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Elzaki transform method for finding solutions to two-dimensional elasticity problems in polar coordinates formulated using Airy stress functions

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ABSTRACT

This paper presents the Elzaki transform method for finding solutions to two-dimensional (2D) elasticity problems in plane polar coordinates formulated using Airy stress potential functions. Airy stress function is used to express the stress compatibility equation as a biharmonic equation. Elzaki transform is applied with respect to the radial coordinate to a modified form of the stress compatibility equation, and the biharmonic equation simplified to a fourth order ordinary differential equation (ODE). The general solution for the Airy stress potential function in the Elzaki transform space is obtained by solving the ODE. By inversion, the general solution for the Airy stress potential function is obtained in the physical domain space variables in terms of four unknown integration constants. Normal stresses and shear stress fields are also determined for the general case of 2D elasticity problems. The Flamant problem is solved as a particular illustration of 2D elasticity problems. The stress boundary conditions and the requirement of equilibrium of the internal stress resultants and the external forces are used simultaneously to determine the four constants of integration. The Airy stress potential function and the normal and shear stress fields were thus completely determined. The principle of superposition is used to obtain the elasticity solutions for the stress fields in the elastic half plane due to strip load of infinite extent, and solutions for horizontal stresses on smooth rigid retaining walls due to strip loads and parallel line load of infinite extent acting on the elastic half plane.

1. Introduction

Three-dimensional (3D) problems of the mathematical theory of elasticity are governed by system of differential equations of equilibrium, constitutive relations of stress and strain, and kinematic equations relating strains and displacements [1 – 20]. Additional constraining equations are provided from the boundary conditions. Such problems which are extremely important in geotechnical and structural engineering are aimed at the determination of fields of stresses, strains and displacements in the elastic material caused by applied loads/forces [1 – 20]. Solutions to such problems are usually very intractable to obtain, and only exist for few problems. Consequently, without loss of the general nature of the problems, some three-dimensional elasticity problems can be simplified to two- dimensional (2D) problems to achieve some reduction in analytical and mathematical rigours of the solution [21 – 26].

In general, 2D elasticity problems are governed by the two dimensional forms of the material constitutive relations, the strain-displacement (kinematic) equations, and the differential equations of equilibrium, subject to the boundary conditions [1 – 20]. The governing equations of 2D elasticity problems are formulated using displacement, stress or mixed (hybrid) methods. In displacement methods of formulation, the governing equations are expressed in terms of the displacement components only as the primary unknown variables [1 – 20]. In stress-based methods, the equations are expressed in terms of stress components which are the unknowns in the formulation [1 – 20].

In hybrid (mixed) methods, the governing equations are expressed such that some components of stress and some components of displacement become the unknown variables to be found [1 – 20]. The mixed method is rarely used.

The stress-based method of formulation of 2D elasticity problems is used in this work. The stress compatibility equation is expressed in terms of Airy stress potential function in the plane

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polar coordinates system to obtain the stress compatibility equation in terms of the Airy stress potential function as a biharmonic equation. Since the Airy stress potential functions are derived from solutions of the differential equations of equilibrium for 2D problems, solutions of the resulting biharmonic equation for the Airy stress potential function would yield equilibrating stress fields.

Recent research studies and publications on elasticity theory include: Danesh et al [27], Mohammadi et al [28], Mohammadi and Rastgoo [29, 30], Safarabadi et al [31] and Barati et al [32].

1.1 Research aim and objectives

The aim of the research is to apply the Elzaki transform method to the solution of 2D elasticity problems in plane polar coordinates. The objectives are:

(i) to present a stress formulation in terms of Airy stress potential function for 2D elasticity problems.

(ii) to apply the Elzaki transform to the governing biharmonic stress compatibility equation to reduce the equation to an ordinary linear differential equation (ODE) in the Elzaki transform space variable.

(iii) to solve the resulting ODE and obtain the Airy stress potential function in the Elzaki transform space variable.

(iv) to use Elzaki inversion to find the general expression for the Airy stress potential function in the physical problem variable.

(v) to obtain the general expressions for the normal and shear stresses in the physical domain variable.

(vi) to apply the results obtained to the particular case of the Flamant problem, and obtain specific solutions for the Airy stress potential function, and the normal and shear stress fields for the Flamant problem.

(vii) to use superposition principle to derive other 2D elasticity solutions from the solutions of Flamant problem used as Kernel functions.

2. Theoretical framework

The governing equations of the elastic half plane are given by the differential equations of equilibrium, the stress-strain relations and the kinematic relations and the equations are solved subject to the boundary conditions imposed by the loading.

Airy derived solutions to two-dimensional (2D) elasticity problems using a stress-based formulation of the problem by finding Airy's stress potential functions $\phi(r,\theta)$ of the two-dimensional coordinates of the problem that identically satisfy the differential equations of equilibrium. Airy found the normal and shear stresses for 2D problems in the plane polar coordinates system to be derivable from the Airy stress function as follows [8,12]:

$$\sigma_{rr}(r,\theta) = \frac{1}{r}\frac{\partial\phi}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}(r,\theta) + V \quad (1)$$

$$\sigma_{\theta\theta}(r,\theta) = \frac{\partial^2\phi}{\partial r^2}(r,\theta) + V \quad (2)$$

where V is the body force potential $f_r = B_r = -\dfrac{\partial V}{\partial r}$ (3)

$$f_\theta = B_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} \quad (4)$$

$$\tau_{r\theta}(r,\theta) = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}(r,\theta)\right) = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta}(r,\theta) - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}(r,\theta) \quad \ldots(5)$$

where σ_{rr} is the radial stress field, $\sigma_{\theta\theta}$ is the circumferential stress, $\tau_{r\theta}$ is the shear stress, f_r (or B_r) is the radial component of the body force, f_θ (or B_θ) is the tangential component of the body force, $\phi(r,\theta)$ is a potential function, and hence satisfies the Laplaces equation in 2D polar coordinates:

$$\nabla^2\phi(r,\theta) = 0 \quad (6)$$

where ∇^2 denotes the Laplacian operator in the radial coordinates system.

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \quad (7)$$

The Airy stress functions are solutions to the differential equations of equilibrium:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \quad (8)$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{2\tau_{r\theta}}{r} = 0 \quad (9)$$

2.1 Compatibility Equation

The stress compatibility equations are given by [24-26]:

$$\nabla^2(\nabla^2\phi) = -(1+\mu)\left(\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{1}{r}\frac{\partial B_\theta}{\partial\theta}\right) \quad (10)$$

(for plane stress)

and

$$\nabla^2(\nabla^2\phi) = \frac{-1}{1+\mu}\left(\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{1}{r}\frac{\partial B_\theta}{\partial\theta}\right) \quad (11)$$

(for plane strain)

where μ is the Poisson's ratio.

$$\nabla^2\nabla^2\phi(r,\theta) + \left(2 - \frac{1}{\alpha}\right)\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}\right) = 0 \quad (12)$$

$$\alpha = \begin{cases} 1-\mu & \text{for plane strain} \\ \frac{1}{1+\mu} & \text{for plane stress} \end{cases} \quad (13)$$

The compatibility equation in terms of stress for 2D problems where body forces are absent is:

$$\nabla^2(\sigma_{\theta\theta} + \sigma_{rr}) = 0 \quad (14)$$

$$\nabla^2\left(\frac{\partial^2\phi}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial\phi}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}(r,\theta)\right) = 0 \quad (15)$$

$$\nabla^2\nabla^2\phi(r,\theta) = \nabla^4\phi(r,\theta) = 0 \quad (16)$$

The Airy stress function for 2D elastostatic problems is biharmonic. Radial, circumferential and normal stress fields which are derived from an Airy stress function that satisfies the biharmonic differential equation of stress compatibility will thus simultaneously satisfy the differential equations of equilibrium and correspond to compatible strain fields.

3. Research methodology

3.1 Definitions

The Elzaki transform of a function $f(x)$ of the independent variable x where $f(x)$ is of exponential order, is considered a modification of the Sumudu and Laplace transforms, and is defined by the integral equation [33-36]:

$$Ef(x) = F(u) = u^2 \int_0^{\infty} f(ux)e^{-x}dx \qquad (17)$$

$$Ef(x) = u^2 \lim_{\alpha\to\infty} \int_0^{\alpha} f(ux)e^{-x}dx \qquad (18)$$

where $u \in (k_1, k_2)$, $k_1 > 0$, $k_2 > 0$, $x > 0$, and $k_1 \le u \le k_2$

E is the Elzaki transform operator, u is the Elzaki transform parameter, k_1 and k_2 are complex numbers which may be finite or infinite.

The Elzaki transform is alternatively defined, by a suitable change of variables, by [33-36]:

$$Ef(x) = u \int_0^{\infty} f(x)e^{-x/u}dx \qquad (19)$$

$$Ef(x) = u \lim_{\alpha\to\infty} \int_0^{\alpha} f(x)e^{-x/u}dx = F(u) \qquad (20)$$

Similarly, the Elzaki transform of the function $f(x, t)$ with respect to the independent variable x is given by [33-36]:

$$E_x f(x,t) = F(u,t) = u \int_0^{\infty} f(x,t)e^{-x/u}dx = u \lim_{\alpha\to\infty} \int_0^{\alpha} f(x,t)e^{-x/u}dx \qquad \ldots(21)$$

The Elzaki transform of $f(x, t)$ with respect to the independent variable t is given by:

$$E_t f(x,t) = F(x,v) = v \int_0^{\infty} f(x,t)e^{-t/v}dt = v \lim_{\beta\to\infty} \int_0^{\beta} f(x,t)e^{-t/v}dt \qquad \ldots(22)$$

3.2 Properties of Elzaki transform

The Elzaki transform is a linear transform and if $f_1(x, t)$, $f_2(x, t)$, … , $f_n(x, t)$ are functions, then for n constants c_1, c_2, …, c_n, we have:

$$E_x\left(c_1 f_1(x,t) + c_2 f_2(x,t) + \ldots + f_n(x,t)\right) = c_1 E_{x_1} f_1(x,t) + c_2 E_{x_2}(x,t) + \ldots + c_n f_{x_n}(x,t) \qquad (23)$$

The Elzaki transforms of the derivatives of $f(x, t)$ are obtained using integration by parts [33 – 36].

4. Results

4.1 Application of Elzaki transform to the stress compatibility equation

The stress compatibility equation expressed in terms of the Airy stress potential function $\phi(r,\theta)$ is given by:

$$\nabla^4\phi(r,\theta) = \frac{\partial^4\phi}{\partial r^4} + \frac{2}{r}\frac{\partial^3\phi}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^3}\frac{\partial\phi}{\partial r} + \frac{2}{r^2}\frac{\partial^4\phi}{\partial r^2\partial\theta^2} - \frac{2}{r^3}\frac{\partial^4\phi}{\partial r\partial\theta^3} + \frac{4}{r^4}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r^4}\frac{\partial^4\phi}{\partial\theta^4} = 0 \qquad (24)$$

Multiplication by r^4 gives:

$$r^4\nabla^4\phi(r,\theta) = r^4\frac{\partial^4\phi}{\partial r^4} + 2r^3\frac{\partial^3\phi}{\partial r^3} - r^2\frac{\partial^2\phi}{\partial r^2} + r\frac{\partial\phi}{\partial r} + 2r^2\frac{\partial^4\phi}{\partial r^2\partial\theta^2} - 2r\frac{\partial^4\phi}{\partial r\partial\theta^3} + 4\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^4\phi}{\partial\theta^4} = 0 \qquad (25)$$

Application of the Elzaki's transform with respect to r to Equation (25) gives:

$$E_r\left(r^4\nabla^2\phi(r,\theta)\right) = 0 \qquad (26)$$

Thus

$$u\int_o^{\infty} r^4\nabla^4\phi(r,\theta)e^{-r/u}dr = 0 \qquad (27)$$

Thus

$$u\int_0^{\infty}\left(r^4\frac{\partial^4\phi}{\partial r^4} + 2r^3\frac{\partial^3\phi}{\partial r^3} - r^2\frac{\partial^2\phi}{\partial r^2} + r\frac{\partial\phi}{\partial r} + 2r^2\frac{\partial^2}{\partial\theta^2}\frac{\partial^2\phi}{\partial r^2} - 2r\frac{\partial^3}{\partial\theta^3}\frac{\partial\phi}{\partial r} + 4\frac{\partial^2\phi}{\partial r^2} + \frac{\partial^4\phi}{\partial r^4}\right)e^{-r/u}dr = 0 \qquad (28)$$

Making use of the linearity property of the Elzaki transform, we have:

$$u\left\{\int_0^{\infty} r^4\frac{\partial^4\phi}{\partial r^4}e^{-r/u}dr + \int_0^{\infty}2r^3\frac{\partial^3\phi}{\partial r^3}e^{-r/u}dr - \int_0^{\infty}r^2\frac{\partial^2\phi}{\partial r^2}e^{-r/u}dr + \int_0^{\infty}r\frac{\partial\phi}{\partial r}e^{-r/u}dr + 2\frac{d^2}{d\theta^2}\int_0^{\infty}r^2\frac{\partial^2\phi}{\partial r^2}e^{-r/u}dr - 2\frac{d^3}{d\theta^3}\int_0^{\infty}r\frac{\partial\phi}{\partial r}e^{-r/u}dr + 4\frac{d^2}{d\theta^2}\int_0^{\infty}\phi(r,\theta)e^{-r/u}dr + \frac{d^4}{d\theta^4}\int_0^{\infty}\phi(r,\theta)e^{-r/u}dr\right\} = 0 \qquad (29)$$

But by definition,

$$u\int_0^{\infty}\phi(r,\theta)e^{-r/u}dr = \overline{\phi}(u,\theta) \qquad (30)$$

where $\overline{\phi}(u,\theta)$ is the Elzaki transform of $\phi(u,\theta)$ with respect to the radial coordinate, r.

Using integration by parts, we have:

$$\frac{d^4\overline{\phi}}{d\theta^4}(u,\theta) + 4\frac{d^2}{d\theta^2}\overline{\phi}(u,\theta) + 2u\frac{d^2\overline{\phi}}{d\theta^2}(u,\theta) + 2u(u+1)\frac{d^2\overline{\phi}}{d\theta^2}(u,\theta) + \left(u(u+1)(u+2)(u+3) - 2u(u+1)(u+2) - u(u+1) - u\right)\overline{\phi}(u,\theta) = 0 \qquad (31)$$

Simplifying,

$$\frac{d^4\overline{\phi}}{d\theta^4}(u,\theta) + (4 + 2u + 2u^2 + 2u)\frac{d^2\overline{\phi}}{d\theta^2} + \left(u(u+1)(u+2)(u+3) - 2u(u+1)(u+2) - u(u+1) - u\right)\overline{\phi}(u,\phi) = 0 \qquad (32)$$

Further simplification gives:

$$\frac{d^4\overline{\phi}}{d\theta^4}(u,\theta) + \left(u^2 + (u+2)^2\right)\frac{d^2\overline{\phi}}{d\theta^2}(u,\theta) + \left(u^2(u+2)^2\right)\overline{\phi}(u,\theta) = 0 \qquad \ldots(33)$$

This differential equation can be expressed alternatively as:

$$\left(\frac{d^2}{d\theta^2}+u^2\right)\left(\frac{d^2}{d\theta^2}+(u+2)^2\right)\bar{\phi}(u,\theta)=0 \quad (34)$$

The general solution to the fourth order ordinary differential equation (33) gives the general solution for the Airy stress function in Elzaki transform space.

4.2 Airy stress function $\bar{\phi}(u,\theta)$ in Elzaki transform space

The Elzaki transformed stress compatibility equation which is the fourth order ODE, Equation (33), is solved using the method of trial functions or the differential operator methods. Using the trial function method, the solution to Equation (33) is assumed in the exponential form:

$$\bar{\phi}(u,\theta)=\exp\lambda\theta \quad (35)$$

where $\lambda(u)$ is an unknown parameter which is sought such that Equation (35) satisfies Equation (33) identically. Then we obtain:

$$\lambda^4 e^{\lambda\theta}+\left(u^2+(u+2)^2\right)\lambda^2 e^{\lambda\theta}+u^2(u+2)^2 e^{\lambda\theta}=0 \quad (36)$$

Simplifying,

$$\left(\lambda^4+(u^2+(u+2)^2)\lambda^2+u^2(u+2)^2\right)e^{\lambda\theta}=0 \quad (37)$$

For nontrivial solutions, $e^{\lambda\theta}\neq 0$, the characteristic polynomial is obtained as:

$$\lambda^4+(u^2+(u+2)^2)\lambda^2+u^2(u+2)^2=0 \quad (38)$$

Factorization yields:

$$(\lambda^2+u^2)\left(\lambda^2+(u+2)^2\right)=0 \quad (39)$$

Solving

$$\lambda^2=-u^2 \quad (40)$$
$$\lambda=\pm iu \quad (41)$$
$$\lambda^2=-(u+2)^2 \quad (42)$$
$$\lambda=\pm(u+2)i \quad (43)$$

where $i=\sqrt{-1}$ (44)

i is the imaginary number.

Using the Euler formula, the general solution becomes:

$$\bar{\phi}(u,\theta)=a_1\sin u\theta+a_2\cos u\theta+a_3\sin(u+2)\theta+a_4\cos(u+2)\theta \quad \ldots(45)$$

$$\bar{\phi}(u,\theta)=c_1e^{iu\theta}+c_2e^{-iu\theta}+c_3e^{i(u+2)\theta}+c_4e^{-i(u+2)\theta} \quad (46)$$

where a_1, a_2, a_3 and a_4 (or c_1, c_2, c_3 and c_4) are four integration constants which can be found using the boundary conditions of the particular 2D elasticity problems.

By inversion, the Airy stress function is obtained in the physical domain variables as:

$$\phi(r,\theta)=E_r^{-1}\bar{\phi}(u,\theta)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{s\theta}s\bar{\phi}\left(\frac{1}{s}\right)ds \quad (47)$$

$$\phi(r,\theta)=E_r^{-1}\left(a_1\sin u\theta+a_2\cos u\theta+a_3\sin(u+2)\theta+a_4\cos(u+2)\theta\right) \quad (48)$$

$$\phi(r,\theta)=a_1r\theta\sin\theta+a_2r\theta\cos\theta+a_3r\ln r\sin\theta+a_4r\ln r\cos\theta \quad \ldots(49)$$

where a_1, a_2, a_3 and a_4 are four constants of integration found by using the boundary conditions.

4.3 Stress fields

The general expressions for the normal and shear stress fields are obtained from the Airy stress function as follows:

$$\sigma_{rr}(r,\theta)=\frac{1}{r}\frac{\partial}{\partial r}(a_1r\theta\sin\theta+a_2r\theta\cos\theta+a_3r\ln r\sin\theta+a_4r\ln r\cos\theta)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(a_1r\theta\sin\theta+a_2r\theta\cos\theta+a_3r\ln r\sin\theta+a_4r\ln r\cos\theta) \quad (50)$$

$$\sigma_{rr}(r,\theta)=a_1\frac{2\cos\theta}{r}+a_2\frac{2\sin\theta}{r}+a_3\frac{\sin\theta}{r}+a_4\frac{\cos\theta}{r} \quad \ldots(51)$$

$$\sigma_{\theta\theta}(r,\theta)=\frac{\partial^2}{\partial r^2}(a_1r\theta\sin\theta+a_2r\theta\cos\theta+a_3r\ln r\sin\theta+a_4r\ln r\cos\theta) \quad (52)$$

$$\sigma_{\theta\theta}(r,\theta)=a_3\frac{\sin\theta}{r}+a_4\frac{\cos\theta}{r} \quad (53)$$

$$\tau_{r\theta}(r,\theta)=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}(a_1r\theta\sin\theta+a_2r\theta\cos\theta+a_3r\ln r\sin\theta+a_4r\ln r\cos\theta\right) \quad (54)$$

$$\tau_{r\theta}(r,\theta)=\left(-\frac{\cos\theta}{r}\right)a_3+\left(\frac{\sin\theta}{r}\right)a_4 \quad (55)$$

4.4 The Flamant problem

The solution of the Flamant problem is given as a specific illustration of the application of the study to practical problems of two-dimensional elasticity theory. The Flamant problem is the problem of determination of the stress fields in an elastic half plane due to line loads of magnitudes P_1 and P_2 of infinite extent applied vertically and horizontally respectively at the origin of the elastic half- plane $(-\infty\le x\le\infty;\ 0\le z\le\infty)$ as shown in Figure 1.

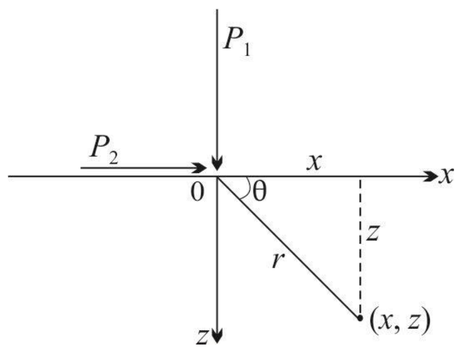

Figure 1: Flamant problem (line loads P_1 and P_2 of infinite extent applied at the origin O of an elastic half-plane)

The Flamant problem is a 2D problem of the theory of elasticity. The governing biharmonic stress compatibility equation has been solved in this work using the Elzaki transform method, and the Airy stress potential function that satisfies the biharmonic stress compatibility equation has been determined in general. The corresponding stresses have also been determined in general. The boundary conditions of the given Flamant problem are:

$$\sigma_{\theta\theta}(r,\theta=0)=\sigma_{\theta\theta}(r,\theta=\pi)=0 \quad (56)$$

$$\tau_{r\theta}(r,\theta=0)=\tau_{r\theta}(r,\theta=\pi)=0 \tag{57}$$

4.4.1 Enforcement of boundary conditions

Using the boundary conditions, we obtain:

$$\tau_{r\theta}(r,\theta=0)=-\frac{a_3}{r}=0 \tag{58}$$

Hence, $a_3 = 0$

$$\tau_{r\theta}(r,\theta=\pi)=\frac{a_3}{r}=0 \tag{59}$$

Hence, $a_3 = 0$ (60)

Similarly,

$$\sigma_{\theta\theta}(r,\theta=0)=\frac{a_4}{r}=0 \tag{61}$$

$$\sigma_{\theta\theta}(r,\theta=\pi)=-\frac{a_4}{r}=0 \tag{62}$$

$\square\ a_4 = 0$ (63)

Then the stresses become:

$$\sigma_{\theta\theta}(r,\theta)=0 \tag{64}$$

$$\tau_{r\theta}(r,\theta)=0 \tag{65}$$

$$\sigma_{rr}(r,\theta)=\frac{2a_1\cos\theta}{r}+\frac{2a_2\sin\theta}{r} \tag{66}$$

4.4.2 Equilibrium equations

For equilibrium of the internal and external forces, we have:

For the vertical direction:

$$P_1=\int_0^{\pi}\left(\sigma_{rr}(r,\theta)\sin\theta+\tau_{r\theta}(r,\theta)\cos\theta\right)rd\theta \tag{67}$$

For the horizontal direction:

$$P_2=\int_0^{\pi}\left(\sigma_{rr}(r,\theta)\cos\theta-\tau_{r\theta}(r,\theta)\sin\theta\right)rdr \tag{68}$$

and

$$\sum M=\int_0^{\pi} r\,\tau_{r\theta}(r,\theta)rd\theta=0 \tag{69}$$

But $\tau_{r\theta}(r,\theta)=0$ everywhere in the elastic half plane, hence the equilibrium equations become: the system of two equations:

$$P_1=\int_0^{\pi}\sigma_{rr}(r,\theta)\sin\theta\, rd\theta \tag{70}$$

$$P_2=\int_0^{\pi}\sigma_{rr}(r,\theta)\cos\theta\, rd\theta \tag{71}$$

Hence

$$P_1=\int_0^{\pi}\left(\frac{2a_1\cos\theta}{r}+\frac{2a_2\sin\theta}{r}\right)\sin\theta\, rd\theta \tag{72}$$

$$P_2=\int_0^{\pi}\left(\frac{2a_1\cos\theta}{r}+\frac{2a_2\sin\theta}{r}\right)\cos\theta\, rd\theta \tag{73}$$

Simplifying,

$$P_1=2\int_0^{\pi}(a_1\cos\theta+a_2\sin\theta)\sin\theta\, d\theta \tag{74}$$

$$P_2=2\int_0^{\pi}(a_1\cos\theta+a_2\sin\theta)\cos\theta\, d\theta \tag{75}$$

$$P_1=a_2\pi \tag{76}$$

$$P_2=a_1\pi \tag{77}$$

Solving, $a_1=\dfrac{P_2}{\pi}$ (78)

$$a_2=\frac{P_1}{\pi} \tag{79}$$

Hence for the Flamant problem, the solutions for the Airy stress potential functions and the stress fields become:

$$\phi(r,\theta)=\frac{P_2}{\pi}r\theta\sin\theta+\frac{P_1}{\pi}r\theta\cos\theta \tag{80}$$

$$\sigma_{rr}(r,\theta)=\frac{P_2}{\pi}\frac{2\cos\theta}{r}+\frac{P_1}{\pi}\frac{2\sin\theta}{r} \tag{81}$$

The solutions are observed to obey the superposition principle (or linearly principle). Solutions can thus be expressed for the loads P_1 and P_2 acting separately or independently. When P_1 acts alone, the solutions become:

$$\phi(r,\theta)=\frac{P_1}{\pi}r\theta\cos\theta \tag{82}$$

$$\sigma_{rr}(r,\theta)=\frac{2P_1\sin\theta}{\pi r} \tag{83}$$

$$\sigma_{\theta\theta}(r,\theta)=\tau_{r\theta}(r,\theta)=0 \tag{84}$$

When P_2 is applied alone, the solutions are:

$$\phi(r,\theta)=\frac{P_2}{\pi}r\theta\sin\theta \tag{85}$$

$$\sigma_{rr}(r,\theta)=\frac{2P_2\cos\theta}{\pi r} \tag{86}$$

$$\tau_{r\theta}(r,\theta)=\sigma_{\theta\theta}(r,\theta)=0 \tag{87}$$

4.4.3 Stress fields in Cartesian coordinates

The stress fields are obtained in terms of the Cartesian coordinates for the Flamant problem where the vertical line load of infinite extent P_1 is applied alone at the origin. Thus,

$$\sigma_{zz}=\sigma_{rr}\sin^2\theta=\frac{2P_1}{\pi r}\sin^3\theta \tag{88}$$

$$\sigma_{zz}(x,z)=\frac{2P_1}{\pi r}\left(\frac{z}{r}\right)^3=\frac{2P_1z^3}{\pi r^4} \tag{89}$$

$$\sigma_{zz}=\frac{2P_1}{\pi z}\frac{z^4}{r^4}=\frac{P_1}{z}\frac{2}{\pi}\frac{z^4}{(x^2+z^2)^2}=\frac{P_1}{z}\frac{2}{\pi}\left(\frac{z^2}{x^2+z^2}\right)^2 \tag{90}$$

$$\sigma_{zz}=\frac{P_1}{z}\frac{2}{\pi}\left(1+\left(x/z\right)^2\right)^{-2}=\frac{P_1}{z}I(x,z) \tag{91}$$

$$I(x,z)=\frac{2}{\pi}\left(1+\left(x/z\right)^2\right)^{-2} \tag{92}$$

$$\sigma_{xx}(x,z)=\frac{2P_1}{\pi z}\cos^2\theta\sin^2\theta=\frac{2P_1x^2z}{\pi(x^2+z^2)^2}=\frac{2P_1}{\pi z}\left(\frac{x^2}{x^2+z^2}\right)\left(\frac{z^2}{x^2+z^2}\right) \tag{93}$$

$$\sigma_{xx}(x,z) = \frac{2P_1}{\pi z}\left(1+\left(z/x\right)^2\right)^{-1}\left(1+\left(x/z\right)^2\right)^{-1} \quad (94)$$

$$\tau_{xz}(x,z) = \frac{2P_1 xz^2}{\pi(x^2+z^2)^2} \quad (95)$$

$$\sigma_{yy}(x,z) = \mu(\sigma_{xx}+\sigma_{zz}) =$$

$$= \mu\left(\frac{2P_1x^2z}{\pi(x^2+z^2)^2} + \frac{2P_1z^3}{\pi(x^2+z^2)^2}\right) \quad (96)$$

$$\sigma_{yy}(x,z) = \frac{2P_1\mu}{\pi}\left(\frac{x^2z+z^3}{(x^2+z^2)^2}\right) =$$

$$\frac{2P_1\mu}{\pi}\left(\frac{z(x^2+z^2)}{(x^2+z^2)^2}\right) = \frac{2P_1\mu}{\pi}\frac{zr^2}{r^4} \quad (97)$$

$$\sigma_{yy}(x,z) = \frac{2P_1\mu z}{\pi r^2} = \frac{2P_1\mu z}{\pi(x^2+z^2)} \quad (98)$$

4.5 Stresses due to strip loads of infinite extent

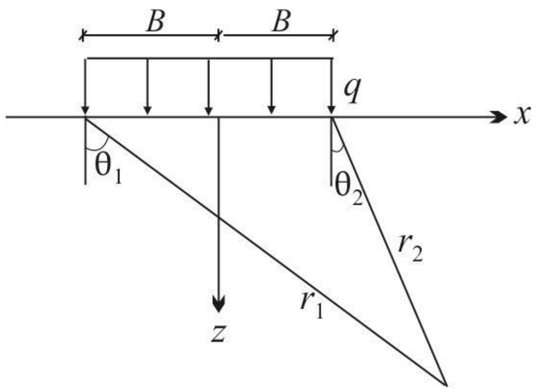

Figure 2: Strip load of infinite extent on elastic half plane

Using the principle of superposition,

$$\sigma_{zz} = \int_{-B}^{B} \frac{2qd\varepsilon z^3}{\pi\left((x-z)^2+z^2\right)^2} \quad (99)$$

where ε is a dummy variable of integration.

$$\sigma_{zz} = \frac{q}{\pi}\left(\tan^{-1}\left(\frac{z}{x-B}\right) - \tan^{-1}\left(\frac{z}{x+B}\right) - \frac{2Bz(x^2-z^2-B^2)}{(x^2+z^2-B^2)^2+4B^2z^2}\right) \quad (100)$$

$$\sigma_{zz} = \frac{q}{\pi}\left((\theta_1-\theta_2)+\sin\theta_1\cos\theta_1 - \sin\theta_2\cos\theta_2\right) \quad (101)$$

$$\sigma_{xx} = \int_{-B}^{B} \frac{2q}{\pi}\frac{(x-\varepsilon)^2 z\,d\varepsilon}{((x-\varepsilon)^2+z^2)^2} \quad (102)$$

$$\sigma_{xx} = \frac{q}{\pi}\left(\tan^{-1}\left(\frac{z}{x-B}\right) - \tan^{-1}\left(\frac{z}{x+B}\right) + \frac{2Bz(x^2-z^2-B^2)}{(x^2+z^2-B^2)^2+4B^2z^2}\right) \quad (103)$$

$$\sigma_{xx} = \frac{q}{\pi}(\theta_1-\theta_2-\sin\theta_1\cos\theta_1+\sin\theta_2\cos\theta_2) \quad (104)$$

$$\tau_{xz} = \int_{-B}^{B} \frac{2q(x-\varepsilon)z^2 d\varepsilon}{((x-\varepsilon)^2+z^2)^2\pi} \quad (105)$$

$$\tau_{xz} = \frac{q}{\pi}\frac{4Bxz^2}{((x^2+z^2-B^2)^2+4B^2z^2)} \quad (106)$$

$$\tau_{xz} = \frac{q}{\pi}(\cos^2\theta_2 - \cos^2\theta_1) \quad (107)$$

4.6 Stresses on smooth rigid retaining wall due to strip load of infinite extent but finite width, B

At the center of the half plane $x = 0$, $\theta_2 = -\theta_1$. The stresses at the center of the half plane are

$$\sigma_{zz}(x=0,z) = \frac{2q}{\pi}(\theta_1+\sin\theta_1\cos\theta_1) \quad (108)$$

$$\sigma_{zz}(x=0,z) = \frac{2q}{\pi}\left(\tan^{-1}\frac{B}{z}+\frac{Bz}{B^2+z^2}\right) \quad (109)$$

$$\frac{\sigma_{zz}(x=0,z)}{q} = \frac{2}{\pi}\left(\tan^{-1}\frac{B}{z}+\frac{Bz}{B^2+z^2}\right) \quad (110)$$

$$\sigma_{xx}(x=0,z) = \frac{2q}{\pi}(\theta_1-\sin\theta_1\cos\theta_1) \quad (111)$$

$$\sigma_{xx}(x=0,z) = \frac{2q}{\pi}\left(\tan^{-1}\frac{B}{z}-\frac{Bz}{B^2+z^2}\right) \quad (112)$$

$$\frac{\sigma_{xx}(x=0,z)}{q} = \frac{2}{\pi}\left(\tan^{-1}\frac{B}{z}-\frac{Bz}{B^2+z^2}\right) \quad (113)$$

$$\tau_{xz}(x=0,z) = 0 \quad (114)$$

The shear stress field vanishes at $x = 0$ plane due to the symmetrical nature of the elastic half-plane and the loading. The stresses σ_{xx} and σ_{zz} can be determined and represented in terms of influence ratio z/B. Expectedly, due to symmetry $u_x(x = 0, z) = 0$. This suggests that the obtained solutions can be used to determine the stresses on a rigid retaining wall of height H, due to a strip load of uniform intensity q, finite width B and infinite length (extent) in the y coordinate direction as shown in Figure 3.

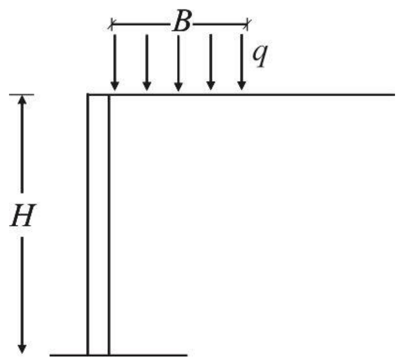

Figure 3: Horizontal stresses on smooth rigid retaining wall due to uniformly distributed strip load of infinite extent

The resultant horizontal force P_r on the smooth rigid wall of height H is found by integration of $\sigma_{xx}(x=0,z)$ as follows:

$$P_r = \int_0^H \sigma_{xx}(x=0,z)dz = \int_0^H \frac{2q}{\pi}\left(\tan^{-1}\frac{B}{z} - \frac{Bz}{B^2+z^2}\right)dz$$

$$= \frac{2q}{\pi}\int_0^H \left(\tan^{-1}\frac{B}{z} - \frac{Bz}{B^2+z^2}\right)dz \quad (115)$$

$$P_r = \frac{2q}{\pi}\left[z\tan^{-1}\frac{B}{z}\right]_0^H = \frac{2qH}{\pi}\tan^{-1}\frac{B}{H} \quad (116)$$

For retaining walls where $H >> B$, and $B/H \to 0$, since for small angles tan x . x radians,

$$\tan^{-1}\frac{B}{H} \approx \frac{B}{H} \quad (117)$$

Then,

$$P_r \simeq \frac{2}{\pi}qH \cdot \frac{B}{H} = \frac{2qB}{\pi} \quad (118)$$

For walls where $H << B$, $B/H \to \infty$, and since tan ∞ = 90E = $\pi/2$ radians,

$$\tan^{-1}\frac{B}{H} \simeq \frac{\pi}{2} \quad (119)$$

$$\therefore P_r(H << B) = \frac{2qH}{\pi}\frac{\pi}{2} = qH \quad (120)$$

For the case of two parallel line loads applied at $x = \forall b$ on the elastic half plane as shown in Figure 4(i), the symmetry of the loading and the problem suggests that the shear stress field and the horizontal displacement will each vanish at x = 0. This suggests and implies that the solution of the problem is identical to the solution of the 2D problem of stresses on a smooth rigid retaining wall due to a line load applied at a known or given distance, $x = b$ from the origin (on the surface) as shown in Figure 4(ii); since the boundary conditions on the wall are identical, $u_x(x = 0, z) = 0$, $\tau_{xz}(x=0,z)=0$

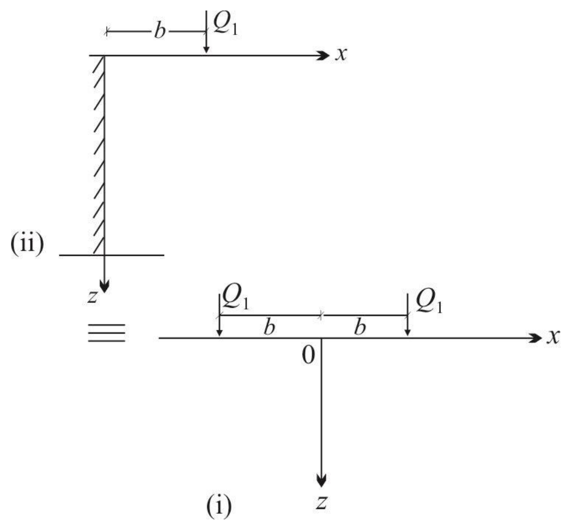

Figure 4: (i) Parallel line load of infinite extent in the y direction acting at $x = \forall b$ on the elastic half plane; (ii) line load of infinite length in y direction acting parallel to smooth rigid retaining wall of height H.

The horizontal stress distribution on the smooth rigid wall is found using the results from Figure 4(i) as:

$$\sigma_{xx}(x=0,z) = \frac{4}{\pi}\frac{Q_1 b^2 z}{(b^2+z^2)^2} \quad (121)$$

$\sigma_{xx}(x=0,z)$ is maximum when $z = 0.577b$ found from

$$\frac{\partial\sigma_{xx}}{\partial z}(x=0,z) = \frac{\partial}{\partial z}\frac{4}{\pi}\frac{Q_1 b^2 z}{(b^2+z^2)^2} = 0 \quad (122)$$

$$\frac{4Q_1 b^2}{\pi}\frac{\partial}{\partial z}\frac{z}{(b^2+z^2)^2} = 0 \quad (123)$$

$$\sigma_{xx_{\max}} = 0.4135\frac{Q_1}{b} \quad (124)$$

The resultant force Q_r on the smooth rigid wall of height H is found by integration over the wall surface:

$$Q_r = \int_0^H \frac{4}{\pi}\frac{Q_1 b^2 z}{(b^2+z^2)^2}dz = \frac{4}{\pi}Q_1 b^2 \int_0^H \frac{z}{(b^2+z^2)^2}dz$$

$$= \frac{4}{\pi}Q_1 b^2\left[\frac{1}{2(b^2+z^2)}\right]_0^H \quad (125)$$

$$Q_r = \frac{2}{\pi}\frac{Q_1}{1+b^2/H^2} \quad (126)$$

For b = 0,

$$Q_r(b=0) = \frac{2}{\pi}Q_1 \quad (127)$$

6. Discussion

Elzaki transform method was used in this work to solve the two dimensional problems in the theory of elasticity for stress fields in elastic half plane due to boundary loads. The 2D elasticity problem was formulated using stress formulation method, and Airy stress function as a biharmonic partial differential equation in terms of the Airy stress potential function $\phi(r,\theta)$ for the case where body forces are disregarded. The 2D elasticity problem then becomes the problem of finding Airy stress function $\phi(r,\theta)$ that solves the stress compatibility equation – Equation (24). By the Elzaki transform method, the Elzaki transformation was applied to a slightly modified version of the stress compatibility equation presented explicitly as Equation (25). The Elzaki transformation of the governing equation then converted the problem to the integral equation given by Equation (28). Using the linearity property of the Elzaki transform and integration by parts, the integral equation problem was simplified to an ordinary differential equation (ODE) which is in terms of $\bar{\phi}(u,\theta)$, the Elzaki transform of the Airy stress potential function $\phi(r,\theta)$. Simplification gave the ODE in Equation (31) as the fourth order ODE in Equation (33). The ODE was solved using the methods of trial functions, differential operator or other methods for solving ODEs to obtain the general solution for $\bar{\phi}(u,\theta)$ as Equation (45) which is in terms of four unknown constants of integration. By inversion, the Airy stress potential function is obtained in the domain variables for the general 2D problem of elasticity as Equation (49), which also contains four unknown constants of integration. The normal and shear stress fields are determined for the general case of 2D elasticity problems as Equations (51), (53) and (55).

Flamant problem, which is the problem of finding normal and shear stress fields in an elastic half plane (for which

$-\infty \leq x \leq \infty$, $0 \leq z \leq \infty$) due to vertical and horizontal line loads of infinite extent was used to provide a specific application of the general solutions obtained. The Flamant problem is also governed by the biharmonic stress compatibility equations for $\phi(r, \theta)$ for which the general solutions were derived using the Elzaki transform method. The stress boundary conditions were used to obtain the two constants of integration a_3, and a_4 as Equations (60) and (63), and the stresses $\sigma_{\theta\theta}(r, \theta)$ and $\tau_{r\theta}(r, \theta)$ were found as Equations (64) and (65) and $\sigma_{rr}(r, \theta)$ was found as Equation (66) which now contains only two constants of integration a_1 and a_2. The requirements of equilibrium of the internal stress resultants and the applied forces in the horizontal and vertical coordinate directions were used to set up two equations of equilibrium which were used to obtain the two constants of integration as Equations (78) and (79). The Flamant problem was thus fully solved and the Airy stress potential function found explicitly as Equation (80). The radial stress was found as Equation (81). The solutions for the vertical line loads which are of more significance in civil engineering were integrated over distributed loaded areas to obtain solutions for strip loads of finite width but of infinite extent (length). The stress fields obtained for strip loads are given by Equations (101), (104) and (107). The solutions were also used to obtain horizontal stress distributions and their resultants on the backface of smooth rigid retaining walls of known height due to strip loads of infinite extent, and due to infinitely long line loads acting parallel to the retaining wall.

7. Conclusion

The following conclusions are made from the study:

(i) 2D elasticity problems which are formulated using stress-based methods and Airy stress potential function $\phi(r, \theta)$ as biharmonic stress compatibility equation in terms of $\phi(r, \theta)$ can be solved using the Elzaki transform method.

(ii) The Elzaki transform of the biharmonic stress compatibility equation expressed in terms of the Airy stress potential function $\phi(r, \theta)$ simplifies the biharmonic equation which is a partial differential equation (PDE) to a fourth order linear ordinary differential equation (ODE) in terms of $\bar{\phi}(u, \theta)$ the Elzaki transformed Airy stress potential function.

(iii) The use of trial function solution or differential operator methods for ODEs further transforms the ODE to fourth degree characteristic (auxiliary) polynomial in $\bar{\phi}(u, \theta)$, and hence the governing biharmonic equation is ultimately transformed by the Elzaki transform to an algebraic equation.

(iv) The general solution for the Airy stress potential function in the Elzaki transform space variable is obtained in terms of four unknown constants of integration which can be determined from the enforcement of boundary conditions.

(v) The general solution for the Airy stress potential function in the physical problem domain is obtained by inversion of the general solution in the Elzaki transform space. The general solution in the problem domain also contains four unknown constants determined by using the boundary conditions.

(vi) The general solutions are obtained for the normal and shear stresses in terms of unknown constants of integration.

(vii) The illustrative case of the Flamant problem used the boundary conditions and equilibrium of internal stress resultants and the applied forces to determine the four constants of integration yielding the solutions for the normal and shear stresses for the Airy stress potential function for the Flamant problem.

(viii) The principle of superposition was applied to extend the solutions for the Flamant problem to obtain the solutions for strip loads of infinite extent and other 2D elasticity solutions for horizontal stresses on smooth rigid retaining walls due to strip loads and parallel line loads of infinite extent.

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