopic structures. In recent decades, the use of nonlocal elasticity theory in mechanical modelling of these structures has seen an inflationary development forces between atoms. Furthermore, many studies have been investigated by using the nonlocal (local) elasticity theory [19-52].

In the last few decades, it has become evident that the next state of the physical system depends not only on its present state but also on all its historical ones. Wang and Li established the concept of memory dependent derivatives in their paper [53]. This new sort of derivative turned out to be a useful mathematical tool and a missing link in a number of physical situations. Memory dependent derivatives (MDD) are now, in addition to fractional ordered derivatives, a significant mathematical tool for understanding many real-world phenomena. In the rate of heat flux, Yu et al. [54] utilized memory dependent derivatives (MDD) in the Lord-Shulman (LS) extended thermoelasticity theory. Recently, [55-62] studied various models related to generalized thermoelasticity theories with MDD.

The present contribution aims to investigate a generalized dual phase-lag model with variable thermal material properties and memory dependent derivative (VMDPL). This model is used to examine the thermoelastic behaviour of a half-space under an external body force and subjected to exponentially varying heat. The effects of the variable thermal material properties and memory dependent derivative on all the physical quantities of a half-space are discussed. The obtained results demonstrate that the physical fields of a half-space depend not only on the distance, but also on the memory time delay and the variable thermal parameter. Furthermore, the variable thermal parameter and the variable thermal parameter has a clear effect on the temperature and the stress but has a negligible effect on the displacement. The Laplace transform technique is used to solve the governing differential equations numerically. The impacts of the variable thermal material characteristics and memory dependent derivative are graphically depicted in our numerical computations. Finally, the obtained results are supported by the previous literature.

2. Thermoelastic Model and Fundamental Equations

The conventional theory of heat conduction based on Fourier's law clearly predicts an infinite heat propagation speed. In addition, the basic Fourier's law [1]

$$\vec{q}(x,t) = -K\nabla\theta(x,t). \tag{1}$$

where $\vec{q}(\boldsymbol{x},t)$ denotes the heat flux vector, $\theta = T - T_0$ signifies the changing temperature, where T is the absolute temperature over the reference temperature T_0 and K is the thermal conductivity.

With dual-phase-lag heat conduction, Tzou [8] introduces the modified Fourier law.

$$\left(1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2!}\frac{\partial^2}{\partial t^2}\right)\vec{q} = -K\left(1 + \tau_\theta \frac{\partial}{\partial t}\right)\nabla\theta \tag{2}$$

Yu et al. [54] generalizes the Lord-Shulman (LS) by using memory dependent derivatives (MDD).

$$\left(1 + \tau\, D_\omega^{(1)}\right)\vec{q} = -K\nabla\theta, \tag{3}$$

Similarly, we analyze the heat conduction equation with memory-dependent derivative as a dual-phase lag model.

$$\left(1 + \tau_q D_\omega + \frac{\tau_q^2}{2} D_\omega^2\right)\vec{q} = -K(1 + \tau_\theta D_\omega)\nabla\theta \tag{4}$$

where $D_\omega^{(m)}$ is the memory dependent derivatives (MDD) of m$^{\text{th}}$ order defined by

$$D_\omega^{(m)} f(r,t) := \frac{1}{\omega}\int_{t-\omega}^{t} \mathcal{K}(t-\xi) f^{(m)}(r,\xi)\, d\xi \tag{5}$$

with the time delay $\omega > 0$ and m-times differentiable function $f(r,t)$ about t, together with the kernel function

$\mathcal{K}(\mathrm{t}-\xi)$ which can chosen freely with $0 \leq \mathcal{K}(t-\xi) \leq 1$ over $\xi \in [t-\omega, t]$.

Here, we apply the memory kernel function $\mathcal{K}(\mathrm{t}-\xi)$ proposed by Ezzat et al [56]:

$$\mathcal{K}(t-\xi) := 1 - \frac{2b}{\tau}(t-\xi) + \frac{a^2(t-\xi)^2}{\tau^2} = \begin{cases} 1, & if\ a=b=0 \\ 1-\frac{(t-\xi)}{\omega}, & if\ a=0, b=1/2 \\ 1-(t-\xi), & if\ a=0, b=\omega/2 \\ \left(1-\frac{(t-\xi)}{\omega}\right)^2, & if\ a=1, b=1 \end{cases} \tag{6}$$

The fixed point dependence can be ignored via the fractional derivative. On the other hand, the memory dependent derivatives have a better chance of capturing the material reaction. As a result, we make use of this derivative in our work.

The energy balance equation in the absence of a heat source Q has the shape

$$\rho C_E \frac{\partial \theta}{\partial t} + \gamma T_0 \frac{\partial}{\partial t}(div\, \vec{u}) = -div\, \vec{q} \tag{7}$$

where C_E denotes specific heat under constant strain, the stress temperature modulus is denoted by $\gamma = (3\lambda + 2\mu)\alpha_t$, where α_t signifies the thermal expansion coefficient, λ, μ Lamé's constants, $\vec{u}$ the displacement vector, ρ is the medium's density.

Using the divergence of Eq. (4) and taking Eq. (6) into account, we obtain

$$\left(1 + \tau_q D_\omega + \frac{\tau_q^2}{2} D_\omega^2\right)\left[\rho C_E \frac{\partial \theta}{\partial t} + \gamma T_0 \frac{\partial}{\partial t}(\mathrm{div}\, \vec{u})\right] = (1 + \tau_\theta D_\omega)[\nabla \cdot (K(\theta)\nabla\theta)] \tag{8}$$

Thermal conductivity is a significant material parameter that is usually regarded as constant. Nevertheless, several experimental and theoretical investigations have shown that thermal conductivity is strongly related to changes in temperature [63, 64]. As a result, the linear relationships between the thermal material characteristics K and C_E and temperature increment are used.

$$K = K(\theta) = k_0(1 + k_1\theta), \quad C_E = C_E(\theta) = k_2(1 + k_1\theta) \tag{9}$$

such that k_0 indicates the value of the thermal conductivity when it independent of temperature and k_1 is a non-positive constant. In this case, the thermal diffusivity has the form $N = \frac{K}{\rho C_E}$, and then

$$\rho C_E(\theta) = \frac{K(\theta)}{N} \tag{10}$$

Using the mapping (Kirchhoff's transformation):

$$\psi := \frac{1}{k_0}\int_0^\theta K(\theta) d\theta, \tag{11}$$

and applying Nabla operator, we obtain

$$\begin{gathered} k_0 \nabla\psi = K(\theta)\nabla\theta \\ k_0 \nabla^2\psi = \mathrm{div}[K(\theta)\nabla\theta], \end{gathered} \tag{12}$$

When both sides of Eq. (10) are differentiated with regard to time, the result is

$$k_0 \frac{\partial \psi}{\partial t} = K \frac{\partial \theta}{\partial t} \tag{13}$$

Due to Kirchhoff's transformation (11) and using Eqs. (12) and (13), Eq. (8) becomes

$$\left(1 + \tau_q D_\omega + \frac{\tau_q^2}{2} D_\omega^2\right)\left[\frac{k_0}{N}\frac{\partial \psi}{\partial t} + \gamma T_0 \frac{\partial}{\partial t}(div\, \vec{u})\right] = k_0(1 + \tau_\theta D_\omega)\nabla^2\psi \tag{14}$$

where $\psi = \theta + \frac{1}{2}k_1\theta^2$. Once ψ is identified θ given by

$$\theta = \frac{1}{k_1}\left[\sqrt{2k_1\psi + 1} - 1\right] \tag{15}$$

On the other hand, the additional basic equations of motion, constitutive equations and strain and displacement relations based on the theory of thermoelasticity for a homogeneous and isotropic thermoelastic solid are

$$\sigma_{lm} = 2\mu e_{lm} + [\lambda e_{rr} - \gamma\theta]\delta_{lm}, \tag{16}$$

$$e_{lm} = (u_{m,l} + u_{l,m})/2, \tag{17}$$

$$\sigma_{lm,l} + F_l = \rho \ddot{u}_l \tag{18}$$

where ρ means the mass density, and F_l represents the component of the external forces. The system described above is totally hyperbolic in the sense that both the equations of motion (18) and the equation of heat transport (16) are of the hyperbolic type.

Now, Eqs. (14) and (9) represent our generalized dual phase-lag thermoelastic model with variable thermal material properties and memory-dependent derivative, which we will refer to as VMDPL.

3. Formulation of the problem

Using our generalized dual phase-lag model, we study an isotropic homogeneous thermoelastic half-space $x \geq 0$ with an external body force and exposed to exponentially varying heat. Also, we supposed that the state of the medium depends only on x, t and that the vector of displacement $\vec{u} = (u(x,t), 0, 0)$.

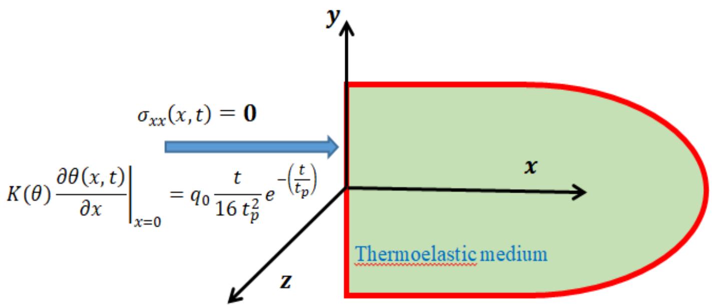

Figure 1 Geometry of the thermoelastic half-space.

The constitutive equation has the form

$$\sigma_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \gamma\theta\ ; \tag{19}$$

Also, the equation of motion in the present external force F_x in the one dimensional case has the form

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \gamma\frac{\partial\theta}{\partial x} + F_x \tag{20}$$

In view of Kirchhoff's transformation (11), Eqs. (19-20) becomes

$$\sigma_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \gamma\psi \tag{21}$$

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \gamma\frac{\partial\psi}{\partial x} + F_x \tag{22}$$

The heat equation in Eq. (14) becomes

$$\left(1 + \tau_q D_\omega + \frac{\tau_q^2}{2} D_\omega^2\right)\left[\frac{k_0}{N}\frac{\partial\psi}{\partial t} + \gamma T_0\frac{\partial^2 u}{\partial t\partial x}\right] = k_0(1 + \tau_\theta D_\omega)\frac{\partial^2\psi}{\partial x^2} \tag{23}$$

By applying the subsequent non-dimensional variables:

$$\{x', u'\} = c_1\eta\{x, u\},\ \{t', \tau'_q, \tau'_\theta\} = c_1^2\eta\{t, \tau_q, \tau_\theta\},\ \eta = \frac{\rho k_2}{k_0} = \frac{1}{N},$$
$$\psi' = \frac{\gamma\psi}{\lambda+2\mu},\ \sigma'_{xx} = \frac{\sigma_{xx}}{\lambda+2\mu},\ \ c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}},\ \ F'_x = \frac{F_x}{\rho c_1^3\eta}, \tag{24}$$

Equations (21)-(23) converts to

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial \psi}{\partial x} + F_x \tag{25}$$

$$\left(1 + \tau_q D_\omega + \frac{\tau_q^2}{2} D_\omega^2\right)\left[\frac{\partial \psi}{\partial t} + \varepsilon \frac{\partial^2 u}{\partial t \partial x}\right] = (1 + \tau_\theta D_\omega)\frac{\partial^2 \psi}{\partial x^2} \tag{26}$$

$$\sigma_{xx} = \frac{\partial u}{\partial x} - \psi \tag{27}$$

where

$$\varepsilon = \frac{\gamma^2 T_0}{\rho^2 k_2 c_1^2} \tag{28}$$

The primes in the above equations have been omitted for clarity and convenience. Besides, we use

$F_x = f(x) = e^{-\alpha x}$, in which α is a constant parameter (decaying parameter).

4. Boundary and initial conditions

In this study, we assume that the medium is initially at rest, and hence the problem's beginning conditions are as follows:

$$u(x,t)|_{t=0} = \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0,\ \ x > 0,$$
$$\theta(x,t)|_{t=0} = \left.\frac{\partial \theta(x,t)}{\partial t}\right|_{t=0} = 0,\ \ x > 0, \tag{29}$$

Also, we suppose that the boundary conditions are :

$$\sigma_{xx}(x,t)|_{x=0} = 0 \tag{30}$$

$$-K(\theta)\left.\frac{\partial \theta(x,t)}{\partial x}\right|_{x=0} = q(x,t)|_{x=0} = q_0 \frac{t}{16\, t_p^2} e^{-\left(\frac{t}{t_p}\right)} \tag{31}$$

where t_p is the time pulse of heat.

In view of Kirchhoff's transformation, the 2nd initial assumption of Eq. (29) and the boundary condition of Eq. (31), become

$$\psi(x,t)|_{t=0} = \left.\frac{\partial \psi}{\partial t}\right|_{t=0} = 0,\ \ x > 0$$
$$\left.\frac{\partial \psi(x,t)}{\partial x}\right|_{x=0} = -\frac{q_0 t}{16 k_0\, t_p^2} e^{-\left(\frac{t}{t_p}\right)} \tag{32}$$

Moreover, the regularity boundary conditions are

$$u(x,t),\ \theta(x,t)\ \&\ \sigma_{xx}(x,t)\ \to 0\ \ \text{when}\ \ x \to \infty \tag{33}$$

5. The Solution of the problem in the Laplace transform domain

Applying Laplace transform $\mathcal{L}$ into Equations (26)-(28), noting that for any function $f(t)$ with i^{th} order MDD [65]:

$$\mathcal{L}[\omega D_\omega^i f(t)] = \mathcal{L}\left[\int_{t-\omega}^{t} \mathcal{K}(t-\xi) f^{(i)}(\xi)\, d\xi\right] = s^{i-1}\mathcal{L}[f(t)]\, G(s,\omega), \tag{34}$$

and using the considered initial conditions (29), we have

$$\frac{d\bar{e}}{dx} - \frac{d\bar{\psi}}{dx} = s^2\bar{u} - \frac{e^{-\alpha x}}{s} \tag{35}$$

$$\mathcal{A}_q\bar{\psi} + \varepsilon\mathcal{A}_q\bar{e} = \mathcal{A}_\theta\nabla^2\bar{\psi} \tag{36}$$

$$\bar{\sigma}_{xx} = \bar{e} - \bar{\psi} \tag{37}$$

where

$$\mathcal{A}_q = s\left(1 + \frac{\tau_q}{\omega}G(s,\omega) + \frac{s\,\tau_q^2}{2\omega}G(s,\omega)\right),\ \mathcal{A}_\theta = \left(1 + \frac{\tau_\theta}{\omega}G(s,\omega)\right),\ \nabla^2 = \frac{d^2}{dx^2}$$
$$G(s,\omega) = (1 - e^{-s\omega})\left[1 - \frac{2b}{\omega\, s} + \frac{2a^2}{\omega^2 s^2}\right] - \left[a^2 - 2b^2 + \frac{2a^2}{\omega s}\right]e^{-s\omega},\ \bar{e} = \frac{d\bar{u}}{dx}. \tag{38}$$

Also, in the absence of MDD, we putting $G(s,\omega) = s\omega$.   .

One can show that Eq. (35) becomes

$$\nabla^2\bar{e} - s^2\bar{e} = \nabla^2\bar{\psi} + \frac{\alpha}{s}e^{-\alpha x} \tag{39}$$

From which together with Eq. (36), we have

$$[\nabla^4 - A\nabla^2 + B]\bar{\psi} = Ce^{-\alpha x} \tag{40}$$

where $A = \frac{\mathcal{A}_q}{\mathcal{A}_\theta}(1+\varepsilon) + s^2,\ B = s^2\frac{\mathcal{A}_q}{\mathcal{A}_\theta}, C = \frac{\alpha\varepsilon\mathcal{A}_q}{s\mathcal{A}_\theta}$

Hence, the general solution of the function $\bar{\psi}$ has the form

$$\bar{\psi}(x,s) = A_1 e^{-m_1 x} + A_2 e^{-m_2 x} + A_3 e^{-\alpha x} \tag{41}$$

where $A_3 = \frac{C}{\alpha^4 - A\alpha^2 + B}$ and $m_r^2, r = 1,2$ are satisfy $m^4 - Am^2 + B = 0,$

From Eqs. (36) and (41), we get

$$\bar{e}(x,s) = \sum_n^2\left[\frac{1}{\varepsilon}\left(\frac{\mathcal{A}_\theta}{\mathcal{A}_q}m_n^2 - 1\right)\right]A_n e^{-m_n x} + \frac{1}{\varepsilon}\left(\frac{\mathcal{A}_\theta}{\mathcal{A}_q}\alpha^2 - 1\right)A_3 e^{-\alpha x} \tag{42}$$

Also, the displacement $\bar{u}$ can be expressed as

$$\bar{u}(x,s) = \sum_n^2\left[\frac{1}{m_n\varepsilon}\left(1 - \frac{\mathcal{A}_\theta}{\mathcal{A}_q}m_n^2\right)\right]A_n e^{-m_n x} + \frac{1}{\alpha\varepsilon}\left(1 - \frac{\mathcal{A}_\theta}{\mathcal{A}_q}\alpha^2\right)A_3 e^{-\alpha x} \tag{43}$$

In view of Eq. (37) and using Eqs. (41) and (42), we have

$$\bar{\sigma}_{xx} = \sum_n^2\left[\frac{1}{\varepsilon}\left(\frac{\mathcal{A}_\theta}{\mathcal{A}_q}m_n^2 - \varepsilon - 1\right)\right]A_n e^{-nx} + \frac{1}{\varepsilon}\left(\frac{\mathcal{A}_\theta}{\mathcal{A}_q}\alpha^2 - \varepsilon - 1\right)A_3 e^{-\alpha x} \tag{44}$$

After applying Laplace transform, the boundary conditions (32) become

$$\left.\frac{d\bar{\psi}(x,s)}{dx}\right|_{x=0} = \bar{H}(s) = -\frac{q_0}{16k_0\,(t_p\, s+1)^2}$$
$$\bar{\sigma}_{xx}(x,s)|_{x=0} = 0 \tag{45}$$

From Eqs. (41) and (44) taking into account the above conditions, we obtain

$$\sum_n^2 m_n A_n(s) + \alpha A_3 = -\bar{H}(s)$$
$$\sum_n^2 \left[\frac{1}{\varepsilon}\left(\frac{\mathcal{A}_\theta}{\mathcal{A}_q} m_n^2 - \varepsilon - 1\right)\right] A_n(s) + \frac{1}{\varepsilon}\left(\frac{\mathcal{A}_\theta}{\mathcal{A}_q}\alpha^2 - \varepsilon - 1\right) A_3 = 0 \tag{46}$$

From the above equations, we determine the unknown parameters A_i, $(i = 1,2,)$. To determine the studied fields in the physical domain, the Riemann-sum approximation method is used to obtain numerical results. [65] has the details of these procedures.

To determine the solutions of the examined fields in the physical domain, we use an appropriate and effective numerical approach based on a Fourier series expansion [65]. Any Laplace domain function $\bar{\mathcal{F}}(x,s)$ can be inverted to the time domain using this method as follows:

$$\mathcal{F}(x,t) = \frac{e^{ct}}{t}\left(\frac{1}{2}\boldsymbol{Re}[\bar{\mathcal{F}}(x,c)] + \boldsymbol{Re}\left[\sum_{n=1}^{k}\bar{\mathcal{F}}\left(x, c + \frac{in\pi}{t}\right)(-1)^n\right]\right) \tag{47}$$

where k is a finite number of terms, $\boldsymbol{Re}$ is the real part and $\boldsymbol{i}$ is imaginary number unit. For faster convergence, numerous numerical experiments have shown that the value of c fulfills the relation $\mathrm{ct} \cong 4.7$ [65].

6. Special cases of a generalized thermoelastic model (VMDPL).

This paper investigates a generalized thermoelastic model with variable thermal material properties and memory dependent derivatives (VMDPL). The investigated model is reduced to several models with (without) variable thermal material properties and memory dependent derivatives. There are four types of reduced models:

[1] Thermoelastic models without both variable thermal material properties and memory dependent derivatives:

- Classical thermoelastic model (CTE): $\tau_q = \tau_\theta = 0, \mathrm{G}(\mathrm{s},\omega) = s\ \omega\ , k_1 = 0$
- Lord-Shulman model (LS): $\tau_q > 0\ (O(\tau_q^2) \equiv 0), \tau_\theta = 0, \mathrm{G}(\mathrm{s},\omega) = \mathrm{s}\ \omega\ , k_1 = 0$
- Dual phase-lag model (DPL): $\tau_q \geq \tau_\theta > 0, \mathrm{G}(\mathrm{s},\omega) = \mathrm{s}\ \omega\ , k_1 = 0$

[2] Thermoelastic models without variable thermal material properties and with memory dependent derivatives:

- MDD Lord-Shulman model (MLS): $\tau_q > 0\ (O(\tau_q^2) \equiv 0), \tau_\theta = 0, \mathrm{G}(\mathrm{s},\omega) \equiv Eq.(38),\ k_1 = 0.$
- MDD Dual phase-lag model including (MDPL): $\tau_q \geq \tau_\theta > 0,\ \mathrm{G}(\mathrm{s},\omega) \equiv Eq.(38),\ \ k_1 = 0.$

[3] Thermoelastic models with variable thermal material properties and without memory dependent derivatives:

- Variable thermal material Classical thermoelastic model (VCTE): $\tau_q = \tau_\theta = 0, \mathrm{G}(\mathrm{s},\omega) = s\ \omega\ , k_1 \neq 0$.
- Variable thermal material Lord-Shulman model (VLS): $\tau_q > 0\ (O(\tau_q^2) \equiv 0), \tau_\theta = 0, \mathrm{G}(\mathrm{s},\omega) = \mathrm{s}\ \omega\ , k_1 \neq 0.$
- Variable thermal material Dual phase-lag model (VDPL): $\tau_q \geq \tau_\theta > 0, \mathrm{G}(\mathrm{s},\omega) = s\ \omega,\ \ k_1 \neq 0.$

[4] Thermoelastic models with both variable thermal material properties and memory dependent derivatives:

- Variable thermal material Lord-Shulman model with MDD (VMLS): $\tau_q > 0\ (O(\tau_q^2) \equiv 0), \tau_\theta = 0, \mathrm{G}(\mathrm{s},\omega) \equiv Eq.(38),\ k_1 \neq 0$.
- Variable thermal material Dual phase-lag model with MDD (VMDPL): $\tau_q \geq \tau_\theta > 0,\ \mathrm{G}(\mathrm{s},\omega) \equiv Eq.(38),\ \ k_1 \neq 0$.

7. Results and Discussion

To confirm and describe the results obtained in the foregoing sections, we investigate the numerical results using the value of the Silicon (Si) material at $\mathrm{T}_0 = 298\ \mathrm{K}$ as [62]

$$\lambda = 2.696 \times 10^{10}\ m^{-1}s^{-2}\ kg,\ \ \mu = 1.639 \times 10^{10}\ m^{-1}s^{-2}kg,\ \ \rho = 1740\ \ m^{-3}kg,$$
$$k_0 = K = 2.510\ \ m^{-1}K^{-1}W,\ \ k_2 = C_E = 1.04 \times 10^3\ K^{-1}J\ kg,$$

The thermoelastic behaviour of a half-space under an external body force and subjected to exponentially varying heat is discussed by our modiefied model (VMDPL). The acquired results are represented visually in Figs. 2-19 for a variety of distance values $x(0 \leq x \leq 4)$ at $t = 0.15$, when the dual phase-lags $\tau_q = 0.05$ and $\tau_\theta = 0.03$ in the present external body forces $F_x = e^{-\alpha x}$ and the time pulse of heat $t_p = 2.0$ Ps. Furthermore, our numerical computations are obtained using the Mathematica programming Language and are prepared for four directions.

7.1 Influence of the memory kernel $\mathcal{K}$ on the physical fields

In this section, we demonstrate how the memory kernel $\mathcal{K}$ acts with the field variables of a half-space corresponding to the generalized model VMDPL. The obtained results are represented in Figs.2-4 for the field quantities corresponding to different values of the distance $x(0 \leq x \leq 4)$ at $t = 0.15$ and different values of the constants a, b, when the phase-lags $\tau_q = 0.05$ and $\tau_\theta = 0.03$, together with variable thermal material properties $k_1 = -0.3$, $\omega = 0.3$.

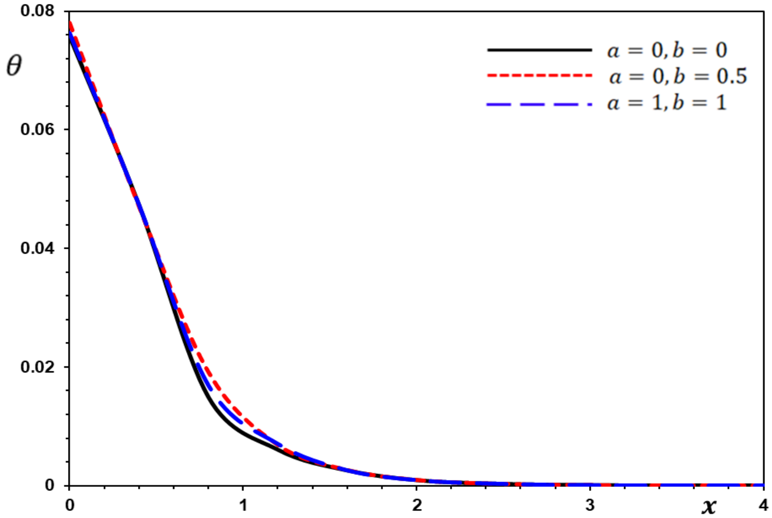

Figure 2: The effect of the memory kernel $\mathcal{K}$ on the temperature θ.

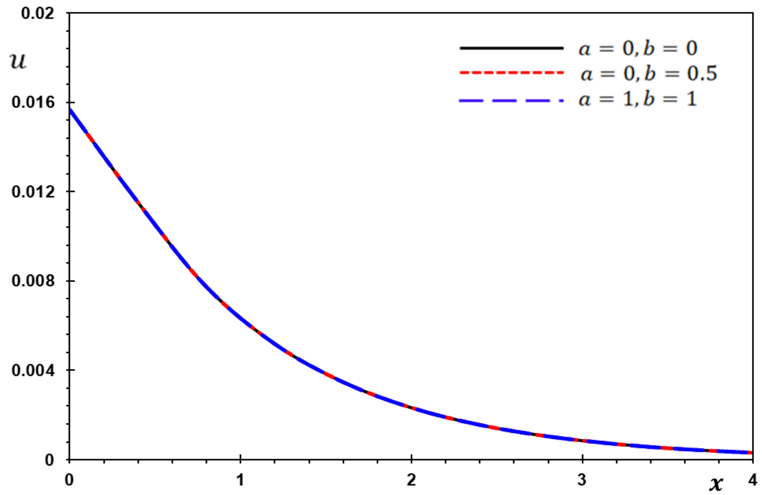

Figure 3: The effect of the memory kernel $\mathcal{K}$ on the displacement u.

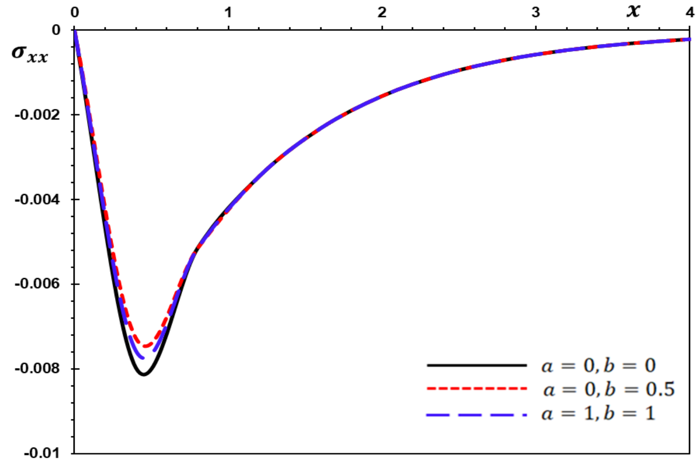

Figure 4: The effect of the memory kernel $\mathcal{K}$ on the stress σ_{xx}.

Figure 2 displays the distribution of the temperature θ of a half-space for distinct values of the constants a, b (the kernel function $\mathcal{K}$ of MDD). This Figure ensures that the values of temperature θ decreases with increasing the distance x for $0 < x < 4$. Also, the magnitude of the temperture curve for VMDPL in the case ($a = 1, b = 0$) is greater than that for the other cases of the kerenel, although they coincide to a constant value from $x = 1.5$. Hence, the different values of the constants a, b (the memory kernel function k), has clearly effect on the temperature θ.

It is evident from Figure 3, that the memory kernel function k has a negligible effect on u.

Figure 4. illustrates that the depth of the stress curves for VMDPL in the case ($a = 0, b = 0$) is greater than that for the other cases of the kerenel. Hence, the kernel function k of MDD has a significant effect on the stress σ_{xx}. From Figures 2-4, we notice that values of the physical fields (temperature θ, the displacement u and the stress σ_{xx}) converge to zero when the distance tends to 4, which is in quite good agreement with the regularity boundary conditions. Finally, we conclude that the kernel $\mathcal{K}$ of MDD has a significant effect on all the fields except the displacement u. This result is consistent with the results obtained by[57, 62].

7.2 The effects of a memory time delay ω on physical fields

In the present case, we introduce the effect of memory time delay ω on the field variables of a half-space. The obtained results are shown in Figs.5-7 for the field quantities corresponding to different values of the distance $x(0 \leq x \leq 4)$ at $t = 0.15$ and different values of the time delay ω of MDD in the case $a = 1, b = 1$, when the dual phase-lags $\tau_q = 0.05, \tau_\theta = 0.03$, together with the variable thermal material properties

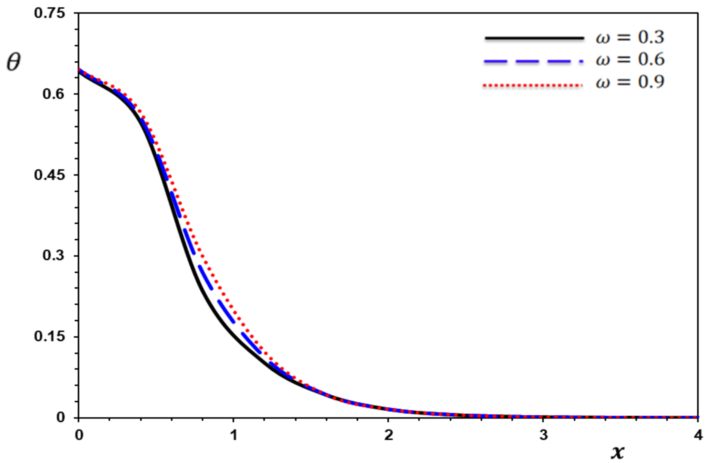

Figure 5: The effect of the memory time delay ω on the temperature θ.

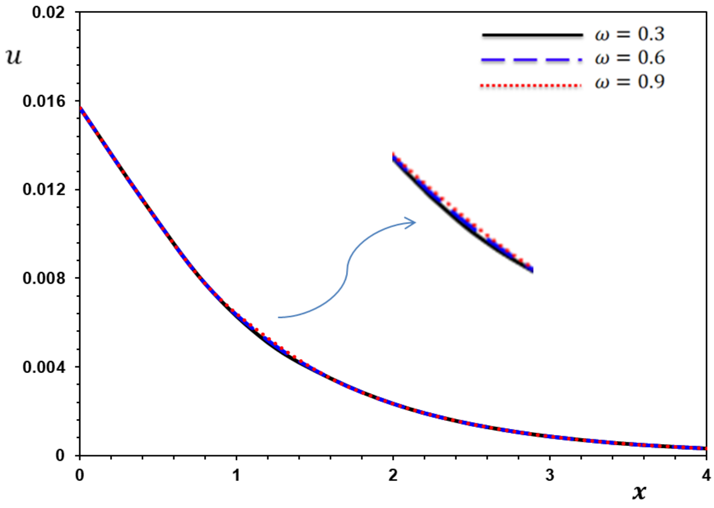

Figure 6: The effect of the memory time delay ω on the displacement u.

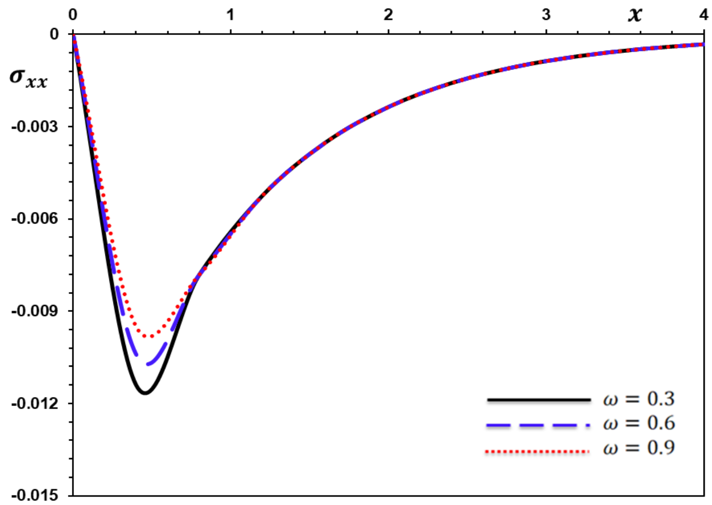

Figure 7: The effect of the memory time delay ω on the stress σ_{xx}.

Figure 5 presents the variations of the temperature θ of a half-space for different values of the memory time delay ω. It is noticed that increasing the amount of the memory time delay ω increases the variation of temperature θ in the interval $0 < x < 4$, even though they eventually coincide to a constant value after $x = 1.5$. Also, the different values of memory time delay ω has obviously effect on the temperature θ. But from Figure 6, it is clear that the memory time delay ω has a weak effect on u .

Figure 7 illustrates that the memory time delay ω has a significant effect on the stress σ_{xx}. On the other hand, these figures ensure the values of the physical fields (temperature θ, the displacement u and the stress σ_{xx}) converge to zero when the distance tends to 4, which agrees with the regularity boundary conditions. Finally, we achieve that the physical quantities depend not only on the distance x, but also on the memory time delay ω. Our findings are in strong accordance with the results of [57-62]

7.3 Different models of thermoelasticity

In this subsection, we study the distributions of the physical fields for two classes of different models of thermoelasticity (LS, VLS & VMLS) and (DPL,VDPL & VMDPL). The achieved results are represented in Figs. 8-13 for the field quantities matching to different values of the distance $x(0 \leq x \leq 4)$ at $t = 0.15$, $k_1 = -0.3$ and the time delay $\omega = 0.3$ of MDD with $a = 1, b = 1$, together with $\tau_q = 0.05$ and $\tau_\theta = 0.03$.

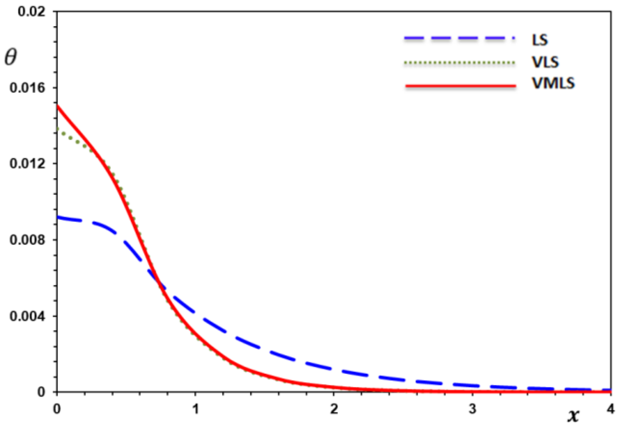

Figure 8: The temperature θ for different models of LS .

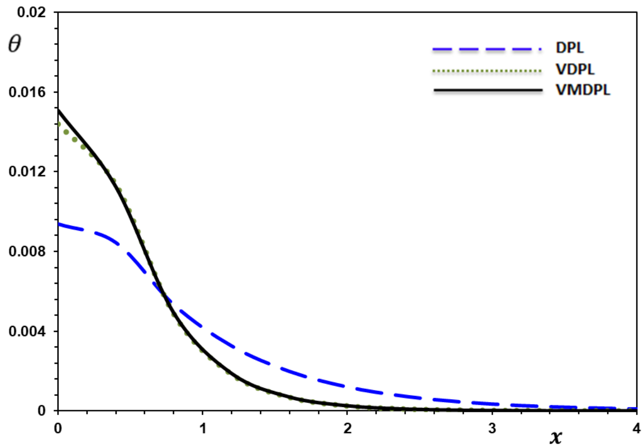

Figure 11: The temperature θ for different models of DPL .

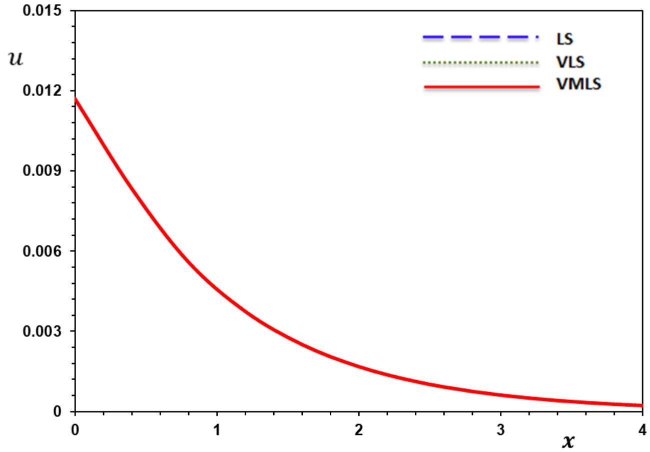

Figure 9: The displacement u for different models of LS .

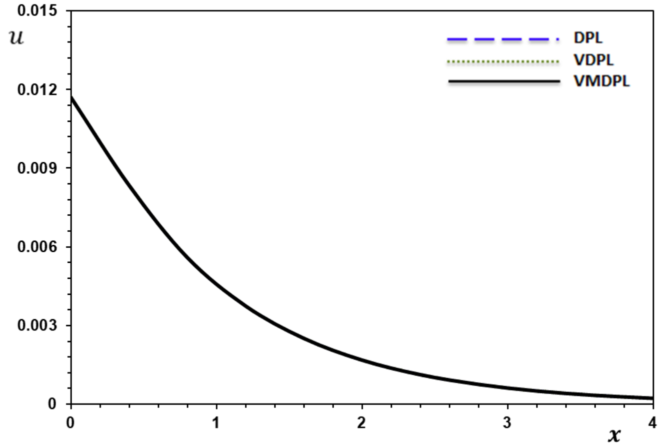

Figure 12: The displacement u for different models of DPL .

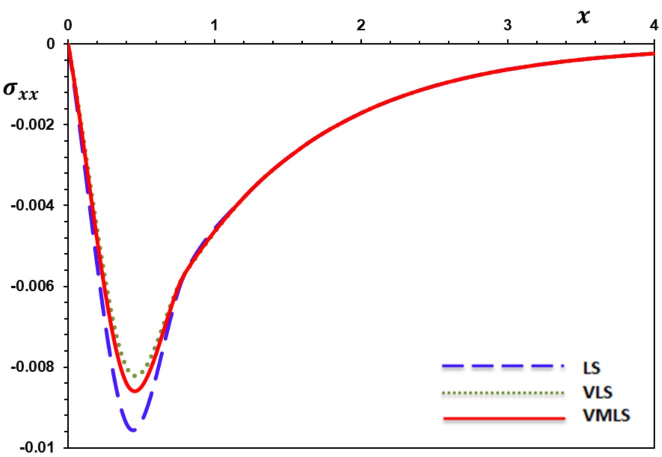

Figure 10: The stress σ_{xx} for different models of LS .

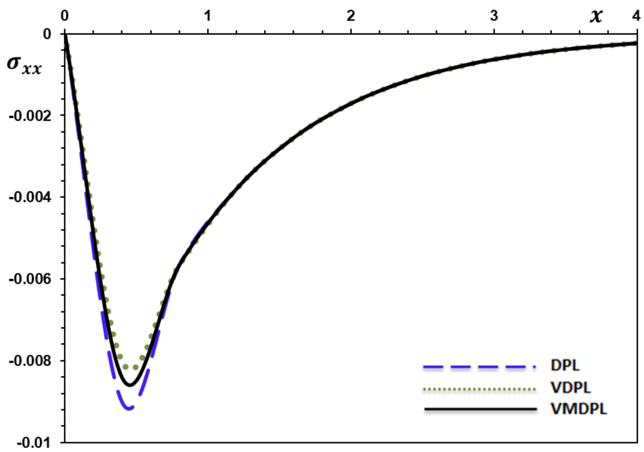

Figure 13: The stress σ_{xx} for different models of DPL .

Figs. 8-13 show that the results of the VMLS and VMDPL models differ from Lord-Shulman model (LS) and the dual phase-lag model (DPL) for the phenomenon of limited velocities of heat wave propagation. It is worth mentioning that the memory dependent derivatives (MDD) serve as an important mathematical tool in describing many real world phenomenon. On the other hand, experimental and theoretical investigations have shown that thermal conductivity is strongly related to changes in temperature. Therefore, the current generalized modified model with variable thermal material properties and memory-dependent derivative (VMDPL or VMLS) is the best.

7.4 Influence of the variable thermal material properties on the physical fields

This section is dedicated to discuss how the variable thermal parameter k_1 effects on the field variables of a half-space using the modified model VMDPL. The acquired results are depicted in Figs.(14-16) for the field quantities corresponding to different values of the radius $x(0 \le x \le 4)$ at $t = 0.15$ and different values of the variable thermal parameter k_1, when the dual phase-lags $\tau_q = 0.05$ and $\tau_\theta = 0.03$, the time delay $\omega = 0.3$ of MDD with $a = 1, b = 1$.

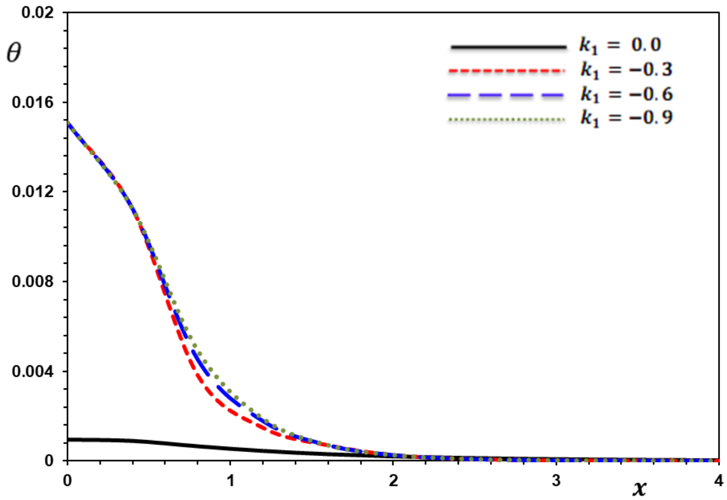

Figure 14: The effect of variable thermal material k_1 on the temperature θ.

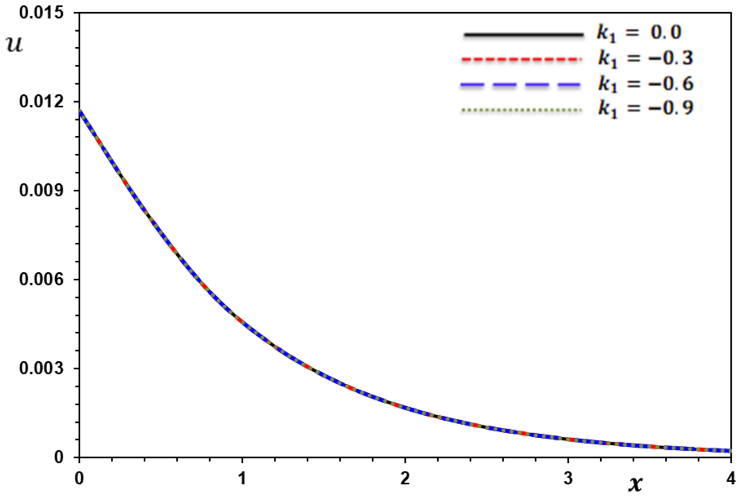

Figure 15: The effect of variable thermal material k_1 on the displacement u.

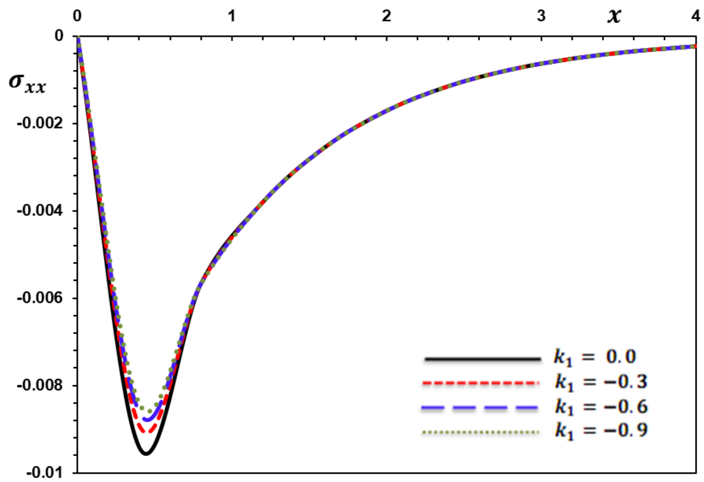

Figure 16: The effect of variable thermal material k_1 on the stress σ_{xx}

Figures 14-16 illustrates the variety of the field variables of a half-space throught distinct values the variable thermal parameter k_1. These figures guarantee that the variable thermal parameter k_1 has a clear effect on the temperature θ and the stress σ_{xx} until $x = 2$ but has a weak effect on u. Therefore, the physical quantities depend not only on the distance x, but also on the variable thermal parameter k_1. This result is consistent with the results obtained by [63, 64, 66].

7.5 The effect of the time on the physical fields via VMDPL model.

Adopting our generalized dual phase-lag model (VMDPL), we exhibit the effect of time t on all field variables (VMDPL). In this case, we put the phase-lags $\tau_q = 0.05$ and $\tau_\theta = 0.03$, when variable thermal material properties $k_1 = -0.3$ and the time delay $\omega = 0.3$ of MDD with $a = 1, b = 1$, together with the time pulse of heat $t_p = 2.0\ Ps$. For a comparison of the results, the temperature, the displacement, and stress are accessible in Figs. 17-19. These distributions are very sensitive to the time instant *t*, as can be seen from the figures. It is also clear from these Figs. that the behavior of the temperature, displacement and stress are the most affected by the change of time.

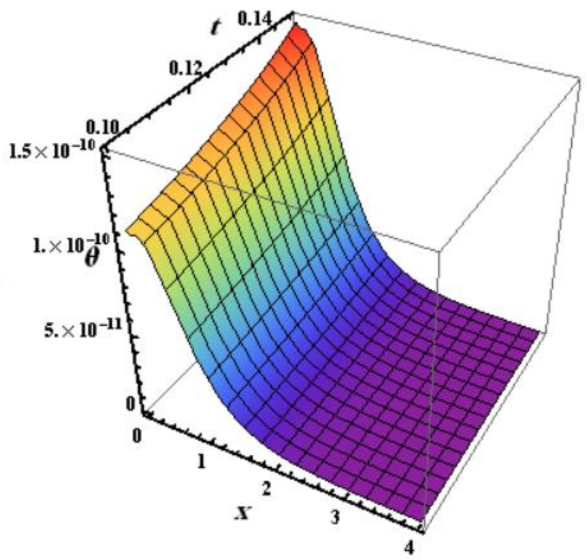

Figure 17: The temperature θ with different time

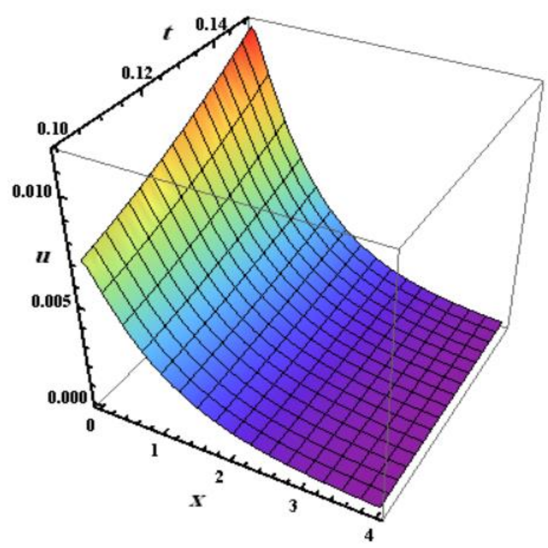

Figure 18: The displacement u with different time.

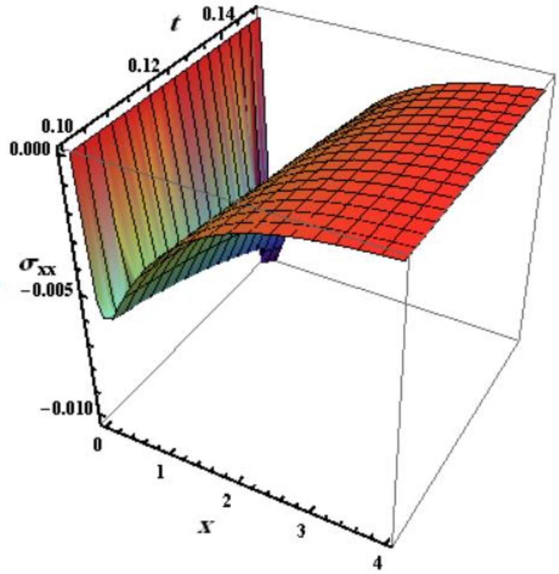

Figure 19: The stress σ_{xx} with different time

Conclusion

A generalized dual phase-lag thermoelastic model with variable thermal material properties and memory-dependent derivative (VMDPL) is investigated. Various classical and generalized thermoelasticity models are extracted from its general model. Via the generalized dual phase-lag thermoelastic model, the distributions of the physical quantities for a half-space under an external body force, are discussed. The results of the numerical simulation lead to the following conclusions:

- The effects of the memory kernel function $\mathcal{K}$ on the physical fields of a half-space are very obvious.
- The physical quantities of a half-space depend not only on the distance x, but also on the kernel function $\mathcal{K}$ of memory dependent derivative.
- The kernel function $\mathcal{K}$ of MDD has a significant effect on all the fields, but has a weak effect on the displace ment u.
- The results of our study (VMLS & VMDPL) differs from Lord-Shulman model (LS) and the dual phase-lag model (DPL) of the phenomenon of limited velocities of the propagation of heat waves. As a result, our modified model is the most effective.
- The variable thermal parameter k_1 has a clear effect on the temperature θ and the stress σ_{xx} but has a negligi ble effect on u.
- The thermoelasticity with memory-dependent type is better than the fractional type at expressing the memory effect.
- MDD is more adaptable. We have adequate option to choose kernel function and delay time based on our pr oblem.

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