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Usage of the Variational Iteration Technique for Solving Fredholm Integro-Differential Equations

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ABSTRACT

In this paper, we present a variational iteration method for solving Fredholm integro-differential equations. This study provides an analytical approximation to determine the behavior of the solution. Moreover, It proves the existence and uniqueness results and convergence of the solution. Finally, some examples are included to demonstrate the validity and applicability of the proposed technique.

1. Introduction

In recent years, there has been a growing interest in the integro-differential equations which are a combination of differential and integral equations. The nonlinear Fredholm integro-differential equations play an important role in many branches of nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, electrostatics, biology, chemistry and economics [1-11]. In this paper, we consider the Fredholm integro-differential equations of the type:

$$\sum_{j=0}^{k}\xi_j(x)Z^{(j)}(x) = f(x)+\gamma\int_a^b K(x,t)G(Z(t))dt\,, \quad (1.1)$$

with the initial conditions

$$Z^{(r)}(a) = b_r, \quad r = 0,1,2,\ldots,(k-1), \quad (1.2)$$

where $Z^{(j)}(x)$ is the j^{th} derivative of the unknown function $Z(x)$ that will be determined, $K(x,t)$ is the kernel of the equation, $f(x)$ and $\xi_j(x)$ are an analytic function, G is nonlinear function of Z and a,b,γ, and b_r are real finite constants. Recently, many authors focus on the development of numerical and analytical techniques for integro-differential equations. For instance, we can mention the following works: Abbasbandy and Elyas [2] studied some applications on variational iteration method for solving system of nonlinear Volterra integro-differential equations, Alao et al. [4] used Adomian decomposition and variational iteration methods for solving integro-differential equations, Behzadi et al. [5] solved some class of nonlinear Volterra-Fredholm integro-differential equations by homotopy analysis method, Hamoud and Ghadle [6] applied the hybrid methods for solving nonlinear Volterra-Fredholm integro-differential equations, Mittal and Nigam [12] applied the Adomian decomposition method to approximate solutions for fractional integro-differential equations, and Yang and Hou [13] applied the Laplace decomposition method to solve the fractional integro-differential equations. Moreover, several authors have applied the Adomian decomposition method and the variational iteration method to find the approximate solutions of various types of integro-differential equations [14-22].

The main objective of the present paper is to study the behavior of the solution that can be formally determined by semi-analytical approximated method as the variational iteration method. Moreover, we proved the existence, uniqueness results and convergence of the solutions of the Fredholm integro-differential equation (1.1).

2. Description of the Method

Some powerful methods have been focusing on the development of more advanced and efficient methods for integro-differential equations such as the variational iteration method [23-26]. We will describe this method in this section:

2.1. Variational Iteration Method (VIM)

This method is applied to solve a large class of linear and nonlinear problems with approximations converging rapidly to exact solutions. The main idea of this method is to construct a correction functional form using general Lagrange multipliers. These multipliers should be chosen such that its correction solution is superior to its initial approximation, called trial function. It is the best within the flexibility of trial functions. Accordingly, Lagrange multipliers can be identified by the variational theory [25]. A complete review of He's variational

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iteration method is available in [27]. To illustrate, we consider the following general differential equation:

$$LZ(x)+NZ(x)=f(x), \quad (2.1)$$

where *L* is a linear operator, *N* is a nonlinear operator and *f*(*x*) is inhomogeneous term. The terms Z_n are calculated by a correction functional as follows:

$$Z_{n+1}(x) = Z_n(x) + \int_0^x \lambda(\tau)(LZ_n(\tau) + N\tilde{y}(\tau) - f(\tau))d\tau. \quad (2.2)$$

The successive approximation $Z_n(x)$, $n \geq 0$ of the solution Z(x) will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function Z_0. The zeroth approximation Z_0 may be selected using any function that just satisfies at least the initial and boundary conditions, with λ determined, several approximations $Z_n(x)$, $n \geq 0$ follow immediately.

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions.

To obtain the approximation solution of IVB (1.1) − (1.2), according to the VIM, the iteration formula (2.2) can be written as follows:

$$Z_{n+1}(x) = Z_n(x) + L^{-1}\Big[\lambda(x)\Big[\sum_{j=0}^{k}\xi_j(x)Z_n^{(j)}(x)$$
$$- f(x) - \gamma\int_a^b K(x,t)G(Z_n(t))dt\Big]\Big],$$

where L^{-1} is the multiple integration operator given as follows:

$$L^{-1}(\cdot) = \int_a^b\int_a^b ... \int_a^b (\cdot)dxdx...dx(k - times).$$

To find the optimal *λ*(*x*), we proceed as follows:

$$\begin{aligned}&\delta Z_{n+1}(x) = \delta Z_n(x) + \\ &\delta L^{-1}\Big[\lambda(x)\Big[\sum_{j=0}^{k}\xi_j(x)Z_n^{(j)}(x) - f(x) \\ &-\gamma\int_a^b K(x,t)G(Z_n(t))dt\Big]\Big] \\ &= \delta Z_n(x) + \lambda(x)\delta Z_n(x) - L^{-1}\big[\delta Z_n(x)\lambda'(x)\big].\end{aligned} \quad (2.3)$$

From Eq. (2.3), the stationary conditions can be obtained as follows:

$$\lambda'(x) = 0, and\ 1 + \lambda(x)|_{x=t} = 0.$$

As a result, the Lagrange multipliers can be identified as *λ*(*x*) = −1 and by substituting in Eq. (2.3), the following iteration formula is obtained:

$$Z_0(x) = L^{-1}\Big[\frac{f(x)}{\xi_k(x)}\Big] + \sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r,$$

$$Z_{n+1}(x) = Z_n(x) - L^{-1}\Big[\sum_{j=0}^{k}\xi_j(x)Z_n^{(j)}(x)$$
$$- f(x) - \gamma\int_a^b K(x,t)G(Z_n(t))dt\Big], n \geq 0.$$

The term $\sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r$ is obtained from the initial conditions, $\xi_k(x) \neq 0$. Relation (2.4) will enable us to determine the components $Z_n(x)$ recursively for $n \geq 0$. Consequently, the approximation solution may be obtained by using

$$Z(t) = \lim_{n\to\infty} Z_n(t). \quad (2.4)$$

3. Main Results

In this section, we shall give the results of existence and uniqueness of Eq. (1.1), with the initial condition (1.2) and prove it. We can write the equation (1.1) in the form of:

$$\begin{aligned}Z(x) &= L^{-1}\Big[\frac{f(x)}{\xi_k(x)}\Big] + \sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r \\ &+\gamma L^{-1}\Big[\int_a^b \frac{1}{\xi_k(x)}K(x,t)G(Z_n(t))dt\Big] \\ &-L^{-1}\Big[\sum_{j=0}^{k-1}\frac{\xi_j(x)}{\xi_k(x)}Z^{(j)}(x)\Big].\end{aligned} \quad (3.1)$$

Such that,

$$L^{-1}\Big[\int_a^b \frac{1}{\xi_k(x)}K(x,t)G(Z_n(t))dt\Big]$$
$$= \int_a^b \frac{(x-t)^k}{k!\xi_k(x)}K(x,t)G(Z_n(t))dt$$
$$\sum_{j=0}^{k-1}L^{-1}\Big[\frac{\xi_j(x)}{\xi_k(x)}\Big]Z^{(j)}(x) =$$
$$\sum_{j=0}^{k-1}\int_a^b \frac{(x-t)^{k-1}\xi_j(t)}{k-1!\xi_k(t)}Z^{(j)}(t)dt.$$

We set,

$$\Psi(x) = L^{-1}\Big[\frac{f(x)}{\xi_k(x)}\Big] + \sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r.$$

Before starting and proving the main results, we introduce the following hypotheses:

(H1) There exist two constants α and $\gamma_j > 0, j = 0,...,k$ such that, for any $Z_1, Z_2 \in C(J,\mathbb{R})$.

$$|G(Z_1)) - G(Z_2))| \leq \alpha|Z_1 - Z_2|$$

and

$$\left|D^j(Z_1) - D^j(Z_2)\right| \leq \gamma_j|Z_1 - Z_2|,$$

we suppose that the nonlinear terms $G(Z(x))$ and $D^j(Z) = (\frac{d^j}{dx^j})Z(x) = \sum_{i=0}^{\infty}\gamma_{i_j}$, ($D^j$ is a derivative operator), $j = 0,1,...,k,$ are Lipschitz continuous.

(H2) We suppose that for all $a \leq t \leq x \leq b,$ and $j = 0,1,...,k,$:

$$\left|\frac{\gamma(x-t)^k K(x,t)}{k!\xi_k(x)}\right| \leq \theta_1, \quad \left|\frac{\gamma(x-t)^k K(x,t)}{k!}\right| \leq \theta_2,$$
$$\left|\frac{(x-t)^{k-1}\xi_j(t)}{(k-1)!\xi_k(t)}\right| \leq \theta_3, \quad \left|\frac{(x-t)^{k-1}\xi_j(t)}{(k-1)!}\right| \leq \theta_4.$$

(H3) There exist three functions θ_3^*, θ_4^*, and $\gamma^* \in C(D,\mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R}\times\mathbb{R} : 0 \leq t \leq x \leq 1\}$ such that:

$\theta_3^* = \max|\theta_3|$, $\theta_4^* = \max|\theta_4|$, and $\gamma^* = \max|\gamma_j|$.

(H4) $\Psi(x)$ is bounded function for all x in $J=[a,b]$.

Theorem 3.1. *Assume that (H1)–(H4) hold. If*

$$0<\psi=(\alpha\theta_1+k\gamma^*\theta_3^*)(b-a)<1.$$

Then there exists a unique solution $Z(x)\in C(J)$ to IVB (1.1) − (1.2).

Proof.

Let Z_1 and Z_2 be two different solutions of IVB (1.1) − (1.2), then

$$\left|Z_1-Z_2\right|=\left|\int_a^b\frac{\gamma(x-t)^kK(x,t)}{\xi_k(x)k!}[G(Z_1)-G(Z_2))]dt\right.$$

$$\left.-\sum_{j=0}^{k-1}\int_a^b\frac{(x-t)^{k-1}\xi_j(t)}{\xi_k(t)(k-1)!}[D^j(Z_1)-D^j(Z_2))]dt\right|$$

$$\leq\int_a^b\left|\frac{\gamma(x-t)^kK(x,t)}{\xi_k(x)k!}\right|\left|G(Z_1)-G(Z_2))\right|dt-$$

$$\sum_{j=0}^{k-1}\int_a^b\left|\frac{(x-t)^{k-1}\xi_j(t)}{\xi_k(t)(k-1)!}\right|\left|D^j(Z_1)-D^j(Z_2))\right|dt$$

$$\leq(\alpha\theta_1+k\gamma^*\theta_3^*)(b-a)\,|Z_1-Z_2|,$$

we get $(1-\psi)|Z_1-Z_2|\leq 0$. Since $0<\psi<1$, so $|Z_1-Z_2|=0$. therefore, $Z_1=Z_2$ and the proof is completed.

Theorem 3.2. *If problem* (1.1) − (1.2) *has a unique solution, then the solution* $Z_n(x)$ *obtained from the recursive relation* (2.4) *using VIM converges when*

$$0<\phi=(\alpha\theta_2+k\gamma^*\theta_4^*)(b-a)<1.$$

Proof. We have from equation (2.4):

$$Z_{n+1}(x)-Z(x)=Z_n(x)-Z(x)-$$

$$-\Big(L^{-1}\Big[\sum_{j=0}^{k}\xi_j(x)[Z_n^{(j)}(x)-Z^{(j)}(x)]\Big]-$$

$$-L^{-1}\Big[\gamma\int_a^bK(x,t)[G(Z_n(t))-G(Z(t))]dt\Big]\Big).$$

If we set, $\xi_k(x)=1$, and $W_{n+1}(x)=Z_{n+1}(x)-Z(x)$, $W_n(x)=Z_n(x)-Z(x)$ since $W_n(a)=0$, then

$$W_{n+1}(x)=W_n(x)+\int_a^b\frac{\gamma K(x,t)(x-t)^k}{k!}\big[G(Z_n(t))-G(Z(t))\big]dt$$

$$-\sum_{j=0}^{k-1}\int_a^b\frac{\lambda_1\xi_j(t)(x-t)^{k-1}}{(k-1)!}\big[D^j(Z_n(t))-D^j(Z(t))\big]dt$$

$$-(W_n(x)-W_n(a)).$$

Therefore,

$$|W_{n+1}(x)|\leq\int_a^b\left|\frac{\gamma K(x,t)(x-t)^k}{k!}\right||W_n|\,\alpha dt$$

$$+\sum_{j=0}^{k-1}\int_a^b\left|\frac{\gamma\xi_j(t)(x-t)^{k-1}}{(k-1)!}\right|\max|\gamma_j|\,\|W_n|\,dt$$

$$\leq|W_n|\Big[\int_a^b\alpha\theta_2dt+\sum_{j=0}^{k-1}\int_a^b\theta_4^*\,|\max|\gamma_j|$$

$$\leq|W_n|(\alpha\theta_2+k\gamma^*\theta_4^*)(b-a)=|W_n|\,\phi.$$

Hence,

$$\|W_{n+1}\|=\max_{\forall x\in J}|W_{n+1}(x)|$$

$$\leq\phi\max_{\forall x\in J}|W_n(x)|=\phi\,\|W_n\|.$$

Since $0<\phi<1$, then $\|W_n\|\to 0$. So, the series converges and the proof is complete.

4. Numerical Results

In this section, we present the semi-analytical technique based on the VIM to solve Fredholm integro-differential equations. To show the efficiency of the present method for our problems in comparison with the exact solution, we report absolute error.

Example 1. Consider the following Fredholm integro-differential equation.

$$Z'(x)=e^x-x+xe^x+\int_0^1xZ(s)ds,$$

with the initial condition

$$Z(0)=0,$$

and the exact solution is $Z(x)=xe^x$.

The numerical comparison has been shown in table 1 for example 1.

Table 1. Numerical Results of the Example 1.

x	Exact solution	VIM	Er VIM
0.1	0.1105170918	0.1105170888	3.00×10^{-9}
0.2	0.2442805516	0.2442805397	1.19×10^{-8}
0.3	0.4049576424	0.4049576156	2.68×10^{-8}
0.4	0.5967298792	0.5967298316	4.76×10^{-8}
0.5	0.8243606355	0.8243605611	7.44×10^{-8}
0.6	1.0932712800	1.0932711730	1.07×10^{-7}
0.7	1.4096268950	1.4096267490	1.46×10^{-7}
0.8	1.7804327420	1.7804325510	1.91×10^{-7}
0.9	2.2136428000	2.2136425590	2.41×10^{-7}
1.0	2.7182818280	2.7182815300	2.98×10^{-7}

Example 2. Consider the following Fredholm integro-differential equation.

$$Z'(x)=1-\frac{x}{3}+\int_0^1xsZ(s)ds,$$

with the initial condition

$$Z(0) = 0,$$

and the exact solution is

$$Z(x) = x\,.$$

The numerical comparison has been shown in table 2 for example 2.

Table 2. Numerical Results of the Example 2.

x	Exact solution	VIM	Er $_{VIM}$
0.1	0.1000000000	0.0999999205	7.950×10^{-8}
0.2	0.2000000000	0.1999996821	3.179×10^{-7}
0.3	0.3000000000	0.2999992847	7.153×10^{-7}
0.4	0.4000000000	0.3999987284	1.272×10^{-6}
0.5	0.5000000000	0.4999980132	1.987×10^{-6}
0.6	0.6000000000	0.5999971390	2.861×10^{-6}
0.7	0.7000000000	0.6999961058	3.894×10^{-6}
0.8	0.8000000000	0.7999949137	5.086×10^{-6}
0.9	0.9000000000	0.8999935627	6.437×10^{-6}
1.0	1.0000000000	0.9999920527	7.947×10^{-6}

5. Conclusion

We present the variational iteration method for solving Fredholm integro-differential equations. From the computational viewpoint, the variational iteration method is more efficient, convenient and easy to use. The method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear Fredholm integro-differential equations. Moreover, we proved the existence and uniqueness results and convergence of the solution. The convergence theorem and the numerical results establish the precision and efficiency of the proposed technique.

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