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RESEARCH PAPER

An analytical solution for nonlinear vibration of floating plate on the fluid by modified multiple scales method

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Abstract

The aim of the present paper is to analytically study the nonlinear forced vibration of a rectangular plate floating on the fluid by Modified Multiple Time Scales method for the first time. It is assumed that the fluid is stationary, incompressible, non-viscous, and non-rotational, and the plate is subjected to transversal excitation. The boundary condition is considered to be simply supported. Using von Karman nonlinear strain displacement relationships, the extended Hamilton principle, and FSTD plate theory, the partial differential equations of motion are derived. The fluid is mathematically modeled by Bernoulli equation and the velocity potential function. Galerkin method is then applied for converting the nonlinear partial differential equations into time-dependent nonlinear ordinary differential equations. The resulted equations are solved analytically by the Modified Multiple Scales Method, thereafter. Despite the large number of derivatives and calculations of the conventional multiple scale method, this approach is very simple and straightforward. The results reveal an excellent agreement with the traditional Multiple Scales method results and existing studies, and are more accurate than other available results. The effect of the presence of fluid near the plate on natural frequency and amplitude of vibration of plate are studied. The effects of some key parameters of the system are also examined.

Keywords: Modified Multiple Scales Method, Multiple Scales method, Nonlinear vibration, nonlinear forced vibration of plate floating on the fluid.

Introduction

Fluid-structure interference problems cover a wide range of engineering applications including shipbuilding, offshore structures, coastal structures, dams, and submarines. The presence of fluid close to a plate significantly increases the kinetic energy of the system and thus reduces the values of the natural frequencies of the plate in contact with the fluid, compared to a plate vibrating in a vacuum. Hence, plates in contact with fluid have been extensively concerned by many researchers.

Many researchers have studied nonlinear responses of a plate. Some of the most recent studies on nonlinear vibration of plate in this paragraph mentioned. Adeli et al. [1], investigated free torsional vibration behavior of a nonlinear nano-cone, based on the nonlocal strain gradient elasticity theory. Ajiri et al. [2] investigated the nonstationary oscillation, secondary resonance and nonlinear dynamic behavior of viscoelastic nanoplates with linear damping based on the modified strain gradient theory extended for viscoelastic materials. Also Ajiri and Seyed Fakhrabadi [3], developed a new viscoelastic size-depended model based on a modified couple stress theory and the for nonlinear viscoelastic material in order to vibration analysis of a

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viscoelastic nanoplate. Mohamadi et al. [4] studied nonlinear free and forced vibration behavior of a porous functionally graded Euler-Bernoulli nanobeam subjected to mechanical and electrical loads based on the nonlocal strain gradient elasticity theory.

Numerous studies have been performed on plates in contact with fluid, many of which investigated the vertical plate in contact with the fluid. Zhou and Cheung [5] considered the vibration of vertical rectangular plate in contact with water analytically. Khorshidi and Farhadi [6] studied the free vibrations of a laminated composite rectangular plate partially in contact with the fluid. They extracted the natural frequency of the plate coupled to the sloshing fluid modes using the Rayleigh Ritz method.

Many studies have been also performed on the analysis of free and forced vibrations of plates partially or completely immersed in the liquid. Robinson and palmer [7] presented modal analysis of a thin horizontal plate floating on liquid. They considered the plate-fluid interaction by adding a mass due to fluid. Kerboua et al. [8] investigated a rectangular plate immersed in the fluid and floating on the fluid. They modeled the interaction of the fluid-structure by an added mass. Also, they developed the mathematical model of the plate using a combination of the Finite Element method and Sanders' shell theory. Hosseini Hashemi et al. [9] presented an exact-closed solution for free vibration of a moderately thick plate submerged in fluid or floating on fluid. They obtained equations of motion of the plate based on Mindlin plate theory. Using modal analysis expansion method, Jafari and Rahmani [10] presented natural frequency and mode shapes of a rectangular CLPT composite plate floating on the fluid. They used modal expansion method for the forced vibration analysis for the first time here. In their paper they considered limited domains for the fluid. Yousefzadeh et al. [11] investigated vibration of thick rectangular functionally graded plate floating on the fluid numerically. They derived governing equations of motion based on the first order shear deformation theory. Thinh et al. [12] studied free vibration of functionally graded rectangular plate submerged horizontally in fluid. They presented Navier's solution for solving their obtained equations of plate.

Very few researchers have worked on the nonlinear vibrations of the horizontal plate in contact with the fluid. A.A. Bukatov and A.A. Bukatov [13] used Multi Scale method to solve nonlinear free vibration equations of thin plate floating on incompressible fluid. They considered a traveling periodic wave of finite amplitude in their paper. Soni et al. [14] analyzed vibration of partially cracked plate submerged in fluid. They used classical plate theory for deriving governing equation. They modeled the fluid based on velocity potential and Bernoulli's equation and considered the geometric nonlinearity due to in-plane forces to be extracted from the equations. Finally, an analytical solution was applied to solve the governing equations of motion for different boundary conditions. Hashemi and jafari [15] investigated nonlinear free vibration of FG plate in contact with fluid. They used von Karman nonlinear strain displacement and FSTD theory to derive equations of motion and then solved these equations by Lindstedt-Poincare method.

Due to the computational difficulties of the traditional perturbation methods and since their application is often limited to the cases where small parameters are involved, other approaches have been considered by some of the researchers. Modified Lindstedt Poincare (MLP) method was applied by He [16] for both small and large parameters. Multiple Scales Lindstedt-Poincare method was used by Pakdemirli et al. [17] for the first time for analyzing a forced vibration problem. A new approach was proposed by Hai-En Du et al. [18] for improving the solutions to strongly nonlinear systems from perturbation methods. The nonlinear free vibration of a simply supported FG rectangular plate was analytically studied by S. Hashemi and A.A. Jafari [19] from Modified Lindstedt Poincare (MLP) method, for the first time. M. A. Razzak et al. [20] used the Modified Multiple Scale (MMTS) method for investigation of the nonlinear forced vibration of systems. While being simple and straightforward, this approach leads to results well compatible with the numerical results and is more accurate than the existing solutions.

The Modified Multiple Scales method is an analytical method that can be considered as closed form solution for nonlinear free and forced vibration of plate. The main advantage of the present method is that it covers all the cases: weak nonlinearities with small damping effect, weak nonlinearities with strong damping effect, and strong nonlinearities with strong damping effect. Very recently, using Modified Multiple Scales method, Rabiee and Jafari [21], the nonlinear forced vibration of a rectangular plate was analytically investigated for the first time. Their research focuses on resonance case with 3:1 internal resonance. Their obtained results were compared with both the traditional Multiple Scales method and previous studies, and excellent compatibility was observed.

Based on a review of the literature, very few researchers have analyzed the vibrations of a nonlinear plate floating on the fluid analytically. Present paper focuses on nonlinear forced vibration of a plate floating on the fluid by a new analytical approach. The plate is subjected to transversal harmonic excitation. Based on Hamilton principle and the first order shear deformation theory, equations of motion are derived, first. The fluid is mathematically modeled using the Bernoulli equation and the velocity potential function. The Modified Multiple Scales method is applied for the first time in this study for solving the nonlinear equations of a plate floating on the fluid. The results are then validated in 3 stages. In step one, the linear natural frequency obtained from this study is compared with the linear frequency from previous works, and a good agreement is noted. In the next step, the first five frequencies of the wet plate are obtained from the presented formulation and the results are compared with the existing studies. And in step 3, the transverse displacement of the dry plate is validated with Runge Kutta method. Finally, effects of some key parameter on the results are investigated and presented.

Geometry of the problem

A rectangular simply supported plate floating on the fluid is concerned, as shown in Figure1. The mid plane is selected as the origin of the Cartesian coordinate system.

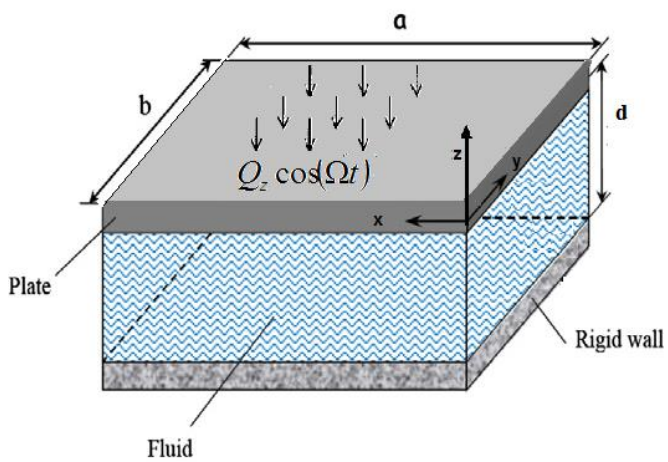

Figure 1. The geometry of the problem

Equations

All equations must be written using Times New Roman font and 10pt size. Number all equations sequentially. Each equation number should be right justified written in parentheses.

$$\left(\rho c_p\right)_{nf}\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\frac{\partial}{\partial x}\left(k_{nf}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial x}\left(k_{nf}\frac{\partial T}{\partial y}\right)$$

Formulation Of The Plate

Based on the FSTD theory of the plate for the large deflections, the displacement field of the plate is considered to be as follows [22]:

$$u(x,y,z,t)=u_0(x,y,t)+z\alpha(x,y,t), \tag{1}$$

$$v(x,y,z,t)=v_0(x,y,t)+z\beta(x,y,t), \tag{2}$$

$$w(x,y,z,t)=w_0(x,y,z). \tag{3}$$

where u, v, and w are the displacements in x, y, z directions, respectively, u_0, v_0, and w_0 are the displacements of the origin, and α, β are rotations about y and x axes, respectively. Due to small thickness of the plate, u_0, v_0 are assumed to be zero,

Von Karman nonlinear strain displacement relationships are written as [15]:

$$\varepsilon_{xx}=z\frac{\partial\alpha(x,y,t)}{\partial x}+0.5\left(\frac{\partial w_0\left(x,y,t\right)\big)}{\partial x}\right)^2, \tag{4-a}$$

$$\varepsilon_{yy}=z\frac{\partial\beta(x,y,t)}{\partial y}+0.5\left(\frac{\partial w_0\left(x,y,t\right)\big)}{\partial y}\right)^2, \tag{4-b}$$

$$\gamma_{xy}=z\left(\frac{\partial\alpha(x,y,t)}{\partial y}+\frac{\partial\beta(x,y,t)}{\partial x}\right)+\frac{\partial w_0\left(x,y,t\right)\big)}{\partial x}\frac{\partial w_0\left(x,y,t\right)\big)}{\partial y}, \tag{4-c}$$

$$\gamma_{xz}=\frac{\partial w_0\left(x,y,t\right)\big)}{\partial x}+\alpha(x,y,t), \tag{4-d}$$

$$\gamma_{yz}=\frac{\partial w_0\left(x,y,t\right)\big)}{\partial y(x,y,t)}+\beta(x,y,t). \tag{4-e}$$

The extended Hamilton principle for plates in contact with fluid is [23]:

$$0=\int\left(\delta T_{Plate}+\delta T_{Fluid}-\delta\Pi+\delta W_{nc}\right)dt \tag{5}$$

Where, body forces are neglected. $W_{nc}, T_{plate}, T_{fluid}$, and Π are the non-conservative energy, kinetic energy of the plate, kinetic energy of the fluid, and elastic energy of plate respectively, and δ is the variation operator. δW_{nc}, δT_{plate} and $\delta\Pi$ are found from [23]:

$$\delta T_{plate} = -\int_A \int_z \rho \ddot{D} \cdot \delta D \, dz dA \tag{6}$$

$$\delta W_{nc} = \delta W_{Nc\_Q} + \delta W_{Nc\_damping} = \int_A \int_z Q_z \cos(\Omega t) \cdot \delta w dz dA + \int_A c \frac{\partial w}{\partial t} \delta w dA \tag{7}$$

$$\delta \Pi = \int_A \int_z \left(\sigma_{11} \delta\varepsilon_{11} + \sigma_{22} \delta\varepsilon_{22} + \sigma_{33} \delta\varepsilon_{33} + \sigma_{23} \delta\varepsilon_{23} + \sigma_{13} \delta\varepsilon_{13} + \sigma_{12} \delta\varepsilon_{12} \right) dz \, dA \tag{8}$$

In which, A is the un-deformed area of the reference plane, ρ is the mass density , D is the displacement vector of an arbitrary point of the subject differential plate element , σ_{ij} and ε_{ij} are the Jaumann stresses and strains, respectively, Q is the external force vector, and c is the damper coefficient. According to [23]:

$$D = (-z\alpha)\hat{j}_1 + (-z\beta)\hat{j}_2 + w\hat{j}_3 \tag{9}$$

1.1. *Formulation of the Fluid*

The fluid is assumed to be stationary, incompressible, non-viscous, and irrotational. The fluid displacement is also considered to be small. ρ_f is the density of fluid, and a, b, and d are the length, width, and depth of the fluid's tank, respectively. According to the assumption, velocity potential function of fluid Φ must satisfy the Laplace equation. In the Cartesian coordinates, Laplace equation can be expressed as [9]:

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{10}$$

By method of separation of variables, the velocity potential function is obtained as follows:

$$\Phi(x, y, z, t) = \varphi(x, y, t) R(z) \tag{11}$$

Where $\varphi(x, y, t)$ and $R(z)$ are two unknown functions that will be obtained by considering the plate-fluid interaction. The kinetic energy of the fluid, T_{Fluid} is [6]:

$$T_{fluid} = \frac{1}{2} \rho_f \int_0^a \int_0^b \Phi(x, y, z, t)\Big|_{z=-h/2} \times \frac{\partial w}{\partial t} dy \, dx \tag{12}$$

Boundary Conditions and Discretization

Boundary Condition of the Plate

Boundary conditions for simply supported plate with movable edges are [15]:

$$\begin{aligned} & w_0 = \beta = 0, \frac{\partial^2 w_0}{\partial x^2} + \nu \frac{\partial^2 w_0}{\partial y^2} = 0, \qquad at\, x = 0, a \\ & w_0 = \alpha = 0, \frac{\partial^2 w_0}{\partial y^2} + \nu \frac{\partial^2 w_0}{\partial x^2} = 0, \qquad at\, y = 0, b \end{aligned} \tag{13}$$

For a harmonic solution, the following expansions are used [15]:

$$w_0(x,y,t)=\sum_{m=1}^{M}\sum_{n=1}^{N}W_{m,n}(t)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$
$$\alpha(x,y,t)=\sum_{m=1}^{M}\sum_{n=1}^{N}A_{m,n}(t)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad (14)$$
$$\beta(x,y,t)=\sum_{m=1}^{M}\sum_{n=1}^{N}B_{m,n}(t)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

1.2. *Boundary Condition of the Fluid*

At the plate-fluid interaction, compatibility condition of the plate can be stated as [9]:

$$\left.\frac{\partial \Phi}{\partial z}\right|_{z=-h/2}=\frac{\partial w_0}{\partial t} \quad (15)$$

From Eq. (15), velocity potential function of the plate is obtained as:

$$\Phi(x,y,z,t)=\frac{R(z)}{\left.\frac{\partial R}{\partial z}\right|_{z=-h/2}}\frac{\partial w_0}{\partial t} \quad (16)$$

By substituting Eq. (16) into (10):

$$\frac{\partial^2 R(z)}{\partial z^2}-\eta^2 R(z)=0 \quad (17)$$

Where η is independent of z and is found in the next Section, after the mode definition. From Eq. (17), R(z) is found as follows:

$$R(z)=C_1e^{\eta z}+C_2e^{-\eta z} \quad (18)$$

Where C_1 and C_2 are unknown coefficients and are derived in this Section. Therefore, by substituting Eq. (18) into (16):

$$\Phi(x,y,z,t)=\frac{C_1e^{\eta z}+C_2e^{-\eta z}}{\eta\left(C_1e^{-\eta h/2}-C_2e^{\eta h/2}\right)}\frac{\partial w_0}{\partial t} \quad (19)$$

Boundary condition at the bottom of the tank is [9]:

$$\left.\frac{\partial \Phi}{\partial z}\right|_{z=-d-h/2}=0 \quad (20)$$

By substituting (20) in (19)

$$\Phi(x,y,z,t)=\frac{e^{\eta(2(d+h/2)+z)}+e^{-\eta z}}{\eta\left(e^{\eta(2d+h/2)}-e^{\eta h/2}\right)}\frac{\partial w_0}{\partial t} \quad (21)$$

The variation of the kinetic energy of the fluid can then be expressed as:

$$\delta T_{fluid}=-M_1\int_0^a\int_0^b\frac{\partial^2 w}{\partial t^2}\times\delta w\,dy\,dx \quad (22)$$

where M_1 is obtained as:

$$M_1 = \frac{\rho_f}{\eta} \frac{\left(e^{\eta(2d+h/2)} + e^{\eta h/2}\right)}{\left(e^{\eta(2d+h/2)} - e^{\eta h/2}\right)} \tag{23}$$

Considering the first term in the series of Eq. (14)

$$\eta^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 \tag{24}$$

Equations of Motion of the Plate in Contact with Fluid

Substituting Equations (6)-(8) and (23) into Equation (5), plate motion equations are obtained:

$$\frac{Eh}{2(1-\nu^2)} \left(\begin{array}{l} (1-\nu)\left(\frac{\partial^2 w_0(x,y,t)}{\partial x^2} + \frac{\partial^2 w_0(x,y,t)}{\partial y^2} + \frac{\partial \alpha(x,y,t)}{\partial x} + \frac{\partial \beta(x,y,t)}{\partial y} \right) \\ +4(1-\nu)\frac{\partial^2 w_0(x,y,t)}{\partial x \partial y} \frac{\partial w_0(x,y,t)}{\partial x} \frac{\partial w_0(x,y,t)}{\partial y} \\ +\frac{\partial^2 w_0(x,y,t)}{\partial y^2}\left(3\left(\frac{\partial w_0(x,y,t)}{\partial y}\right)^2 + \left(\frac{\partial w_0(x,y,t)}{\partial x}\right)^2 \right) \\ +\frac{\partial^2 w_0(x,y,t)}{\partial x^2}\left(3\left(\frac{\partial w_0(x,y,t)}{\partial x}\right)^2 + \left(\frac{\partial w_0(x,y,t)}{\partial y}\right)^2 \right) \end{array} \right) \tag{25}$$

$$+\left(\rho_{plate} h + \mu_{fluid}\right)\frac{\partial^2 w(x,y,t)}{\partial t^2} + c\left(\frac{\partial w}{\partial t}\right) + q_z \cos(\Omega t) = 0$$

$$\frac{E}{(1-\nu^2)} \left(\begin{array}{l} 2\frac{\partial^2 \alpha(x,y,t)}{\partial x^2} + (1-\nu)\frac{\partial^2 \alpha(x,y,t)}{\partial y^2} + (1+\nu)\frac{\partial^2 \beta(x,y,t)}{\partial x \partial y} \\ -12(1-\nu)\frac{\partial w_0(x,y,t)}{\partial x} - 12(1+\nu)\alpha(x,y,t) \end{array} \right) - 2\rho_{plate}\frac{\partial^2 \alpha(x,y,t)}{\partial t^2} = 0 \tag{26}$$

$$\frac{E}{(1-\nu^2)} \left(\begin{array}{l} 2\frac{\partial^2 \beta(x,y,t)}{\partial y^2} + (1-\nu)\frac{\partial^2 \beta(x,y,t)}{\partial x^2} + (1+\nu)\frac{\partial^2 \alpha(x,y,t)}{\partial y \partial x} \\ -12(1-\nu)\frac{\partial w_0(x,y,t)}{\partial y} - 12(1+\nu)\beta(x,y,t) \end{array} \right) - 2\rho_{plate}\frac{\partial^2 \beta(x,y,t)}{\partial t^2} = 0 \tag{27}$$

Considering the first term in the series of Eq. (14) and substituting into Equations (25-27), and applying Galerkin method, the non-dimensional nonlinear time dependent equations are found as below:

$$C_{11}\ddot{A}+C_{12}W+C_{13}A+C_{14}B=0$$

$$C_{21}\ddot{B}+C_{22}W+C_{23}A+C_{24}B=0 \tag{28}$$

$$\left(\rho_{plate}h+\mu_{fluid}\right)\ddot{W}+C_{31}W^3+C_{32}W+c\dot{W}+C_{33}A+C_{34}B+Q=0$$

Where C_{ij} are defined in **APPENDIX A1**. Due to small thickness of the plate, in plane inertia effects and rotary inertia effects can be neglected [24](i.e. $C_{11}=C_{21}=0$). Also, by defining the non-dimensional parameters:

$$\bar{w}=\frac{w}{h}$$

$$\bar{Q}=\frac{12\left(ab\right)^{\frac{7}{2}}\left(1-\nu^2\right)}{Eh^7}\times Q \tag{29}$$

$$\bar{c}=\frac{\left(ab\right)^2}{h^4}\left(\frac{1}{\rho E}\right)^{0.5}\times c,$$

and dimensionless time, $t^*=th\sqrt{E/\rho ab}$, the nonlinear time dependent equation in the z direction is obtained:

$$\ddot{\bar{W}}+\omega_0{}^2\bar{W}+\bar{c}\dot{\bar{W}}+b_4\bar{W}^3+\bar{Q}=0 \tag{30}$$

Where coefficients are defined in **APPENDIX A2.**

Modified Multiple Scale Method

Forced vibration of damped plate floating on the fluid

In order to find the natural frequencies and transverse mode shapes of the plate, Equation (30) must be solved by either of, numerical, Finite Elements, or analytical solutions such as perturbation methods. The Modified Multiple Scales method is applied in current study.

According to Appendix B, the coefficient b_4 is a small value. Thus, the small non-dimensional parameter $\in$ is defined as:

$$\in=\left(h/a\right)^2$$

And Equation (30) will be become:

$$\ddot{W}+\omega_0{}^2W+2k\dot{W}=\in\left(B_4W^3+p\cos\left(\Omega t\right)\right) \tag{31}$$

It should be noted that the bar has been omitted for simplification. In which, due to the presence of $\in$, weak nonlinear terms appear in the equation (Appendix B) and ω_0 is the undamped linear natural frequency, $\bar{Q}=\in p\cos(\Omega t)$, $\bar{c}=2k$, $\omega_0\geq 0$ and $k<\omega_0$.

For $\in=0$, the two Eigen values of Equations (31) are obtained as [19]:

$$\lambda_1=-k+i\omega, \lambda_2=-k-i\omega \tag{32}$$

Where $\omega=\sqrt{\omega_0^{\,2}-k^2}$, For $\in\neq 0$:

$$(D-\lambda_1)(D-\lambda_2)W=\in\left(B_4W^3+p\cos(\Omega t)\right) \tag{33}$$

Where:

$$D=D_0+\in D_1+\in^2 D_2+... \ , D_i(\ )=\frac{d^i(\ )}{dt^i} \qquad i=0,1,2,... \tag{34}$$

Assuming the first approximate solution of Equation (31) in form of:

$$W=a_1(t)+a_2(t)+\in u_1(t)+... \tag{35}$$

The right hand side term of Equion (31) can be extended as below:

$$-B_4W^3=-B_4.\left(a_1^{\,3}+3a_1^{\,2}a_2+3a_1a_2^{\,2}+a_2^{\,3}+3\in(a_1+a_2)^2u_1+...\right) \tag{36}$$

Substituting Eqs. (35) and (36) into Eq. (31) and applying the separation rule [22], the following linear equations is resulted:

$$(D_0-\lambda_2)(D_1a_1)=-3B_4a_1^{\,2}a_2+\frac{1}{2}pe^{i\Omega t} \tag{37}$$

$$(D_0-\lambda_1)(D_1a_2)=-3B_4a_1a_2^{\,2}++\frac{1}{2}pe^{-i\Omega t} \tag{38}$$

$$(D_0-\lambda_1)(D_0-\lambda_2)u_1=-B_4\left(a_1^{\,3}+a_2^{\,3}\right) \tag{39}$$

The solution of Equation (37) will be [19]:

$$D_1a_1=l_1a_1^{\,2}a_2 \tag{40}$$

Since $D_0a_1=\lambda_1a_1$ and $D_0a_2=\lambda_1a_2$, Equation (37) leads to:

$$(D_0-\lambda_2)\left(l_1a_1^{\,2}a_2\right)=-3B_4a_1^{\,2}a_2 \tag{41}$$

And from Equation (41):

$$l_1=-\frac{3B_4}{2\lambda_1} \tag{42}$$

The equations can then be written as follows:

$$\dot{a}_1=Da_1=(D_0+\in D_1+...)a_1=\lambda_1a_1+\in D_1a_1+O\left(\in^2\right). \tag{43}$$

The second and higher orders of $\in$ from Equation (43) are neglected and thus:

$$\dot{a}_1 = (-k+i\omega)a_1 + \in \left(\frac{3B(k+i\omega)a_1^2 a_2}{2(k^2+\omega^2)} + \frac{p\left(k-i(\Omega+\omega)e^{i\Omega t}\right)}{\left(2\left(k^2+(\Omega+\omega)^2\right)\right)} \right). \tag{44}$$

Assuming $a_1 = \Lambda e^{i\psi}/2$ and $a_2 = \Lambda e^{-i\psi}/2$, and separating the imaginary and real parts on both sides of Equation (44):

$$\dot{\Lambda} = -k\Lambda + \frac{3 \in B\Lambda^3 k}{8(k^2+\omega^2)} + \frac{\in p\left(k\cos(\psi)-(\Omega+\omega)\sin(\psi)\right)}{\left(k^2+(\Omega+\omega)^2\right)} \tag{45}$$

$$\Lambda\dot{\phi} = \omega\Lambda + \frac{3 \in B\Lambda^3 \omega}{8(k^2+\omega^2)} + \frac{\in p\left(-k\sin(\psi)-(\Omega+\omega)\cos(\psi)\right)}{\left(k^2+(\Omega+\omega)^2\right)} \tag{46}$$

Where $\phi = \psi + \Omega t$ For the steady-state $\dot{\Lambda} = 0$ and $\dot{\phi} = \Omega$, then Equations (45)-(46) become

$$\Lambda\left(k - \frac{3 \in B\Lambda^2 k}{8(k^2+\omega^2)} \right) = \frac{\in p\left(k\cos(\psi)-(\Omega+\omega)\sin(\psi)\right)}{\left(k^2+(\Omega+\omega)^2\right)} \tag{47}$$

$$\Lambda\left(\Omega - \omega - \frac{3 \in B\Lambda^2 \omega}{8(k^2+\omega^2)} \right) = \frac{\in p\left(-k\sin(\psi)-(\Omega+\omega)\cos(\psi)\right)}{\left(k^2+(\Omega+\omega)^2\right)} \tag{48}$$

By squaring both sides of Equations (32)-(33) and adding these equations , frequency response equation is given by:

$$\Lambda^2\left(k - \frac{3 \in B\Lambda^2 k}{8(k^2+\omega^2)} \right)^2 + \Lambda^2\left(\Omega - \omega - \frac{3 \in B\Lambda^2 \omega}{8(k^2+\omega^2)} \right)^2 = \frac{\in^2 p^2}{\left(k^2+(\Omega+\omega)^2\right)} \tag{49}$$

If Ω is given, Λ is obtained from Equation (49), by substituting obtained Λ into Equation (47) or (48) , ψ is obtained. Therefore a_1 and a_2 is obtained. From Equation (39) particular solution of u_1 obtained as follow

$$u_1 = C_1 a_1^3 + C_2 a_2^3 = C_1\left(\frac{\Lambda}{2}e^{i(\Omega t-\psi)}\right)^3 + C_2\left(\frac{\Lambda}{2}e^{-i(\Omega t-\psi)}\right)^3 \tag{50}$$

And according to Equation (39), C_1 and C_2 are

$$C_1 = -\frac{B}{2\lambda_1(3\lambda_1-\lambda_2)},\ C_2 = -\frac{B}{2\lambda_2(3\lambda_2-\lambda_1)} \tag{51}$$

Equation (51) can be rewritten as follow

$$C_1 = \mathrm{Re}^{i\theta},\ C_2 = \mathrm{Re}^{-i\theta} \tag{52}$$

Where

$$R = \frac{B}{2}\frac{\sqrt{\left(2k^2-4\omega^2\right)^2+\left(6k\omega\right)^2}}{\left(2k^2-4\omega^2\right)^2+\left(6k\omega\right)^2},\ \theta = \arctan\left(\frac{6k\omega}{\left(2k^2-4\omega^2\right)}\right) \tag{53}$$

Therefore

$$u_1 = R\frac{\Lambda^3}{8}\left(e^{i(\theta+3(\Omega t-\psi))}+e^{-i(\theta+3(\Omega t-\psi))}\right) \tag{54}$$

Equation (35) can be rewritten as follow

$$W = \Lambda\cos\left(\Omega t-\psi\right)+\varepsilon R\frac{\Lambda^3}{4}\cos\left(3\left(\Omega t-\psi\right)+\theta\right) \tag{55}$$

As regards that θ is a small parameter, by neglecting θ

$$W = \Lambda\cos\left(\Omega t-\psi\right)+\varepsilon R\frac{\Lambda^3}{4}\cos\left(3\left(\Omega t-\psi\right)\right) \tag{56}$$

Free vibration of un-damped plate floating on the fluid

and Equation (31) will be become:

$$\ddot{W}+\omega_0^2 W = \in\left(B_4 W^3\right) \tag{57}$$

In which, due to the presence of $\in$, weak nonlinear terms appear in the equation **(Appendix A)** and ω_0 is the undamped linear natural frequency.

For $\in = 0$, the two Eigen values of Equations (57) are obtained as [19]:

$$\lambda_1 = i\omega_0, \lambda_2 = -i\omega_0 \tag{58}$$

For $\in \neq 0$:

$$\left(D-\lambda_1\right)\left(D-\lambda_2\right)W = \in\left(B_4 W^3\right) \tag{59}$$

Assuming the first approximate solution of Equation (57) in form of:

$$W = a_1\left(t\right)+a_2\left(t\right)+\in u_1(t)+\ldots \tag{60}$$

Substituting Eqs. (36) and (60) into Eq. (58) and applying the separation rule [24], the following linear equations is resulted:

$$\left(D_0-\lambda_2\right)\left(D_1a_1\right)=-3B_4a_1^{\ 2}a_2 \tag{61}$$

$$\left(D_0-\lambda_1\right)\left(D_1a_2\right)=-3B_4a_1a_2^{\ 2} \tag{62}$$

$$\left(D_0-\lambda_1\right)\left(D_0-\lambda_2\right)u_1=-B_4\left(a_1^{\ 3}+a_2^{\ 3}\right) \tag{63}$$

In the same approach, $\dot{a}_1$ obtained as follows:

$$\dot{a}_1=i\omega a_1+\in\left(-\frac{3B_4}{2\left(i\omega\right)}a_1^{\ 2}a_2\right). \tag{64}$$

Assuming $a_1=\Lambda e^{i\psi}/2$ and $a_2=\Lambda e^{-i\psi}/2$, and separating the imaginary and real parts on both sides of Equation (64):

$$\dot{\Lambda}=0 \tag{65}$$

$$\dot{\psi}=\omega_0+\frac{3\in B_4}{8\omega_0}\Lambda^2 \tag{66}$$

Therefore Λ is derived as a constant value, and:

$$a_1=\frac{\Lambda}{2}e^{i\left(\omega_0+\frac{3\in B_4}{8\omega_0}\Lambda^2\right)t} \tag{67}$$

$$a_2=\frac{\Lambda}{2}e^{i\left(\omega_0+\frac{3\in B_4}{8\omega_0}\Lambda^2\right)t} \tag{68}$$

From Equation (63), the particular solution of u_1 is found as follows:

$$u_1=C_1a_1^{\ 3}+C_2a_2^{\ 3}=C_1\frac{\Lambda}{2}e^{3i\left(\omega_0+\frac{3\in B_4}{16\omega_0}\Lambda^2\right)t}+C_2\frac{\Lambda}{2}e^{-3i\left(\omega_0+\frac{3\in B_4}{16\omega_0}\Lambda^2\right)t} \tag{69}$$

And according to Equation (63), C_1 and C_2 are:

$$C_1=-\frac{B_4}{2\lambda_1\left(3\lambda_1-\lambda_2\right)},C_2=-\frac{B_4}{2\lambda_2\left(3\lambda_2-\lambda_1\right)} \tag{70}$$

Therefore,

$$u_1=C_1a_1^{\ 3}+C_2a_2^{\ 3}=\frac{\Lambda^3B_4}{64\omega^2}e^{3i\left(\omega_0+\frac{3\in B_4}{16\omega_0}\Lambda^2\right)t}+\frac{\Lambda B_4}{64\omega^2}e^{-3i\left(\omega_0+\frac{3\in B_4}{16\omega_0}\Lambda^2\right)t} \tag{71}$$

Equation (60) can be presented in the following form:

$$W=\Lambda\cos\left(\left(\omega_0+\frac{3\in B_4}{16\omega_0}\Lambda^2\right)t\right)+\frac{\Lambda^3\in B_4}{32\omega^2}\cos\left(3\left(\omega_0+\frac{3\in B_4}{16\omega_0}\Lambda^2\right)t\right) \tag{72}$$

Nonlinear frequency is then obtained as:

$$\omega_{NL}=\omega_0+\frac{3\in B_4}{16\omega_0}\Lambda^2 \tag{73}$$

Validation and Results

The derived relations in the present study are validated in three steps. First, the natural frequency of the simply supported plate is compared with existing researches. Thereafter, the first five frequencies of the plate in contact with fluid are validated as the second step, in order to verify the accuracy of the fluid formulation. And finally, the solution obtained from this study is compared with both the exact solution and current Multiple Scale Method.

Non-dimensional linear natural frequency of a simply supported isotropic square plate ($E = 380GPa,\ \rho = 3800Kg/m^3, h = 0.1m, a/h = 20$) obtained from the proposed approach is compared with the published results by Hosseieni [24] and Leissa [25] in Table 1. As can be seen, the results show an excellent compatibility.

Table 1. Non dimensional linear natural frequency for the square plate

Method	Current Study	Hosseini Hashemi [24]	Leissa [25]
	0.01484	0.0148	0.01493

To validate the fluid relations, the first five frequencies of the plate (,) in contact with the fluid and in vacuum are compared with Hosseini's results in Table 2. The results are in good agreement with the results in the referenced article [9]. In Hosseini Hashemi's article, a submerged plate in the fluid is considered, where the parameter is the height of the fluid above the plate.

Table 2. The first five frequencies of the plate in contact with water

	In vacuum			
Mode (m,n)	Present Study	Reference [9]	Present Study	Reference [9]
(1,1)	48.63	48.30	41.663	41.429
(2,1)	77.16	76.33	68.029	64.525
(3,1)	123.669	121.632	111.734	110.369
(1,2)	159.984	156.685	146.24	149.690
(2,2)	186.726	182.338	171.791	171.969

To validate the fluid relations, the first five frequencies of the plate ($h/a = 0.05, a/b = 2$) in contact with the fluid and in vacuum are compared with Hosseini's results in Table 2. The results are in good agreement with the results in the referenced article [9]. In Hosseini Hashemi's article, a submerged plate in the fluid is considered, where the h_1 parameter is the height of the fluid above the plate.

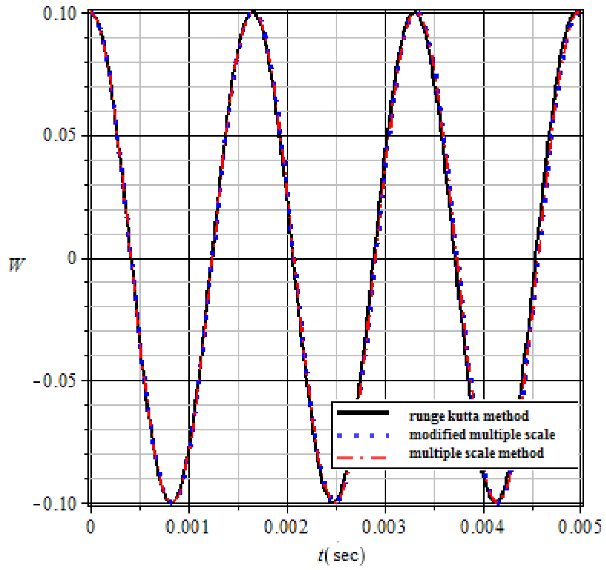

Figure 2. Non-dimensional transverse displacement of dry plate for Modifed multiple scales method, multiple scales method and runge kutta method

Transverse non-dimensional displacement of the plate floating on the water ($E = 207GPa$, $\rho = 7850Kg/m^3$, $\rho_f = 1000kg/m^3$ $h = 0.05m$, $a = 1m$, $a/b = 2$) is shown in Figure 3. A good agreement is noticed in this Figure, as well.

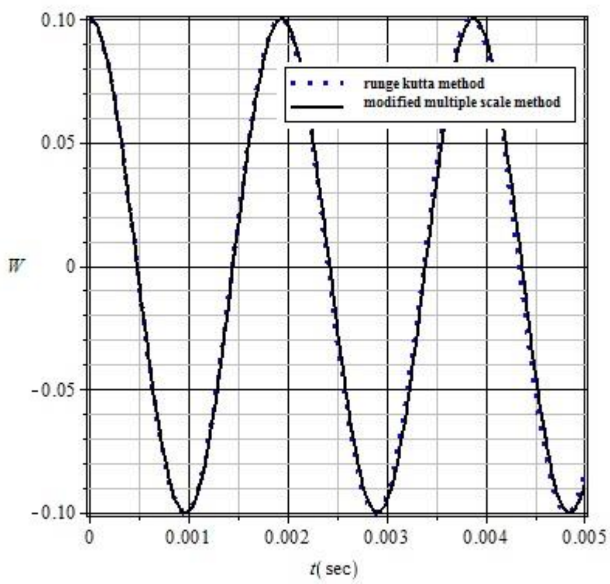

Figure 3. Non dimensional transverse displacement of plate floating on the water

The nonlinear frequency ratio of the simply supported plate ($E = 207GPa$, $\rho = 7850Kg/m^3$, $\rho_f = 1000kg/m^3$ $a = 1m$, $a/b = 2$) is shown in Figure 4. As can be seen, the nonlinear frequency ratio is increased as the plate thickness increases and it can be concluded that thickness has a considerable effect on the ratio of frequency response and natural frequency.

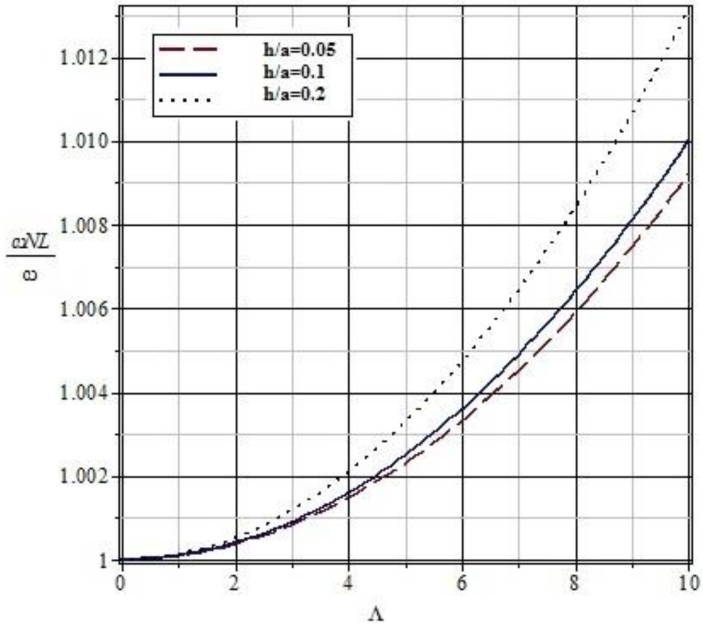

Figure 4. Thickness effects on nonlinear frequency ratios for isotropic simply supported plate floating on the water

The dimensions of the plate have a great effect on the flexibility of the plate and consequently its frequency. Figure 5 shows the effect of decreased plate width (b) on the nonlinear frequency ratio of the plate for a constant plate length ($E = 207GPa$, $\rho = 7850 Kg/m^3$, $\rho_f = 1000 kg/m^3$, $a = 1m$, $h = 0.05m$).

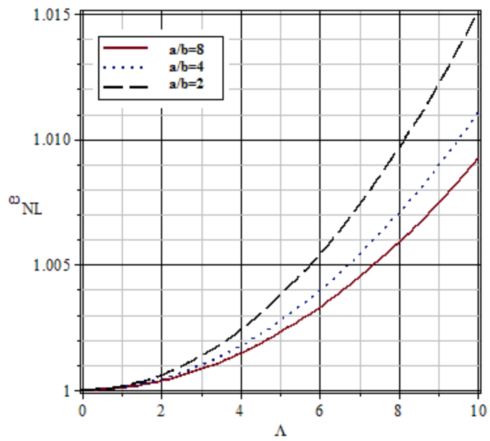

Figure 5. Effects of aspect ratio on nonlinear frequency ratios for isotropic simply supported plate floating on the water

Figure 6 shows the effect of the presence of fluid near the plate on the dimensionless nonlinear frequency for free vibrations of a square plate ($h = 0.05m$, $a = 1m$, $E = 207GPa$, $\rho = 7850 Kg/m^3$). As can be seen in Figure 6, the contact of the plate with the fluid significantly decreases the nonlinear frequency of the plate. Thus, considering an additional mass to account for the fluid effect is a good way to simplify the complex fluid equations.

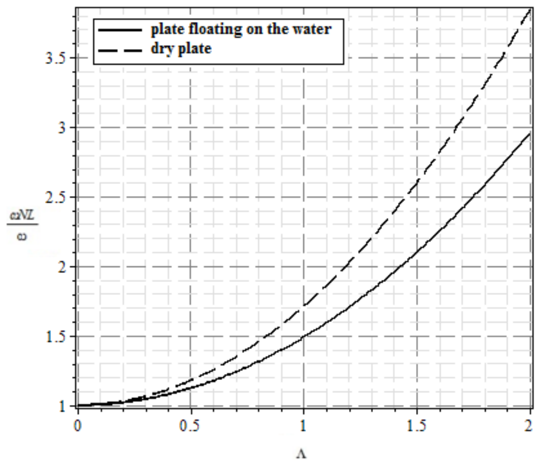

Figure 6. Effect of the presence of fluid near the plate on the non-dimensional frequency ratio of the plate

In Figure 7, the frequency response of forced vibration of dry plate and plate floating on the water for $q_z = 5N/m^2$ and $\xi = C/2\omega_0 = 0.0001$ are shown. The plate properties is $E = 70GPa,\ \rho = 2778Kg/m^3, h = 0.001m, a = 0.6m, b = 0.3m$. Results obtained from Modified multiple scale indicated that presence of fluid close to a plate significantly decreases displacement of plate.

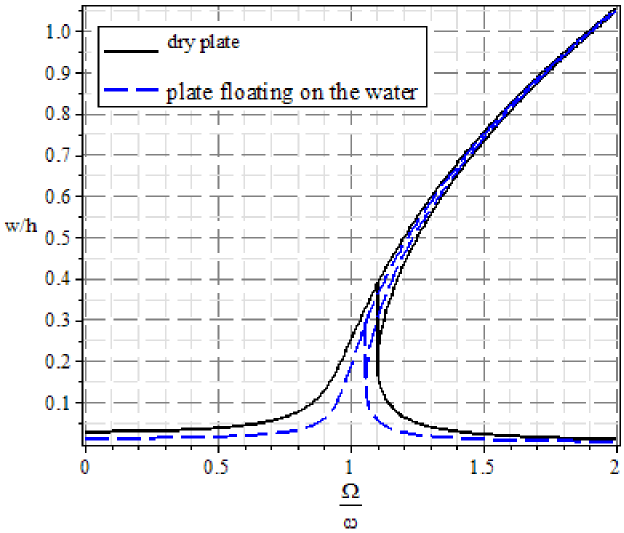

Figure 7. Effects of presence of fluid in frequency response of forced vibration of plate for and

Figure 8 shows transverse non-dimensional displacement of the plate subjected to transverse harmonic excitation ($\Omega = 0.8\omega_0$). Clearly, in figure 8 indicated that presence of fluid close to the plate decrease amplitude of vibration and reduced the natural frequency of plate.

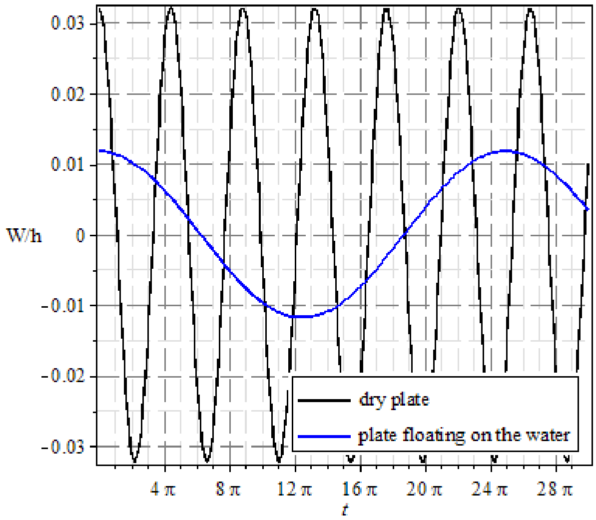

Figure 8. Effects of presence of fluid in transverse displacement of forced vibration of plate for and,

Conclusion

The nonlinear vibration of a rectangular plate floating on the fluid investigated. Nonlinear partial differential equations of motion are derived based on the first order shear deformation theory of plates and von Karman nonlinear strain-displacement relations. The fluid mathematical model is created by Bernoulli equation and the velocity potential function. According to the boundary conditions of the plate, and from Galerkin method, ordinary nonlinear differential equations are obtained. The nonlinear system is analytically solved by the Modified Multiple Scale method. The natural frequency and non-linear natural frequency of the plate were confirmed in comparison with the previous studies. Also, the non-dimensional transverse displacement of the plate in vacuum and in contact with fluid was verified by numerical results. The effect of the presence of fluid near the plate on the natural frequency of the plate and the amplitude of forced vibrations of the sheet compared to the plate in vacuum is also shown.

Clearly, it is indicated that the Modified Multiple Scales method is very simple and leads to better results compared with other available ones and thus, can be used as a powerful means for solving inhomogeneous nonlinear equations such as forced vibration equations of structures.

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