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Least squares weighted residual method for finding the elastic stress fields in rectangular plates under uniaxial parabolically distributed edge loads

Charles C. Ike [a,], Clifford U. Nwoji [b], Benjamin O. Mama [c], Hyginus N. Onah [d], Michael E. Onyia [e]*

[a] *Department of Civil Engineering, Enugu State University of Science and Technology, Enugu State, Nigeria*
[b, c, d, e] *Department of Civil Engineering, University of Nigeria Nsukka, Enugu State, Nigeria*

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ABSTRACT

In this work, the least squares weighted residual method is used to solve the two-dimensional (2D) elasticity problem of a rectangular plate of in-plane dimensions $2a \times 2b$ subjected to parabolic edge tensile loads applied at the two edges $x = \pm a$. The problem is expressed using Beltrami–Michell stress formulation. Airy's stress function method is applied to the stress compatibility equation, and the problem is expressed as a boundary value problem (BVP) represented by a non-homogeneous biharmonic equation. Airy's stress functions are chosen in terms of one and three unknown parameters and coordinate functions that satisfy both the domain equations and the boundary conditions on the loaded edges. Least squares weighted residual integral formulations of the problems are solved to determine the unknown parameters and thus the Airy stress function. The normal and shear stress fields are determined for the one-parameter and the three-parameter coordinate functions. The solutions for the stress fields are found to satisfy the stress boundary conditions as well as the domain equation. The presented solutions for the Airy stress function and the normal stresses and shear stress fields are identical with solutions obtained by using variational Ritz methods, Bubnov–Galerkin methods and agree with results obtained by Timoshenko and Goodier.

1. Introduction

1.1 Background

The governing equations of the theory of elasticity for three-dimensional (3D) problems in general consists of a system of fifteen equations in terms of fifteen unknowns which are: six Cauchy stress components, six strain components and three displacement components [1 – 2].

Mathematical or analytical solutions of the system of governing equations are difficult to obtain in closed form and this difficulty has motivated the development of modified ways of formulation and solutions of the problems of the theory of elasticity. There are three basic procedures of the formulation, namely: displacement-based formulation, stress-based formulation and hybrid/mixed formulation. In the displacement formulation, stresses and strains are eliminated from the governing field equations and the system of fifteen equations expressed in terms of the three Cartesian components of the displacement. This yields a system of three coupled partial differential equations in three unknowns. In the stress-based formulation, the system of field equations are re-expressed such that the strains and displacements are eliminated and the equations are expressed only in terms of the six Cauchy stress components. This yields a system of six coupled partial differential equations in six unknowns. In the mixed formulation, the system of governing equations are expressed in terms of some stress components and some displacement components as the primary unknowns.

The general problem of elasticity is made simpler by further assumptions regarding the dimensional character of the problem. Such simplifications yield two-dimensional elasticity problems which can be further classified as plane stress or plane strain.

The two-dimensional elasticity problems of rectangular plates subjected to non-uniformly distributed edge tensile

* Author Email: ikecc2007@yahoo.com; charles.ike@esut.edu.ng

loads, which are the subject matter of this work, are frequently encountered in engineering applications. Such problems are encountered in the elastic analysis of aircraft panels, spacecraft panels and machine panels, where the panels are idealized as plates carrying in-plane forces. The accurate determination of the distribution of normal stresses and shear stresses in such structural panels modelled as plates is crucial in their elastic analysis and design.

There is no mathematically exact solution so far for such problems of thin rectangular plates subjected to non-uniformly distributed edge tensile loads due to the complex nature of such elasticity problems [3 – 5]. This has necessitated the development of approximate methods for solving such complex two-dimensional problems.

Two-dimensional elasticity problems have been formulated and solved using stress functions in stress-based formulations. The stress function method/technique is based on the general concept of developing a mathematical representation for the normal and shear stress fields in the elastic body such that the differential equations of equilibrium are satisfied, and a single governing equation is obtained from the compatibility equation [4 – 8].

Stress potential functions are scalar or vector potential functions that are derived to satisfy all the differential equations of equilibrium and the compatibility equations; and from which the normal stresses and shear stress field could be derived. The normal stresses and shear stress fields are derivable from the stress functions in two-dimensional elastostatic problems by taking partial derivatives of the stress potential functions with respect to the space coordinate variables [9].

Airy stress potential functions are the most frequently encountered stress functions for two-dimensional elasticity problems of plane stress or plane strain. The merit of the Airy stress potential function use is that it simplifies the general formulation of 2D (plane) elasticity problem to a single governing stress compatibility equation expressed in terms of a single scalar function. This resulting governing equation can then be solved using the methods and procedures of applied mathematics to generate either analytical or numerical solutions. The two-dimensional elasticity problem then reduces to one of solving the fourth order biharmonic equation in terms of the Airy stress potential function and finding the normal stresses and shear stresses from the Airy stress function [10].

Yu-hua Yang and Xin-Wei Wang [3] obtained an approximate solution to the elastic stress analysis of thin rectangular plates under non-uniformly distributed edge loads by using the Ritz method. They used a stress-based formulation of the two-dimensional elasticity problem. They chose Chebyshev polynomials as the stress function which satisfy the boundary conditions, and then proceeded to apply the Ritz techniques in order to determine the unknown in-plane normal and shear stress fields in the rectangular plate. They studied the elastic stress distribution in plates under uniaxial and biaxial parabolically distributed edge loads with the aid of the mathematical computational software, Mathematica [3 – 5].

Their solutions for stress fields satisfy the normal stresses and shear stress boundary conditions on the four edges and agree with stress field solutions obtained using the numerical tools of finite element method and differential quadrature method [3 – 5].

Nwoji *et al*. [5] solved the two-dimensional elasticity problem of rectangular plates $(2a \times 2b)$ subjected to parabolically distributed edge tensile loads applied uniaxially at the two faces $x = \pm a$. They adopted a stress-based formulation of the elasticity problem and used the Airy stress potential function and energy principles to express the problem as a variational problem in terms of the unknown Airy stress function. The total potential energy functional was thus obtained in terms of the Airy stress potential function. Suitable Airy stress potential functions were chosen in terms of known coordinates shape functions that satisfied the boundary conditions and unknown parameters, and the total potential energy functional found for one-parameter and three-parameter Airy stress function. They then applied the variational Ritz method to obtain the minimized total potential energy functional with respect to the unknown parameters in the one- and three- parameter formulations. This yielded the unknown parameters, and hence the Airy stress potential functions for the one and three-parameter variational formulation. They obtained the corresponding normal and shear stress fields for the one-parameter and three-parameter formulation from the corresponding Airy stress potential function. Their solutions also satisfied the stress boundary conditions at the four edges and agreed with the solutions from literature.

Mama *et al*. [4] used the Bubnov – Galerkin method to solve the two-dimensional elasticity problem of rectangular plates $(2a \times 2b)$ under uniaxial in-plane parabolically distributed edge loads at $x = \pm a$. They adopted the stress-based formulation, and used Airy stress potential function to express the 2D elasticity problem as a boundary value problem represented by the inhomogeneous biharmonic equation in terms of Airy stress function. They found Airy stress potential functions in terms of one and three unknown parameters and coordinate shape functions that satisfied both the domain equations and the boundary conditions along the edges. They then formulated and solved the Bubnov–Galerkin variational integral statements for both the one-parameter and the three-parameter Airy stress potential function. They thus determined the unknown parameters of the Airy stress functions and hence the Airy stress functions for the one and three-parameters cases studied. They further determined the normal stresses and shear stress fields for the two cases of one-parameter, and three-parameter Airy stress function considered. Their stress field solutions agreed with the results from previous studies.

Considerable research work has been done on the theme of elasticity of plates, shells and beams. Some of them are reported by: Nejad and Hadi [11 – 12], Nejad et al [13 – 16], Doneshmehr et al [17], Hadi et al [18 – 19], Dahsharihri, et al [20], Barati et al [21], Noroozi et al [22] and Zarezadeh, et al [23].

The novelty of the present work is the use of the least squares weighted residual methodology in a systematic and first principles manner to present and solve the elasticity problem of finding the stresses in a rectangular plate due to parabolic variations of loads on the two opposite edges.

In this work, the least squares weighted residual method is used to solve the two-dimensional elasticity problem of rectangular plates ($2a \times 2b$) subjected to in-plane uniaxial parabolically distributed edge loads applied at the edges $x = \pm a$, while the other edges $y = \pm b$ are free of loads.

1.2 Research aim and objectives

The general aim of this work is to use the least squares weighted residual method to solve the two-dimensional elasticity problem of a rectangular plate ($2a \times 2b$) subjected to in-plane loads distributed parabolically on the two faces $x = \pm a$ of the plate. The specific objectives are:

(i) to formulate the problem in terms of stresses using the Beltrami – Michell stress compatibility equation.

(ii) to apply Airy stress potential function to the Beltrami – Michell stress compatibility equation, and hence express the problem as a boundary value problem (BVP) in terms of Airy stress potential function.

(iii) to formulate the least squares weighted residual integral statements for the governing boundary value problem.

(iv) to solve the least squares weighted residual integral statements obtained for the cases of a one-parameter, and for a three-parameter method.

(v) to obtain the normal stresses and shear stress fields from the Airy stress potential function found for the cases of one-parameter and three-parameter least squares weighted residual formulations.

2. Theoretical background/framework

2.1 Stress formulation of plane-stress elasticity problems

The governing field equations for plane-stress elasticity problems are obtained from the stress-strain equations, the differential equations of equilibrium and the compatibility equation. The stress-strain relations for homogeneous, isotropic materials are given as:

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \mu\sigma_{yy}) \quad (1)$$

$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \mu\sigma_{xx}) \quad (2)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\mu)\tau_{xy}}{E} \quad (3)$$

where σ_{xx}, σ_{yy} are normal stresses as τ_{xy} is the shear stress, ε_{xx}, ε_{yy} are normal strains, γ_{xy} is the shear strain. G is the shear modulus which is related to the Young's modulus of elasticity denoted by E, and the Poisson's ratio denoted by μ as:

$$G = \frac{E}{2(1+\mu)} \quad (4)$$

The differential equations of static equilibrium are given by:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0 \quad (5)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0 \quad (6)$$

$$f_z = 0 \quad (7)$$

where f_x, f_y and f_z are the body force components in the x, y and z coordinate directions respectively.

The compatibility equation expressed in terms of strain is

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad (8)$$

The stress compatibility equation is obtained by substitution of the stress-strain relations in the strain compatibility equation to obtain:

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -(1+\mu)\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) \quad (9)$$

2.2 Beltrami – Michell stress compatibility equation

The Beltrami – Michell stress compatibility equation for two-dimensional (2D) problems of elasticity is given by the partial differential equation:

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -\alpha_1\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) \quad (10)$$

where f_x and f_y are body force components in the x and y coordinate directions respectively, and α_1 can be given, for plane stress, by:

$$\alpha_1 = 1 + \mu \quad (11)$$

μ is the Poisson's ratio.
∇^2 is the two-dimensional Laplacian for 2D problems in the xy coordinate plane.

The research problem considered is given by the 2D elasticity problem of a rectangular plate of dimensions $2a \times 2b$ shown in Figure 1. The plate occupies the domain $-a \le x \le a$; $-b \le y \le b$ on the xy plane. The origin of the 2D Cartesian coordinate system is at the centre of the plate and uniaxial tensile load is distributed on the two faces $x = \pm a$ according to the parabolic function:

$$\sigma_{xx}(x = \pm a, y) = q\left(1 - \frac{y^2}{b^2}\right) \quad (12)$$

where $\sigma_{xx}(x = \pm a, y = 0) = q$ (13)

$$\sigma_{xx}(x = \pm a, y = \pm b) = 0 \quad (14)$$

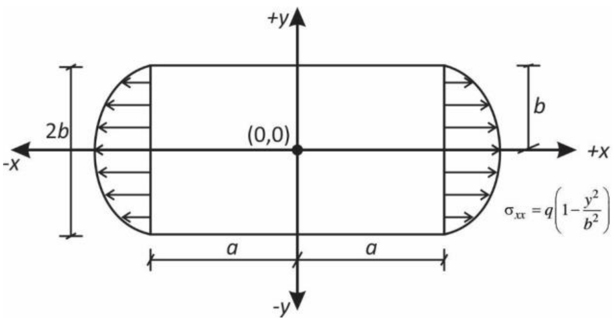


Figure 1. Rectangular plate of dimensions $2a \times 2b$ subject to in-plane parabolic distribution of edge loads at $x = \pm a$

In the absence of the body force components, $f_x = 0,\ f_y = 0$ and the partial differential equation governing the stress compatibility equation simplifies to become:

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = 0 \qquad (15)$$

2.3 Airy stress potential function formulation

The Airy stress potential function formulation of 2D elasticity problems are given by the following equations:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}(x, y) \qquad (16)$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}(x, y) \qquad (17)$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}(x, y) \qquad (18)$$

where $\phi(x, y)$ is the Airy stress potential function, $\sigma_{xx}(x, y)$ and $\sigma_{yy}(x, y)$ are normal stress fields, and $\tau_{xy}(x, y)$ is the shear stress field.

The governing equations of Beltrami–Michell stress compatibility equation becomes in terms of the Airy stress potential function $\phi(x, y)$:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial^2 \phi}{\partial y^2} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial^2 \phi}{\partial x^2} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)q\left(1 - \frac{y^2}{b^2}\right) = 0 \qquad (19)$$

Simplifying,

$$\text{or, } \nabla^2\nabla^2\phi - \frac{2q}{b^2} = \nabla^4\phi(x, y) - \frac{2q}{b^2} = 0 \qquad (20)$$

where ∇^4 is the biharmonic partial differential operator:

$$\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) = \nabla^2\nabla^2 \qquad (21)$$

3. Methodology

3.1 The least squares weighted residual method

The weighted residual method seeks to obtain approximate solutions to partial or ordinary differential equations of the general form:

$$L\varphi(x, y) + f(x, y) = 0 \qquad (22)$$

in the domain D, where $\varphi(x, y)$ is the unknown dependent variable and $f(x,y)$ is the known function, L represents a linear differential operator involving spatial derivatives of φ, which describes the exact form of the differential equation.

An approximate solution to φ is assumed as $\bar{\varphi}(x, y)$ such that $\bar{\varphi}(x, y)$ satisfies the boundary conditions imposed on the solution. The least squares weighted residual method seeks to obtain a solution to the boundary value problem such that the integral

$$I = \iint_D (L\bar{\varphi}(x, y) + f(x, y))^2 dxdy \qquad (23)$$

is minimized on the domain D with respect to the unknown parameters (a_i) of the trial function.

This yields the least squares weighted residual integral given for

$$\bar{\varphi}(x, y) = \sum_{i=1}^{n} a_i \bar{\varphi}_i(x, y) \qquad (24)$$

as:

$$\frac{\partial}{\partial a_i}\iint_D \left(L\sum_{i=1}^{n} a_i \bar{\varphi}_i(x, y) + f(x, y)\right)^2 dxdy = 0 \qquad (25)$$

3.2 Application of the Least Squares Weighted Residual Method to A Rectangular Plate under Uniaxial Parabolic In-Plane Load

For the general 2D elasticity problem of a rectangular plate under uniaxial parabolic in-plane load shown in Figure 1, if the approximate solution for the Airy stress potential function is denoted by $\bar{\phi}(x, y)$ the least squares weighted residual function F to be minimized is given by:

$$F = \iint_{D_{xy}} \left(\nabla^4\bar{\phi}(x, y) - \frac{2q}{b^2}\right)^2 dxdy \qquad (26)$$

where D_{xy} is the plate domain on the xy coordinate plane.

Let the approximate solution $\bar{\phi}(x, y)$ be expressed in terms of a linear combination of basis (shape) functions that are chosen to apriori satisfy all the boundary conditions of the plate, and in terms of n number of unknown generalized parameters c_i, where $i = 1, 2, 3, \ldots, n$ as follows:

$$\bar{\phi}(x, y) = \varphi_0(x, y) + \sum_{i=1}^{n} c_i \varphi_i(x, y) \qquad (27)$$

The least squares weighted residual integral statements become the system of n equations:

$$\frac{\partial}{\partial c_1}\iint_{D_{xy}} \left(\nabla^4\bar{\phi}(x, y) - \frac{2q}{b^2}\right)^2 dxdy = 0 \qquad (28)$$

$$\frac{\partial}{\partial c_2}\iint_{D_{xy}} \left(\nabla^4\bar{\phi}(x, y) - \frac{2q}{b^2}\right)^2 dxdy = 0 \qquad (29)$$

$$\frac{\partial}{\partial c_n}\iint_{D_{xy}}\left(\nabla^4\bar{\phi}(x,y)-\frac{2q}{b^2}\right)^2 dxdy=0 \quad (30)$$

Simplifying, we have:

$$\iint_{D_{xy}}\left\{\varphi_1\left(\nabla^4(c_1\varphi_1+c_2\varphi_2+\ldots+c_n\varphi_n)-\frac{2q}{b^2}\right)\right\}dxdy=0 \quad \ldots(31)$$

$$\iint_{D_{xy}}\left\{\varphi_2\left(\nabla^4(c_1\varphi_1+c_2\varphi_2+\ldots+c_n\varphi_n)-\frac{2q}{b^2}\right)\right\}dxdy=0 \quad \ldots(32)$$

$$\iint_{D_{xy}}\left\{\varphi_n\left(\nabla^4(c_1\varphi_1+c_2\varphi_2+\ldots+c_n\varphi_n)-\frac{2q}{b^2}\right)\right\}dxdy=0 \quad \ldots(33)$$

Expanding, we obtain:

$$c_1\iint_{D_{xy}}\varphi_1\nabla^4\varphi_1\,dxdy+c_2\iint_{D_{xy}}\varphi_1\nabla^4\varphi_2\,dxdy+\ldots+ c_n\iint_{D_{xy}}\varphi_1\nabla^4\varphi_n\,dxdy=\iint_{D_{xy}}\frac{2q}{b^2}\varphi_1\,dxdy \quad (34)$$

$$c_1\iint_{D_{xy}}\varphi_2\nabla^4\varphi_1\,dxdy+c_2\iint_{D_{xy}}\varphi_2\nabla^4\varphi_2\,dxdy+\ldots+ c_n\iint_{D_{xy}}\varphi_2\nabla^4\varphi_n\,dxdy=\iint_{D_{xy}}\frac{2q}{b^2}\varphi_2\,dxdy \quad (35)$$

$$c_1\iint_{D_{xy}}\varphi_n\nabla^4\varphi_1\,dxdy+c_2\iint_{D_{xy}}\varphi_n\nabla^4\varphi_2\,dxdy+\ldots+ c_n\iint_{D_{xy}}\varphi_n\nabla^4\varphi_n\,dxdy=\iint_{D_{xy}}\frac{2q}{b^2}\varphi_n\,dxdy \quad (36)$$

The equations are expressed as:

$$\begin{aligned} c_1k_{11}+c_2k_{12}+\ldots+c_nk_{1n}&=F_1\\ c_1k_{21}+c_2k_{22}+\ldots+c_nk_{2n}&=F_2\\ &\vdots\\ c_1k_{n1}+c_2k_{n2}+\ldots+c_nk_{nn}&=F_n \end{aligned} \quad (37)$$

where

$$k_{ij}=\iint_{D_{xy}}\varphi_1(x,y)\nabla^4\varphi_j(x,y)\,dxdy \quad (38)$$

and $k_{11}=\iint_{D_{xy}}\varphi_1\nabla^4\varphi_1\,dxdy$ (39)

$$k_{1n}=\iint_{D_{xy}}\varphi_1\nabla^4\varphi_n\,dxdy \quad (40)$$

$$F_1=\iint_{D_{xy}}\frac{2q}{b^2}\varphi_1(x,y)\,dxdy=\frac{2q}{b^2}\iint_{D_{xy}}\varphi_1\,dxdy \quad (41)$$

$$F_2=\iint_{D_{xy}}\frac{2q}{b^2}\varphi_2(x,y)\,dxdy=\frac{2q}{b^2}\iint_{D_{xy}}\varphi_2\,dxdy \quad (42)$$

$$F_n=\iint_{D_{xy}}\frac{2q}{b^2}\varphi_n(x,y)\,dxdy=\frac{2q}{b^2}\iint_{D_{xy}}\varphi_n(x,y)\,dxdy \quad (43)$$

4. Results

4.1 Shape functions

The Beltrami – Michell stress compatibility equation in terms of Airy's stress potential function for the 2D elasticity problem of a rectangular plate $(2a\times 2b)$ subject to uniaxial in-plane parabolic distribution of tensile load in the *xx* direction, given by Equation (12) is written as Equation (20) on the two-dimensional domain D_{xy} expressed as $|x|<a,\ |y|<b$ (or $-a\le x\le a,\ -b\le y\le b$).

The boundary conditions are:

$$\tau_{xy}(x=\pm a,y)=0$$

$$\tau_{xy}(x,y=\pm b)=0$$

$$\sigma_{xx}(x=\pm a,y)=q\left(1-\frac{y^2}{b^2}\right) \quad (44)$$

$$\sigma_{yy}(x,y=\pm b)=0$$

$$\sigma_{xx}(x,y=\pm b)=0$$

$$\sigma_{yy}(x=\pm a,y)=0$$

Using the Airy's stress potential functions the boundary conditions can be expressed as:

$$\left.\frac{\partial^2\phi}{\partial x\partial y}\right|_{(x=\pm a,y)}=0$$

on $|x|=a$ for any *y*.

$$\left.\frac{\partial^2\phi}{\partial x\partial y}\right|_{(x,y=\pm b)}=0 \quad (45)$$

on $|y|=b$ for any *x*.

$$\left.\frac{\partial^2\phi}{\partial y^2}\right|_{(x,y=\pm b)}=0$$

on $|y|=b$ for any *x*.

$$\left.\frac{\partial^2\phi}{\partial x^2}\right|_{(x=\pm a,y)}=0$$

on $|x|=a$ for any *y*.

Shape functions that are biharmonic functions and thus qualify as suitable Airy stress potential functions, and also satisfy the boundary conditions equations are given for rectangular plates $2a\times 2b$ as follows:

$$\varphi_1(x,y) = (x^2 - a^2)^2(y^2 - b^2)^2$$

$$\varphi_2(x,y) = x^2(x^2 - a^2)^2(y^2 - b^2)^2 = x^2\varphi_1(x,y)$$

$$\varphi_3(x,y) = y^2(y^2 - b^2)^2(x^2 - a^2)^2 = y^2\varphi_1(x,y) \quad (46)$$

$$\varphi_4(x,y) = x^4(x^2 - a^2)^2(y^2 - b^2)^2 = x^4\varphi_1(x,y)$$

$$\varphi_5(x,y) = y^4(y^2 - b^2)^2(x^2 - a^2)^2 = y^4\varphi_1(x,y)$$

4.2 Results for one parameter least squares weighted residual method

For a one-parameter least squares weighted residual method, the unknown Airy stress function approximated assuming one unknown parameter is given by:

$$\bar{\phi}(x,y) = \varphi_0(x,y) + c_1\varphi_1(x,y)$$

$$= \varphi_0(x,y) + c_1(x^2 - a^2)^2(y^2 - b^2)^2 \quad (47)$$

$$\varphi_1(x,y) = (x^2 - a^2)^2(y^2 - b^2)^2$$

The least squares weighted residual integral statement becomes:

$$c_1\int_{-b}^{b}\int_{-a}^{a}(x^2 - a^2)^2(y^2 - b^2)^2\nabla^4(x^2 - a^2)(y^2 - b^2)^2 dxdy = \int_{-b}^{b}\int_{-a}^{a}\frac{2q}{b^2}(x^2 - a^2)^2(y^2 - b^2)^2 dxdy \quad (48)$$

$$c_1\int_{-b}^{b}\int_{-a}^{a}\varphi_1(x,y)\nabla^4(\varphi_1(x,y))dxdy = \int_{-b}^{b}\int_{-a}^{a}\frac{2q}{b^2}\varphi_1(x,y)dxdy \quad \ldots(49)$$

Hence,

$$c_1\int_{-b}^{b}\int_{-a}^{a}\left[24(y^2 - b^2)^2 + 32(3x^2 - a^2)(3y^2 - b^2) + 24(x^2 - a^2)^2\right](y^2 - b^2)^2(x^2 - a^2)^2 dxdy = \frac{2q}{b^2}\int_{-b}^{b}\int_{-a}^{a}(x^2 - a^2)^2(y^2 - b^2)^2 dxdy \quad (50)$$

Hence,

$$c_1\int_{-b}^{b}\int_{-a}^{a}\left[24(y^2 - b^2)^4(x^2 - a^2)^2 + 32(3x^2 - a^2)(x^2 - a^2)^2(3y^2 - b^2)(y^2 - b^2)^2 + 24(x^2 - a^2)^4(y^2 - b^2)^2\right]dxdy = \frac{2q}{b^2}\int_{-b}^{b}\int_{-a}^{a}(x^2 - a^2)^2(y^2 - b^2)^2 dxdy \quad (51)$$

$$c_1k_{11} = c_1\left\{24\left(\frac{256}{315}\right)\frac{16}{15}b^9a^5 + 11.8886167a^7b^7 + 24\left(\frac{256}{315}\right)\frac{16}{15}a^9b^5\right\} = \frac{2q}{b^2}\frac{16}{15}\frac{16}{15}a^5b^5 = F_1 \quad (52)$$

Solving,

$$c_1 = \frac{\frac{512}{225}qa^5b^3}{\left(\frac{98304}{4725}(b^9a^5 + a^9b^5)\right) + 11.8886167a^7b^7} = \frac{F_1}{k_{11}} \quad \ldots(53)$$

Simplifying,

$$c_1 = \frac{qa^{-6}\alpha^{-2}}{\frac{64}{7}(1+\alpha^4) + \frac{256}{49}\alpha^2} \quad (54)$$

$$\text{where } \alpha = \frac{b}{a} \quad (55)$$

α is the plate aspect ratio

$$c_1 = F_1(\alpha)qa^{-6} \quad (56)$$

$$F_1(\alpha) = \left(\frac{64}{7}(1+\alpha^4) + \frac{256}{49}\alpha^2\right)^{-1}\cdot\alpha^{-2} \quad (57)$$

Then the solution for ϕ is:

$$\phi(x,y) = \phi_0(x,y) + \phi_1(x,y)$$

where $\phi_0(x,y)$ is the Airy stress potential function calculated to satisfy the boundary conditions on $\sigma_{xx}(x = \pm a, y)$ and $\phi_1(x,y)$ is the Airy stress potential function that is made to satisfy the domain governing partial differential equation. Thus, for a one parameter solution,

$$\nabla^4\phi_1(x,y) - \frac{2q}{b^2} = 0$$

Hence,

$$\sigma_{xx}(x = \pm a, y) = q\left(1 - \frac{y^2}{b^2}\right) = \frac{\partial^2\phi_0}{\partial y^2} \quad (58)$$

Integrating with respect to y,

$$\phi_0(x,y) = q\left(\frac{y^2}{2} - \frac{y^4}{12b^2}\right) = \frac{qy^2}{2}\left(1 - \frac{y^2}{6b^2}\right) \quad (59)$$

Hence,

$$\phi(x,y) = \frac{qy^2}{2}\left(1 - \frac{y^2}{6b^2}\right) + F_1(\alpha)qa^{-6}\varphi_1(x,y) \quad (60)$$

$$\phi(x,y) = \frac{qy^2}{2}\left(1 - \frac{y^2}{6b^2}\right) + F_1(\alpha)qa^{-6}(x^2 - a^2)^2(y^2 - b^2)^2 \quad \ldots(61)$$

The Airy's stress potential function for a one-parameter solution by the least squares weighted residual method is thus given by Equation (61).

4.3 One-parameter least squares weighted residual solution for normal stresses and shear stress fields

The normal stresses $\sigma_{xx}(x,y)$, $\sigma_{yy}(x,y)$ and shear stress $\tau_{xy}(x,y)$ fields are found for a one-parameter least squares weighted residual method from the Airy's stress potential function as follows:

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(F_1(\alpha)qa^{-6}(x^2-a^2)^2(y^2-b^2)^2\right)$$

$$= F_1(\alpha)qa^{-6}(y^2-b^2)^2\frac{d^2}{dx^2}(x^2-a^2)^2 \quad (62)$$

$$\sigma_{yy} = F_1(\alpha)qa^{-6}(y^2-b^2)^2(12x^2-4a^2) \quad (63)$$

$$\sigma_{yy} = -4F_1(\alpha)\left(1-\frac{3x^2}{a^2}\right)q\left(\frac{b}{a}\right)^4\left(\frac{y^2}{b^2}-1\right)^2 \quad (64)$$

$$\sigma_{yy}(x=0,y) = -4F_1(\alpha)\left(\frac{y^2}{b^2}-1\right)^2 q\left(\frac{b}{a}\right)^4$$

$$= -4F_1(\alpha)\left(\frac{y^2}{b^2}-1\right)^2 q\alpha^4 \quad (65)$$

$$\sigma_{yy}(x,y=0) = -4F_1(\alpha)\left(1-\frac{3x^2}{a^2}\right)q\left(\frac{b}{a}\right)^4$$

$$= -4F_1(\alpha)\left(1-\frac{3x^2}{a^2}\right)q\alpha^4 \quad (66)$$

$$\sigma_{yy}(x=0,y=0) = -4F_1(\alpha)q\alpha^4 \quad (67)$$

$$\sigma_{xx}(x,y) = \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2}{\partial y^2}\left\{\left(\frac{qy^2}{2}\left(1-\frac{y^2}{6b^2}\right)+\right.\right.$$

$$\left.\left.F_1(\alpha)qa^{-6}(x^2-a^2)^2(y^2-b^2)^2\right)\right\} \quad (68)$$

$$\sigma_{xx} = q\left(1-\frac{y^2}{b^2}\right) + F_1(\alpha)qa^{-6}(x^2-a^2)^2\frac{d^2(y^2-b^2)^2}{dy^2} \quad \ldots(69)$$

$$\sigma_{xx} = q\left(1-\frac{y^2}{b^2}\right) + 4F_1(\alpha)qa^{-6}(x^2-a^2)^2\cdot(3y^2-b^2) \quad \ldots(70)$$

$$\sigma_{xx}(x=0,y) = q\left(1-\frac{y^2}{b^2}\right) + 4F_1(\alpha)qa^{-6}(a^4)(3y^2-b^2) \quad \ldots(71)$$

$$\sigma_{xx}(x,0) = q - 4F_1(\alpha)qa^{-6}b^2(x^2-a^2)^2 \quad (72)$$

$$\sigma_{xx}(0,0) = q - 4F_1(\alpha)q\alpha^2 \quad (73)$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{\partial^2}{\partial x\partial y}\left(\left(\frac{qy^2}{2}\left(1-\frac{y^2}{6b^2}\right)+\right.\right.$$

$$\left.\left.F_1(\alpha)qa^{-6}(x^2-a^2)^2(y^2-b^2)^2\right)\right) \quad (74)$$

$$\tau_{xy} = -16F_1(\alpha)qa^{-6}(x^3-a^2x)(y^3-b^2y) \quad (75)$$

$$\tau_{xy}(x=0,y=0) = 0$$

$$\tau_{xy}(x=0,y) = 0$$

$$\tau_{xy}(x,y=0) = 0$$

$$\tau_{xy}(x=a,y) = 0 \quad (76)$$

$$\tau_{xy}(x=-a,y) = 0$$

$$\tau_{xy}(x,y=b) = 0$$

$$\tau_{xy}(x,y=-b) = 0$$

For square plates, the one-parameter least squares weighted residual solutions for the stress fields become: since $\alpha = 1$, $a = b$,

$$c_1 = \left(\frac{64}{7}(1+1)+\frac{256}{49}\right)^{-1}\cdot(1)^{-2}qa^{-6} = \frac{49}{1152}qa^{-6} \quad (77)$$

$$F_1(\alpha) = \frac{49}{1152} = F_1(\alpha=1) \quad (78)$$

$$\sigma_{xx} = q\left(1-\frac{y^2}{a^2}\right) + \frac{49}{288}qa^{-6}(x^2-a^2)^2(3y^2-a^2) \quad (79)$$

$$\sigma_{xx} = q\left(1-\frac{y^2}{a^2}\right) - \frac{49}{288}q\left(\frac{x^2}{a^2}-1\right)^2\left(1-\frac{3y^2}{a^2}\right) \quad (80)$$

$$\sigma_{xx}(x=0,y) = q\left(1-\frac{y^2}{a^2}\right) - \frac{49}{288}q\left(1-\frac{3y^2}{a^2}\right) \quad (81)$$

$$\sigma_{xx}(x,y=0) = q - \frac{49}{288}q\left(\frac{x^2}{a^2}-1\right)^2 \quad (82)$$

$$\sigma_{xx}(x=a,y) = q\left(1-\frac{y^2}{a^2}\right) \quad (83)$$

$$\sigma_{xx}(x=-a,y) = q\left(1-\frac{y^2}{a^2}\right) \quad (84)$$

$$\sigma_{xx}(x,y=a) = q(1-1) - \frac{49}{288}q\left(\frac{x^2}{a^2}-1\right)^2(-2)$$

$$= 0 + \frac{49}{144}q\left(\frac{x^2}{a^2}-1\right)^2 \quad (85)$$

$$\sigma_{xx}(x,y=-a) = \frac{49}{144}q\left(\frac{x^2}{a^2}-1\right)^2 \quad (86)$$

$$\sigma_{yy}(x,y) = -4F_1(\alpha)\left(1-\frac{3x^2}{a^2}\right)q\alpha^4\left(\frac{y^2}{a^2}-1\right)^2 \quad (87)$$

$$\sigma_{yy}(x,y) = -\frac{49}{288}q\left(1-\frac{3x^2}{a^2}\right)\left(\frac{y^2}{a^2}-1\right)^2 \quad (88)$$

$$\sigma_{yy}(x=0,y=0) = -\frac{49}{288}q = -0.17013889q \quad (89)$$

$$\sigma_{yy}(x=0,y) = -\frac{49q}{288}\left(\frac{y^2}{a^2}-1\right)^2 \quad (90)$$

$$\sigma_{yy}(x,y=0) = -\frac{49q}{288}\left(1-\frac{3x^2}{a^2}\right) \quad (91)$$

$$\sigma_{yy}(x=\pm a,y=0) = \frac{49}{288}q\left(\frac{y^2}{a^2}-1\right)^2 \quad (92)$$

$$\sigma_{yy}(x=\pm a,y=0) = \frac{49}{144}q = 0.34028q \quad (93)$$

$$\tau_{xy}(x,y) = -16F_1(\alpha)qa^{-6}xy(x^2-a^2)(y^2-a^2)$$

$$= -16\left(\frac{49}{1152}\right)xyqa^{-6}xy(x^2-a^2)(y^2-a^2) \quad (94)$$

$$\tau_{xy}(x,y) = \frac{-49}{72}xyqa^{-2}\left(\frac{x^2}{a^2}-1\right)\left(\frac{y^2}{a^2}-1\right) \quad (95)$$

$$\tau_{xy}(x=0,y=0) = 0$$

$$\tau_{xy}(x=0,y) = 0$$

$$\tau_{xy}(x=\pm a,y) = 0 \quad (96)$$

$$\tau_{xy}(x,y=\pm a) = 0$$

4.4 Results for three-parameter least squares weighted residual method

The approximate solution for the Airy stress function in a three-parameter least squares weighted residual method is given by:

$$\bar{\phi}_3(x,y) = c_1(x^2-a^2)^2(y^2-b^2)^2 + c_2x^2(x^2-a^2)^2(y^2-b^2)^2 + c_3y^2(y^2-b^2)^2(x^2-a^2)^2 \quad (97)$$

or, $\bar{\phi}_3(x,y) = c_1\varphi_1(x,y) + c_2\varphi_2(x,y) + c_3\varphi_3(x,y)$ (98)

where φ_1, φ_2 and φ_3 are given in Equation (46) and $\varphi_1(x,y)$, $\varphi_2(x,y)$ and $\varphi_3(x,y)$ are the shape functions of the three parameter least squares weighted residual method.

The least squares weighted residual statements are obtained as the system of three equations in three unknowns:

$$\frac{\partial}{\partial c_i}\int_{-b}^{b}\int_{-a}^{a}\left(\nabla^4\bar{\phi}_3(x,y)-\frac{2q}{b^2}\right)^2 dxdy = 0 \quad (99)$$

$i = 1, 2, 3$

Thus,

$$\frac{\partial}{\partial c_1}\int_{-b}^{b}\int_{-a}^{a}\left(\nabla^4\bar{\phi}_3(x,y)-\frac{2q}{b^2}\right)^2 dxdy = 0 \quad (100)$$

$$\frac{\partial}{\partial c_2}\int_{-b}^{b}\int_{-a}^{a}\left(\nabla^4\bar{\phi}_3(x,y)-\frac{2q}{b^2}\right)^2 dxdy = 0 \quad (101)$$

$$\frac{\partial}{\partial c_3}\int_{-b}^{b}\int_{-a}^{a}\left(\nabla^4\bar{\phi}_3(x,y)-\frac{2q}{b^2}\right)^2 dxdy = 0 \quad (102)$$

Hence,

$$\int_{-b}^{b}\int_{-a}^{a}\varphi_1\left(\nabla^4(c_1\varphi_1+c_2\varphi_2+c_3\varphi_3)-\frac{2q}{b^2}\right)dxdy = 0 \quad (103)$$

$$\int_{-b}^{b}\int_{-a}^{a}\varphi_2\left(\nabla^4(c_1\varphi_1+c_2\varphi_2+c_3\varphi_3)-\frac{2q}{b^2}\right)dxdy = 0 \quad (104)$$

$$\int_{-b}^{b}\int_{-a}^{a}\varphi_3\left(\nabla^4(c_1\varphi_1+c_2\varphi_2+c_3\varphi_3)-\frac{2q}{b^2}\right)dxdy = 0 \quad (105)$$

Alternatively,

$$c_1\int_{-b}^{b}\int_{-a}^{a}\varphi_1\nabla^4\varphi_1 dxdy + c_2\int_{-b}^{b}\int_{-a}^{a}\varphi_1\nabla^4\varphi_2 dxdy + c_3\int_{-b}^{b}\int_{-a}^{a}\varphi_1\nabla^4\varphi_3 dxdy = \int_{-b}^{b}\int_{-a}^{a}\frac{2q}{b^2}\varphi_1 dxdy \quad (106)$$

$$c_1\int_{-b}^{b}\int_{-a}^{a}\varphi_2\nabla^4\varphi_1 dxdy + c_2\int_{-b}^{b}\int_{-a}^{a}\varphi_2\nabla^4\varphi_2 dxdy + c_3\int_{-b}^{b}\int_{-a}^{a}\varphi_2\nabla^4\varphi_3 dxdy = \int_{-b}^{b}\int_{-a}^{a}\frac{2q}{b^2}\varphi_2 dxdy \quad (107)$$

$$c_1\int_{-b}^{b}\int_{-a}^{a}\varphi_3\nabla^4\varphi_1 dxdy + c_2\int_{-b}^{b}\int_{-a}^{a}\varphi_3\nabla^4\varphi_2 dxdy + c_3\int_{-b}^{b}\int_{-a}^{a}\varphi_3\nabla^4\varphi_3 dxdy = \int_{-b}^{b}\int_{-a}^{a}\frac{2q}{b^2}\varphi_3 dxdy \quad (108)$$

In matrix form,

$$\begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_{33} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \quad (109)$$

where $k_{mn} = \int_{-b}^{b}\int_{-a}^{a}\varphi_m\nabla^4\varphi_n\, dxdy$ (110)

Evaluation of the integrals and simplification yields the system of three equations:

$$c_1\left(\frac{64}{7}+\frac{256}{49}\frac{b^2}{a^2}+\frac{64}{7}\frac{b^4}{a^4}\right)+c_2a^2\left(\frac{64}{77}+\frac{64}{49}\frac{b^4}{a^4}\right)+ c_3a^2\left(\frac{64b^2}{49a^2}+\frac{64b^6}{77a^6}\right)=\frac{q}{a^4b^2} \quad (111)$$

$$c_1\left(\frac{64}{11}+\frac{64}{7}\frac{b^4}{a^4}\right)+c_2a^2\left(\frac{192}{143}+\frac{256b^2}{77a^2}+\frac{192b^4}{7a^4}\right)+ c_3a^2\left(\frac{64b^2}{77a^2}+\frac{64b^6}{77a^6}\right)=\frac{q}{a^4b^2} \quad (112)$$

$$c_1\left(\frac{64}{7}+\frac{64b^4}{11a^4}\right)+c_2a^2\left(\frac{64}{77}+\frac{64b^4}{77a^4}\right)+ c_3a^2\left(\frac{192b^2}{7a^2}+\frac{256b^4}{77a^4}+\frac{192b^6}{146a^6}\right)=\frac{q}{a^4b^2} \quad (113)$$

Let $\frac{b}{a}=\alpha$

Then the system of equations becomes:

$$c_1\left(\frac{64}{7}(1+\alpha^4)+\frac{256}{49}\alpha^2\right)+c_2a^2\frac{64}{7}\left(\frac{1}{11}+\frac{\alpha^4}{7}\right)+ c_3a^2\frac{64}{7}\left(\frac{\alpha^2}{7}+\frac{\alpha^6}{11}\right)=\frac{q}{a^4b^2} \quad (114)$$

$$c_1 64\left(\frac{1}{11}+\frac{\alpha^4}{7}\right)+c_2a^2\left(\frac{192}{143}+\frac{256}{77}\alpha^2+\frac{192}{7}\alpha^4\right)+ c_3a^2\frac{64}{77}(\alpha^2+\alpha^6)=\frac{q}{a^4b^2} \quad (115)$$

$$c_1 64\left(\frac{1}{7}+\frac{\alpha^4}{11}\right)+c_2a^2\frac{64}{77}(1+\alpha^4)+ c_3a^2\left(\frac{192}{7}\alpha^2+\frac{256}{77}\alpha^4+\frac{192}{143}\alpha^6\right)=\frac{q}{a^4b^2} \quad (116)$$

For square plates, $a = b$ and $\alpha = 1$, and we obtain:

$$23.5102c_1+2.1373c_2a^2+2.1373c_3a^2=\frac{q}{a^6} \quad (117)$$

$$14.9610c_1+32.0959c_2a^2+1.6623c_3a^2=\frac{q}{a^6} \quad (118)$$

$$14.9610c_1+1.6623c_2a^2+32.0505c_3a^2=\frac{q}{a^6} \quad (119)$$

In matrix form,

$$\begin{pmatrix}23.5102 & 2.1373a^2 & 2.1373a^2\\ 14.9610 & 32.0959a^2 & 1.6623a^2\\ 14.9610 & 1.6623a^2 & 32.0505a^2\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}qa^{-6}\\qa^{-6}\\qa^{-6}\end{pmatrix} \quad \ldots(120)$$

This is solved by Cramer's rule to obtain:

$$c_1=\frac{\Delta_1}{\Delta_0} \quad (121)$$

$$c_2=\frac{\Delta_2}{\Delta_0} \quad (122)$$

$$c_3=\frac{\Delta_3}{\Delta_0} \quad (123)$$

where,

$$\Delta_1=\begin{vmatrix}qa^{-6} & 2.1373a^2 & 2.1373a^2\\ qa^{-6} & 32.0959a^2 & 1.6623a^2\\ qa^{-6} & 1.6623a^2 & 32.0505a^2\end{vmatrix} \quad (124)$$

$$\Delta_2=\begin{vmatrix}23.5102 & qa^{-6} & 2.1373a^2\\ 14.9610 & qa^{-6} & 1.6623a^2\\ 14.9610 & qa^{-6} & 32.0505a^2\end{vmatrix} \quad (125)$$

$$\Delta_3=\begin{vmatrix}23.5102 & 2.1373a^2 & qa^{-6}\\ 14.9610 & 32.0959a^2 & qa^{-6}\\ 14.9610 & 1.6623a^2 & qa^{-6}\end{vmatrix} \quad (126)$$

$$\Delta_0=\begin{vmatrix}23.5102 & 2.1373a^2 & 2.1373a^2\\ 14.9610 & 32.0959a^2 & 1.6623a^2\\ 14.9610 & 1.6623a^2 & 32.0959a^2\end{vmatrix} \quad (127)$$

Then,

$$c_1=0.04040qa^{-6} \quad (128)$$

$$c_2=c_3=0.01174qa^{-8} \quad (129)$$

The Airy stress potential function for three-parameter least squares weighted residual solution for a square plate under the given parabolic in-plane load distribution is given by:

$$\phi(x,y)=\frac{qy^2}{2}\left(1-\frac{y^2}{6b^2}\right)+0.04040qa^{-6}(x^2-a^2)^2(y^2-b^2)^2+ 0.01174qa^{-8}(x^2+y^2)(x^2-a^2)^2(y^2-b^2)^2 \quad (130)$$

4.5 Three-Parameter Least Squares Weighted Residual Solution for Normal Stresses and Shear Stress Fields

The normal stresses and shear stress fields for a three-parameter least squares weighted residual method are obtained by substitution of Equation (130) for the Airy stress potential function into Equations (16) – (18) to obtain:

$$\sigma_{xx}=\frac{\partial^2\phi}{\partial y^2}=\frac{\partial^2}{\partial y^2}\left\{\frac{qy^2}{2}\left(1-\frac{y^2}{6b^2}\right)+0.04040qa^{-6}(x^2- a^2)^2(y^2-b^2)^2+0.01174qa^{-8}(x^2+\right.$$

$$y^2)(x^2-a^2)^2(y^2-b^2)^2\} \quad (131)$$

$$\sigma_{xx} = \frac{\partial^2}{\partial y^2}\left(\frac{qy^2}{2}\left(1-\frac{y^2}{6b^2}\right)\right)+$$

$$0.04040qa^{-6}(x^2-a^2)^2\frac{d^2}{dy^2}(y^2-b^2)^2+$$

$$0.01174qa^{-8}(x^2-a^2)^2\frac{d^2}{dy^2}(x^2+y^2)(y^2-b^2)^2 \quad (132)$$

$$\sigma_{xx} = q\left(1-\frac{y^2}{b^2}\right)+0.1616qa^{-6}(x^2-$$

$$a^2)^2(3y^2-b^2)+0.01174qa^{-8}(x^2-a^2)^2(2x^2-$$

$$4b^2x^2+30y^4-24b^2y^2+2b^4) \quad (133)$$

$$\sigma_{xx}(x=0,y) = q\left(1-\frac{y^2}{b^2}\right)+0.1616qa^{-2}(3y^2-b^2)+$$

$$0.01174qa^{-4}(30y^4-24b^2y^2+2b^4) \quad (134)$$

$$\sigma_{xx} = q\left(1-\frac{y^2}{b^2}\right)+0.1616qa^{-2}\left(\frac{3y^2-b^2}{b^2}\right)b^2+$$

$$0.01174qa^{-4}\left(\frac{30y^4-24b^2y^2+2b^4}{b^4}\right)b^4 \quad (135)$$

$$\sigma_{xx} = q\left(1-\frac{y^2}{b^2}\right)+0.1616q\alpha^2\left(\frac{3y^2}{b^2}-1\right)+$$

$$0.01174q\alpha^4\left(30\frac{y^4}{b^4}-24\frac{y^2}{b^2}+2\right) \quad (136)$$

$$\sigma_{xx}(0,0) = (1+0.02348\alpha^4-0.1616\alpha^2)q \quad (137)$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left\{\frac{qy^2}{2}\left(1-\frac{y^2}{6b^2}\right)+\right.$$

$$0.04040qa^{-6}(x^2-a^2)^2(y^2-b^2)^2+$$

$$0.01174qa^{-8}(x^2+y^2)(x^2-a^2)^2(y^2-b^2)^2\} \quad (138)$$

$$\sigma_{yy} = 0.04040qa^{-6}(y^2-b^2)^2\frac{d^2}{dx^2}(x^2-a^2)^2+$$

$$0.01174qa^{-8}(y^2-b^2)^2\frac{d^2}{dx^2}(x^2+y^2)(x^2-a^2)^2 \quad (139)$$

$$\sigma_{yy} = 0.04040qa^{-6}(y^2-b^2)^2(12x^2-4a^2)+$$

$$0.01174qa^{-8}(y^2-b^2)^2(30x^4+$$

$$12x^2y^2-24a^2x^2+2a^4-4a^2y^2) \quad (140)$$

$$\sigma_{yy}(0,y) = -0.13812qa^{-4}(y^2-b^2)^2 \quad (141)$$

$$\sigma_{yy}(0,0) = -0.13812qa^{-4}b^4$$

$$= -0.13812q\left(\frac{b}{a}\right)^4 = -0.13812q\alpha^4 \quad (142)$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{\partial^2}{\partial x\partial y}\left(\frac{qy^2}{2}\left(1-\frac{y^2}{6b^2}\right)+\right.$$

$$0.04040qa^{-6}(x^2-a^2)^2(y^2-b^2)^2+$$

$$\left.0.01174qa^{-8}(x^2+y^2)(x^2-a^2)^2(y^2-b^2)^2\right) \quad (143)$$

$$\tau_{xy} = -0.04040qa^{-6}(4x^3-4a^2x)(4y^3-4b^2y)-$$

$$0.01174qa^{-8}\left((6x^5-8a^2x^3+2a^4x)(4y^3-4b^2y)+\right.$$

$$\left.(6y^5-8b^2y^3+2b^4y)(4x^3-4a^2x)\right) \quad (144)$$

$$\tau_{xy}(x=0,y) = 0$$

$$\tau_{xy}(x,y=0) = 0 \quad (145)$$

$$\tau_{xy}(0,0) = 0$$

The results obtained for the stress fields for one-parameter and three-parameter least squares weighted residual methods for square and rectangular plates for the x = 0 plane and at the centre (x = 0, y = 0) are given in tabular form in Tables 1, 2, 3, 4 and 5.

5. Discussion

The least squares weighted residual method has been successfully used in this work to solve the two-dimensional elasticity problem of a rectangular plate of in-plane dimensions $2a\times 2b$ subjected to a parabolic distribution of tensile loads on the two edges $x=\pm a$ where the origin of the Cartesian coordinates is the plate centre. The parabolic load distribution on the edges $x=\pm a$ was given as Equation (12) while the other edges were considered free of normal and shear stresses as given by the stress boundary conditions represented by Equations (44). Two-dimensional theory of elasticity was used to express the stress – strain relations, differential equations of equilibrium and the strain compatibility equation in terms of stresses as the primary unknowns and obtain the stress compatibility equations for plane stress and for plane strain conditions given as Equation (9). Airy stress potential functions were then used to obtain the governing stress compatibility equation in terms of the Airy stress function as a non-homogeneous biharmonic partial differential equation given as Equation (20). The stress boundary conditions on the four edges $x=\pm a$, $x=\pm b$ were expressed in terms of Airy stress potential functions as Equation (45). Shape functions for the unknown Airy stress function in the inhomogeneous biharmonic partial differential equation were chosen as Equation (46) to be the biharmonic functions of the two dimensional space variables x and y and to satisfy the boundary conditions. The Airy stress potential functions were constructed as linear combinations of the shape functions in a one-parameter and a three-parameter technique as Equations (47) and (97). For a one-parameter Airy stress potential function, the least squares weighted residual integral statement was formulated as Equation (48). The unknown parameter of the Airy stress function was obtained as Equation (54), hence the Airy stress function for a one-parameter solution was obtained as Equation (61).

The resulting normal stresses and shear stress fields were obtained from the one-parameter Airy stress function given in Equation (61) and Equations (16 – 18) to obtain the normal stress fields as Equations (64) and (70) and the shear stress field as Equation (75). The variation of the normal stress σ_{yy} on the cross-section $x = 0$, was obtained as Equation (65). The value of the normal stress σ_{yy} at the plate centre (0,0) was obtained as Equation (67). The variation of the normal stress σ_{xx} on the cross-section $x = 0$ was obtained as Equation (71) while the variation of σ_{xx} on the $y = 0$ plane ($y = 0$) was obtained as Equation (72). The magnitude of σ_{xx} at the plate centre was obtained as Equation (73). The shear stresses were found to vanish at the centre and at all the edges of the plate as given in Equations (75) and (76). The shear stresses were similarly found to vanish at the $x = 0$ and $y = 0$ planes of the plates. The normal stresses and shear stress fields were found for square plates as Equations (80), (88) and (95). The normal stress σ_{xx} at the $x = 0$ plane was found as Equation (81) and σ_{xx} at $y = 0$ plane was found as Equation (82).

The normal stress σ_{xx} at $x = \pm a$ was obtained as Equations (83) and (84); and this agrees with the loading/boundary condition of the problem solved using the least squares weighted residual method. The normal stress σ_{yy} was obtained as Equation (88) and it was found to be given at the plate centre by Equation (89). The normal stress σ_{yy} at the $x = 0$ plane was obtained as Equation (90) and on the $y = 0$ plane σ_{yy} was obtained as Equation (88). The shear stresses for a square plate at the centre and the edges are found to vanish.

The 2D elasticity problem was similarly solved using a three-parameter Airy stress potential function given by Equation (97). The least squares weighted residual integral statements were formulated as the system of three equations given by Equations (100 – 102) or alternatively by Equations (106 – 108). Evaluation of the multiple integrals yielded the three-parameter least squares residual integral statements as the system of algebraic equations given by Equations (111 – 113) or in simplified form by Equations (114 – 116). For square plates, the system of algebraic equations simplified to Equations (117 – 119) which was presented in matrix form as Equation (120). Equation (120) was solved by Cramer's rule to obtain the three unknown parameters, as Equations (128) and (129), and hence the three-parameter Airy stress potential function as Equation (130). The corresponding normal stresses were found from Equations (16 – 18) and (120) as Equations (133), (140) and the shear stress as Equation (144). The normal stress σ_{xx} was found at the plate centre as Equation (137).

The normal stress σ_{yy} was found at the $x = 0$ plane as Equation (141) and at the centre of the plate as Equation (142). The shear stress was found to vanish at the $x = 0$, and $y = 0$ planes, and at the plate centre. The shear stress was found to also vanish at all the plate edges.

The solutions obtained for the stress fields for the one-parameter and the three-parameter Airy stress potential functions are presented in tables. Table 1 shows the one-parameter least squares weighted residual solution for the variation of σ_{xx} on the $x = 0$ plane. Table 2 shows the three-parameter least squares weighted residual solution of the same problem for the variation of σ_{xx} on the $x = 0$ plane of the rectangular plate under parabolically varying edge load on $x = \pm a$. The three-parameter least squares weighted residual solution for σ_{xx}at $x = 0$ for the case of a rectangular plate with $a/b = 2$ is presented in Table 3. The table shows that as the plate aspect ratio increases the normal stress distribution σ_{xx} on the $x = 0$ plane becomes more uniform. Table 3 shows an average value of σ_{xx} at $x = 0$ to be about $(2/3)q$. Table 4 shows the three-parameter least squares weighted residual solution for σ_{yy} at the plate centre for various values of the plate aspect ratio $\alpha = b/a$. Table 5 shows the three-parameter least squares weighted residual solution for σ_{xx} at the plate centre for various values of the plate aspect ratio.

Table 1. One-parameter least squares weighted residual solution for the variation of normal stress σ_{xx} on the $x = 0$ plane for square plates under parabolic load $\sigma_{xx} = q\left(1-\frac{y^2}{b^2}\right)$ on $x = \pm a$ for different values of y/a

y/a	$\sigma_{xx}(x=0,y) = q\left(1-\frac{y^2}{a^2}\right) - \frac{49q}{288}\left(1-\frac{3y^2}{a^2}\right)$	$\sigma_{xx}(x=0,y)/q$
–1.0	0.3404q	0.3404
–0.8	0. 5166q	0.5166
–0.6	0.6536q	0.6536
–0.4	0.7515q	0.7515
–0.2	0.8102q	0.8102
0	0.8298q	0.8298
0.2	0.8102q	0.8102
0.4	0.7515q	0.7515
0.6	0.6536q	0.6536
0.8	0. 5166q	0.5166
1.0	0.3404q	0.3404

Table 2. Three-parameter least squares weighted residual solution for normal stress σ_{xx} distribution on the $x = 0$ plane for square plates under parabolic load $\sigma_{xx} = q\left(1-\frac{y^2}{b^2}\right)$ on $x = \pm a$ for different values of y/a

y/a	$\sigma_{xx}(x=0,y)$	$\sigma_{xx}(x=0,y)/q$
–1.0	0.4172q	0.4172
–0.8	0. 4961q	0.4961
–0.6	0.6206q	0.6206
–0.4	0.7434q	0.7434
–0.2	0.8306q	0.8306
0	0.8619q	0.8619
0.2	0.8306q	0.8306
0.4	0.7434q	0.7434
0.6	0.6206q	0.6206
0.8	0. 4961q	0.4961
1.0	0.4172q	0.4172

Table 3. Three-parameter least squares weighted residual solution for normal stress σ_{xx} distribution on the x = 0 plane for a rectangular plate a/b = 2 for the parabolic load $\sigma_{xx} = q\left(1-\frac{y^2}{b^2}\right)$ on $x = \pm a$ for different values of y/b

y/b	$\sigma_{xx}(x=0,y)$	$\sigma_{xx}(x=0,y)/q$
–1.0	0.675q	0.675
–0.8	0. 649q	0. 649
–0.6	0.653q	0.653
–0.4	0.669q	0.669
–0.2	0.684q	0.684
0	0.690q	0.690
0.2	0.684q	0.684
0.4	0.669q	0.669
0.6	0.653q	0.653
0.8	0. 649q	0. 649
1.0	0.675q	0.675

Table 4. Three-parameter least squares weighted residual solution for normal stress $\sigma_{yy}(0,0)$ for rectangular plates under parabolic load $\sigma_{xx} = q\left(1-\frac{y^2}{b^2}\right)$ on $x = \pm a$ for different values of the plate aspect ratio b/a

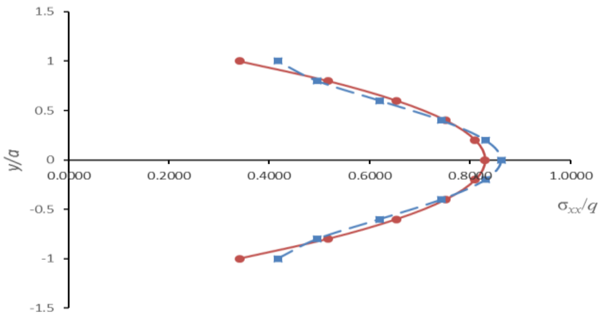
$\alpha = b/a$	$\sigma_{yy}(0,0)$	$\sigma_{yy}(0,0)/q$
0.25	-5.395 × $10^{-4}q$	-5.395 × 10^{-4}
0.50	-8.6325 × $10^{-3}q$	-8.6325 × 10^{-3}
0.75	-0.0437q	-0.0437
1.0	-0.13812q	-0.13812
1.25	-0.3372q	-0.3372
1.50	-0.6992q	-0.6992
1.75	-1.2954q	-1.2954
2	-2.20992q	-2.20992
3	-11.18772q	-11.18772

Table 5. Three-parameter least squares weighted residual solution for σ_{xx} at the centre of a rectangular plate under parabolic load $\sigma_{xx} = q\left(1-\frac{y^2}{b^2}\right)$ on $x = \pm a$ for different values of the plate aspect ratio b/a

$\alpha = b/a$	$\sigma_{xx}(0,0)$	$\sigma_{xx}(0,0)/q$
0.25	0.98999q	0.98999
0.50	0.9610675q	0.9610675
1.0	0.86188q	0.86188
1.50	0.7552675q	0.7552675
2	0.72928q	0.72928
3	1.44748q	1.44748
5	11.635q	11.635

A graphical comparison of the one-parameter and three-parameter least squares weighted residual solutions for the variation of non-dimensional normal stress (σ_{xx}/q) on the x = 0 plane for a square plate $2a \times 2b$ which is subjected to a parabolic load variation given by:

$\sigma_{xx}(x=\pm a, y) = q\left(1-\frac{y^2}{b^2}\right)$ on the two opposite edges $x = \pm a$ is shown in Figure 2.



Continuous line - One parameter least squares weighted residual solution for $\sigma_{xx}(0, y)$
Dotted lines - Three parameter least squares weighted residual solution for $\sigma_{xx}(0, y)$

Figure 2. Distribution of non-dimensional normal stress in the x direction (σ_{xx}/q) on the x = 0 plane of a rectangular plate under edge load $\sigma_{xx}(x=\pm a, y) = q\left(1-\frac{y^2}{b^2}\right)$ for one-parameter and three-parameter least squares weighted residual solutions

Figure 2 is a graphical illustration of Equation (71) for one-parameter solution and Equation (134) for three-parameter solution. It also represents Tables 1 and 2 graphically. The figure shows that the two results show quadratic variation of σ_{xx} on the x = 0 axis.

Figure 3 presents a graphical illustration of the distribution of dimensionless normal stress (σ_{xx}/q) on the x = 0 plane for a rectangular plate of aspect ratio a/b = 2 subject to parabolic load $\sigma_{xx}(x=\pm a, y) = q\left(1-\frac{y^2}{b^2}\right)$ or the two opposite edges $x = \pm a$

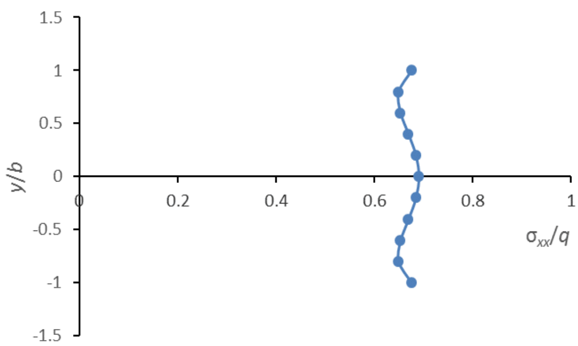


Figure 3. Distribution of normal stress σ_{xx} on the x =0 plane for rectangular plate (with aspect ratio a/b = 2) subject to parabolic variation of edge load at $x = \pm a$, where $\sigma_{xx}(x=\pm a, y) = q\left(1-\frac{y^2}{b^2}\right)$

Figure 3 illustrates an approximately by uniform variation of normal stress σ_{xx} on the x = 0 plane for the case of rectangular plates under parabolic edge loads for the case of aspect ratio a/b = 2.

Figure 4 presents a graphical illustration of the variation with aspect ratio of $\sigma_{yy}(0,0)$ – normal stress in the *y* direction at the plate centre.

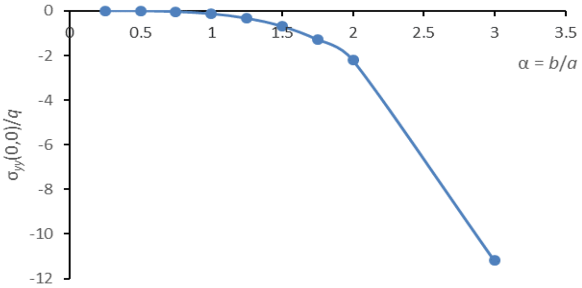


Figure 4. Variation of $\sigma_{yy}(0,0)$ at the plate centre with plate aspect ratio

The variation of normal stress in the *x* direction at the centre $\sigma_{xx}(0,0)$ with various values of plate aspect ratio α is shown graphically displayed on Figure 5.

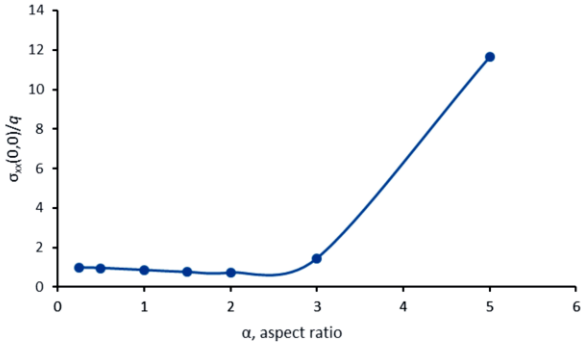


Figure 5. Variation of normal stress $\sigma_{yy}(0,0)$ at the plate centre with plate aspect ratio α for three-term least squares weighted residual solution

Figure 5 illustrates that $\sigma_{yy}(0,0)$ shows approximately little variation for values of plate aspect ratio lying between 0.25 and 2 but increases significantly as the plate aspect ratio exceeds 2.

6. Conclusion

The following conclusion can be made from this study:

(i) The two-dimensional elasticity problem of rectangular plates $(2a \times 2b)$ subject to uniaxial parabolically distributed edge loads applied at $x = \pm a$ (where the origin is considered at the plate centre) is represented mathematically by a non-homogeneous biharmonic fourth order partial differential equation in terms of the Airy stress potential function in a Beltrami–Michell stress-based formulation.

(ii) The least squares weighted residual method is an effective mathematical analysis tool for the approximate/numerical solution of the Airy stress potential function and consequently, the normal stresses and shear stress fields in rectangular plates subjected to a parabolic distribution of edge tensile loads at the two faces $x = \pm a$.

(iii) The least squares weighted residual method simplifies the 2D elasticity problem from a boundary value problem represented by an inhomogeneous biharmonic partial differential equation to an algebraic problem involving a system of algebraic equations whose unknowns are the undetermined parameters of the Airy stress potential function.

(iv) A one-parameter approximation of the Airy stress potential function in the least squares weighted residual integral statement simplified to a simple algebraic equation with one unknown variable and yielded sufficiently accurate results for practical purposes.

(v) A three-parameter approximation of the Airy stress potential function in the least squares weighted residual method/formulation simplified to an algebraic problem given by a system of three equations in three unknown parameters, and the resulting three parameter least squares weighted residual solutions for the stress fields yielded more accurate results.

(vi) As the plate aspect ratio increases and the plate becomes very long in one coordinate direction relative to the other, the normal stress σ_{xx} distribution over the $x = 0$ cross sectional plane becomes uniform, a result that is consistent with logical reasoning.

(vii) The one-parameter and three-parameter least squares weighted residual solutions obtained for normal stresses and shear stresses satisfies both the domain governing equations at all points in the domain $-a \le x \le a,\ -b \le y \le b,$ and the stress boundary conditions at the four edges $x = \pm a,\ y = \pm b.$

(viii) The normal stress fields obtained for both the one parameter and three parameter least squares weighted residual formulations satisfy the loading conditions of the 2D elasticity problem of the plate at the loaded edges $x = \pm a.$

Nomenclature/Notation

x,y,z	Cartesian coordinates
σ_{xx}, σ_{yy}	normal stresses
τ_{xy}	shear stress
ε_{xx}, ε_{yy}	normal strains
γ_{xy}	shear strain
G	shear modulus
E	Young's modulus of elasticity
μ	Poisson's ratio
f_x, f_y, f_z	body force components in the *x*, *y* and *z* coordinate directions respectively
α_1	parameter defined in terms of Poisson's ratio
$2a \times 2b$	in-plane dimensions of the rectangular plate
q	intensity of parabolic tensile load distribution at ($x = \pm a$, $y = 0$)

$\phi(x, y)$ Airy stress potential function
D Domain of plate
D_{xy} plate domain on the xy coordinate plane
$\varphi(x, y)$ unknown dependent variable
$f(x, y)$ known function of x and y
a_i unknown parameters of a trial function
F Least squares weighted residual function to be minimized
n number of unknown generalized parameters
c_i unknown generalized parameters of the trial Airy stress potential function
$\bar{\phi}(x, y)$ approximate (trial) Airy stress potential function
k_{ij} stiffness coefficients or coefficient of stiffness matrix
F_i element of force matrix
$\varphi_i(x, y)$ shape function of the Airy stress potential function
α plate aspect ratio
Δ_i determinant of a matrix
Δ_0 determinant of coefficient matrix
$\frac{\partial}{\partial x}$ partial derivative with respect to x
∇^2 Laplace operator (two-dimensional Laplace operator)
∇^4 biharmonic operator
L linear differential operator
$\sum$ summation
$\int$ integral
$\iint$ double integral
$| \ |$ determinant
2D two-dimensional
3D three-dimensional

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