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Solution of Elasticity Problems in Two Dimensional Polar Coordinates using Mellin Transform

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ABSTRACT

In this work, the Mellin transform method was used to obtain solutions for the stress field components in two dimensional (2D) elasticity problems in terms of plane polar coordinates. the Mellin transformation was applied to the biharmonic stress compatibility equation expressed in terms of the Airy stress potential function, and the boundary value problem transformed to an algebraic problem which was solved to obtain the Mellin transformed Airy stress potential function. The Mellin transform was similarly used to obtain the Mellin transformed stress field components. The use of Mellin transform inversion formula yielded the solutions to the 2D elasticity problem in the physical space domain variables. Specific illustration was considered of the solution by using the Mellin transform method for the Flamant problem and the Mellin transform solutions found to agree with solutions from the literature.

1. Introduction

The governing equations of the theory of elasticity for three dimensional (3D) problems are very complicated, regardless of the reference coordinate system used. Thus, analytical solutions to 3D elasticity problems are very difficult to obtain. Hence, some 3D problems are sometimes simplified by introducing two dimensionality assumptions [1 – 4].

The fundamental two dimensional (2D) problems in the theory of elasticity are formulated using the theories of plane strain, and plane stress elasticity. The simplification of 3D problems to 2D problems permit the solution of many problems in elasticity theory.

Generally, 2D elasticity problems are governed by the requirements of stress – strain laws (material constitutive behaviour), the strain – displacement relations (kinematic relations) and the differential equations of equilibrium subject to the boundary conditions [5 – 7].

The governing equations can be presented using displacement formulation, stress formulation or mixed formulation methods. In a displacement formulation of the 2D elasticity problem, the displacement components are the primary unknowns, and the system of governing equations expressed in terms of displacement components [8 – 11]. In a stress formulation, the governing equations of 2D elasticity are expressed in terms of stress components which are the unknowns [8 – 11].

In mixed formulation, the governing equations are expressed in terms of some stress components and some displacement components as the primary unknowns [8 – 11]. This study adopted the stress – based formulation.

Specifically, the Airy stress function expressed in 2D polar coordinates (r, θ) was used in this study to express the 2D elasticity formulation in terms of a single unknown function of the space coordinate variables. The Airy stress function formulation/approach is based on the general philosophy of developing a solution to the partial differential equations of equilibrium for the unknown stress fields and thus seeking to obtain a single governing partial differential equation for the 2D elasticity problem from the equations of compatibility.

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1.1 Research aim and objectives

The main aim of this study is to use the Mellin transform method to obtain solutions to the 2D problem of the theory of elasticity in plane polar coordinates. The specific objectives are:

(i) to present a stress formulation of the problem of 2D elasticity in plane polar coordinate system.
(ii) to present Airy stress functions in the 2D polar coordinate system as the solutions of the 2D elasticity problem provided the stress functions are harmonic.
(iii) to present the stress – compatibility equations for 2D elasticity problems.
(iv) to apply the Mellin transform to the stress compatibility equation, and obtain solutions for the Airy stress functions in the Mellin transform space.
(v) to apply Mellin transform to the stress field components and obtain the stress field components in Mellin transform space.
(vi) to apply the Mellin transform inversion formula to the stress field components to obtain the solutions to the stress field components in terms of the physical domain space variables.
(vii) to apply the Mellin transform method to the specific case of the solution of the Flamant problem.

2.0 Theoretical framework

The differential equations of equilibrium for 3D elasticity problems in r, θ, z cylindrical polar coordinates system are:

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\partial\tau_{rz}}{\partial z}+f_r=0 \quad (1)$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{2\tau_{r\theta}}{r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\partial\tau_{\theta z}}{\partial z}+f_\theta=0 \quad (2)$$

$$\frac{\partial\tau_{zr}}{\partial r}+\frac{\tau_{zr}}{r}+\frac{1}{r}\frac{\partial\tau_{z\theta}}{\partial\theta}+\frac{\partial\sigma_{zz}}{\partial z}+f_z=0 \quad (3)$$

where $\tau_{r\theta}=\tau_{\theta r}$ (4)

$$\tau_{z\theta}=\tau_{\theta z} \quad (5)$$

$$\tau_{rz}=\tau_{zr} \quad (6)$$

where f_r, f_θ and f_z are body forces in the r, θ and z coordinate directions, σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} are normal stresses, $\tau_{r\theta}$, $\tau_{z\theta}$ and τ_{rz} are shear stresses.

The stresses at any point in a body for a two dimensional (2D) elasticity problem are uniquely defined by the three stress components σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ where σ_{rr} is the radial normal stress, $\sigma_{\theta\theta}$ is the circumferential normal stress, and $\sigma_{r\theta}$ ($\tau_{r\theta}$) is the shear stress on the $r\theta$ plane. 2D elasticity problems are governed by the three simultaneous requirements of the differential equations of equilibrium, the stress – strain laws and the strain – displacement relations. The governing equations are given by:

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+f_r=0 \quad (7)$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{2\tau_{r\theta}}{r}+f_\theta=0 \quad (8)$$

which are the differential equations of equilibrium when body forces f_r and f_θ are present. The stress – strain equations are given by:

$$\sigma_{rr}=\frac{E}{1-\mu^2}(\varepsilon_{rr}+\mu\varepsilon_{\theta\theta}) \quad (9)$$

$$\sigma_{\theta\theta}=\frac{E}{1-\mu^2}(\varepsilon_{\theta\theta}+\mu\varepsilon_{rr}) \quad (10)$$

$$\tau_{r\theta}=\sigma_{r\theta}=G\gamma_{r\theta} \quad (11)$$

where ε_{rr} is the radial strain, $\gamma_{r\theta}$ is the shear strain, $\varepsilon_{\theta\theta}$ is the tangential (circumferential) strain, μ is the Poisson's ratio and E is the Young's modulus of elasticity, G is the shear modulus.

Hooke's law for plane stress conditions is given by:

$$\varepsilon_{rr}=\frac{1}{E}(\sigma_{rr}-\mu\sigma_{\theta\theta}) \quad (12)$$

$$\varepsilon_{\theta\theta}=\frac{1}{E}(\sigma_{\theta\theta}-\mu\sigma_{rr}) \quad (13)$$

$$\varepsilon_{zz}=\frac{-\mu}{E}(\sigma_{rr}+\sigma_{\theta\theta}) \quad (14)$$

$$\varepsilon_{r\theta}=\frac{1+\mu}{E}\tau_{r\theta} \quad (15)$$

Hooke's law for plane strain is:

$$\varepsilon_{rr}=\frac{1+\mu}{E}((1-\mu)\sigma_{rr}-\mu\sigma_{\theta\theta}) \quad (16)$$

$$\varepsilon_{\theta\theta}=\frac{1+\mu}{E}(-\mu\sigma_{rr}+(1-\mu)\sigma_{\theta\theta}) \quad (17)$$

$$\varepsilon_{r\theta}=\frac{1+\mu}{E}\tau_{r\theta} \quad (18)$$

The strain – displacement equations are given by:

$$\varepsilon_{rr}=\frac{\partial u}{\partial r}=\frac{du_r}{dr} \quad (19)$$

$$\varepsilon_{\theta\theta}=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} \quad (20)$$

where $u=u_r$ is the radial displacement, u_θ is the displacement in the θ direction

$$\gamma_{r\theta}=2\varepsilon_{r\theta}=\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r} \quad (21)$$

Airy solved the 2D elasticity problems in plane polar coordinates in a stress-based formulation by finding Airy stress potential (harmonic) functions $\varphi(r,\theta)$ of the space coordinates (r, θ) that identically satisfied the differential equations of equilibrium. The stress fields that solve any 2D elasticity problem in 2D polar coordinates become derivable from the Airy stress potential functions as follows:

$$\sigma_{rr}=\frac{1}{r}\frac{\partial\varphi}{\partial r}(r,\theta)+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}(r,\theta) \quad (22)$$

$$\sigma_{\theta\theta}=\frac{\partial^2\varphi}{\partial r^2}(r,\theta) \quad (23)$$

$$\sigma_{r\theta} = \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}(r,\theta)\right) = \frac{1}{r^2}\frac{\partial \varphi(r,\theta)}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \varphi(r,\theta)}{\partial r \partial \theta} \quad \ldots(24)$$

2.1 Compatibility equations

The compatibility equation expressed in terms of strain for the case of polar coordinates is

$$\frac{1}{r^2}\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} + r\frac{\partial^2 \varepsilon_{\theta\theta}}{\partial r^2} - \frac{2}{r}\frac{\partial^2 \varepsilon_{r\theta}}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{2}{r}\frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{2}{r^2}\frac{\partial \varepsilon_{r\theta}}{\partial \theta} = 0 \quad \ldots(25)$$

The compatibility equations in 2D elasticity problems are expressed in terms of Airy stress potential functions $\varphi(r,\theta)$ as:

$$\nabla^2\nabla^2\varphi(r,\theta) = \nabla^4\varphi(r,\theta) = 0 \quad (26)$$

where $$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad (27)$$

and $$\nabla^4 = \nabla^2\nabla^2 \quad (28)$$

∇^2 is the Laplacian operator while ∇^4 is the biharmonic operator.

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 = \left(\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r} + \frac{2}{r^2}\frac{\partial^4}{\partial r^2 \partial \theta^2} - \frac{2}{r^3}\frac{\partial^3}{\partial r \partial \theta^2} + \frac{4}{r^4}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^4}\frac{\partial^4}{\partial \theta^4}\right) \quad (29)$$

Thus the compatibility equation is given by:

$$\left(\frac{\partial^4 \varphi}{\partial r^4} + \frac{2}{r}\frac{\partial^3 \varphi}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r^3}\frac{\partial \varphi}{\partial r} + \frac{2}{r^2}\frac{\partial^4 \varphi}{\partial r^2 \partial \theta^2} - \frac{2}{r^3}\frac{\partial^3 \varphi}{\partial r \partial \theta^2} + \frac{4}{r^4}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{1}{r^4}\frac{\partial^4 \varphi}{\partial \theta^4}\right) = 0 \quad (30)$$

3.0 Research methodology

The Mellin transform is used to solve the biharmonic stress compatibility equation which is given in terms of the Airy stress potential function. The solution is sought subject to the boundary conditions of the particular 2D elasticity problem. The Mellin transformation of the Airy stress potential function $\varphi(r,\theta)$ is an integral transformation denoted by $\bar{\varphi}(k,\theta)$ and defined as [12 – 14]

$$\bar{\varphi}(k,\theta) = \int_{r=0}^{r=\infty} r^{k-1}\varphi(r,\theta)\,dr \quad (31)$$

where k is the parameter of the Mellin transform or the Mellin transform parameter, and r^{k-1} is the kernel (or nucleus) of the Mellin transform.

Similarly, the Mellin transform of the n^{th} derivatives of the Airy stress potential function are given by the integral transform:

$$\int_{r=0}^{r=\infty} r^{k-1}r^n\frac{\partial^n \varphi}{\partial r^n}(r,\theta)\,dr = \int_0^\infty r^{k+n-1}\frac{\partial^n \varphi(r,\theta)}{\partial r^n}\,dr$$

$$= \frac{(-1)^n\Gamma(k+n)}{\Gamma(k)}\int_0^\infty r^{k-1}\varphi(r,\theta)\,dr$$

$$= \frac{(-1)^n\Gamma(k+n)}{\Gamma(k)}\bar{\varphi}(k,\theta) \quad (32)$$

For $n = 1, 2, 3, 4, \ldots$

where $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) = (\alpha-1)!$ (33)

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ (34)

$\Gamma(x)$ is the gamma function, defined as:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-x}dt; \quad x > 0 \quad (35)$$

From Equation (32) we obtain the Mellin transforms of the first, second, third and fourth partial derivatives of $\varphi(r,\theta)$ with respect to r as follows:

$$\int_0^\infty r^{k-1}r\frac{\partial \varphi}{\partial r}(r,\theta)\,dr = \int_0^\infty r^k\frac{\partial \varphi}{\partial r}\,dr = \frac{(-1)^1\Gamma(k+1)\bar{\varphi}}{\Gamma(k)}$$

$$= \frac{(-1)(k)!\bar{\varphi}}{(k-1)!} = -k\bar{\varphi}(k,\theta) \quad (36)$$

$$\int_0^\infty r^{k-1}r^2\frac{\partial^2 \varphi}{\partial r^2}(r,\theta)\,dr = \int_0^\infty r^{k+1}\frac{\partial^2 \varphi}{\partial r^2}(r,\theta)\,dr$$

$$= \frac{(-1)^2\Gamma(k+2)}{\Gamma(k)}\bar{\varphi}(k,\theta) \quad (37)$$

$$\int_0^\infty r^{k+1}\frac{\partial^2 \varphi}{\partial r^2}(r,\theta)\,dr = \frac{(k+1)!}{(k-1)!}\bar{\varphi}(k,\theta) = k(k+1)\bar{\varphi}(k,\theta)$$

$$= \frac{(k+1)(k)(k-1)!}{(k-1)!}\bar{\varphi}(k,\theta) \quad (38)$$

$$\int_0^\infty r^{k-1}r^3\frac{\partial^3 \varphi}{\partial r^3}(r,\theta)\,dr = \int_0^\infty r^{k+2}\frac{\partial^3 \varphi}{\partial r^3}(r,\theta)\,dr$$

$$= \frac{(-1)^3\Gamma(k+3)}{\Gamma(k)}\bar{\varphi}(k,\theta)$$

$$= \frac{(-1)(k+2)!}{(k-1)!}\bar{\varphi}(k,\theta)$$

$$= \frac{(-1)(k+2)(k+1)k(k-1)!\bar{\varphi}(k,\theta)}{(k-1)!}$$

$$= (-1)k(k+1)(k+2)\bar{\varphi}(k,\theta) \quad (39)$$

$$\int_0^\infty r^{k-1}r^4\frac{\partial^4 \varphi}{\partial r^4}(r,\theta)\,dr = \int_0^\infty r^{k+3}\frac{\partial^4 \varphi}{\partial r^4}(r,\theta)\,dr$$

$$= \frac{(-1)^4\Gamma(k+4)}{\Gamma(k)}\bar{\varphi}(k,\theta) = \frac{(k+3)!}{(k-1)!}\bar{\varphi}(k,\theta) \quad (40)$$

$$\frac{(k+3)!}{(k-1)!}\bar{\varphi}(k,\theta) = \frac{(k+3)(k+2)(k+1)k(k-1)!\bar{\varphi}(k,\theta)}{(k-1)!}$$

$$= k(k+1)(k+2)(k+3)\bar{\varphi}(k,\theta) \quad (41)$$

3.1 Advantages of the Mellin transform

The Mellin transform is chosen as the tool for this research because of the obvious simplifications it offers in dealing with problems described using the Laplacians in the cylindrical or spherical coordinates given as:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \quad (42)$$

or $$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \quad (43)$$

The simplifications arise because the differential given as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} = f_{rr} + \frac{1}{r}f_r \quad (44)$$

which is part of the Laplacian is readily transformed as:

$$\int_{r=0}^{r=\infty} r^{k-1}\left(f_{rr} + \frac{1}{r}f_r\right)r^2 dr = k^2 F(k) \quad (45)$$

where $$F(k) = \int_{r=0}^{\infty} r^{k-1} f(r)\, dr \quad (46)$$

$F(k)$ is the Mellin transform of $f(r, \theta)$.

Another advantage is that the Mellin transform is a linear operator and can be readily applied to linear differential equations.

4.0 Results

4.1 Application of the Mellin transform

The Mellin transform is applied to both sides of the stress compatibility equation to obtain:

$$\int_0^{\infty} (\nabla^4 \varphi(r,\theta)) r^{k-1} dr = 0 \quad (47)$$

The Mellin transformation is expressed as:

$$\int_0^{\infty}\left(r^4\frac{\partial^4\varphi}{\partial r^4} + 2r^3\frac{\partial^3\varphi}{\partial r^3} - r^2\frac{\partial^2\varphi}{\partial r^2} + r\frac{\partial\varphi}{\partial r} + 2r^2\frac{\partial^4\varphi}{\partial r^2\partial\theta^2} - 2r\frac{\partial^3\varphi}{\partial r\partial\theta^2} + 4\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^4\varphi}{\partial\theta^4}\right) r^{k-1} dr = 0 \quad (48)$$

The linearity property of the Mellin transformation is used to simplify the Mellin transformation of the stress compatibility equation to the ordinary differential equation (ODE) in terms of the Mellin transform of the Airy stress potential function $\bar{\varphi}(k,\theta)$ given by:

$$[k(k+1)(k+2)(k+3) - 2k(k+1)(k+2) - k(k+1) - k]\bar{\varphi}(k,\theta) + 2k(k+1)\frac{d^2\bar{\varphi}}{d\theta^2}(k,\theta) + 2k\frac{d^2\bar{\varphi}}{d\theta^2}(k,\theta) + 4\frac{d^2\bar{\varphi}(k,\theta)}{d\theta^2} + \frac{d^4\bar{\varphi}(k,\theta)}{d\theta^4} = 0 \quad (49)$$

Simplifying,

$$\frac{d^4\bar{\varphi}(k,\theta)}{d\theta^4} + (4 + 2k + 2k^2 + 2k)\frac{d^2\bar{\varphi}(k,\theta)}{d\theta^2} + (k(k+1)(k+2)(k+3) - 2k(k+1)(k+2) - k(k+1) - k)\bar{\varphi}(k,\theta) = 0 \quad \ldots(50)$$

Further simplification yields:

$$\frac{d^4\bar{\varphi}}{d\theta^4}(k,\theta) + (k^2 + (k+2)^2)\frac{d^2\bar{\varphi}}{d\theta^2}(k,\theta) + k^2(k+2)^2\bar{\varphi}(k,\theta) = \left(\frac{d^2}{d\theta^2} + k^2\right)\times$$

$$\left(\frac{d^2}{d\theta^2} + (k+2)^2\right)\bar{\varphi}(k,\theta) = 0 \quad (51)$$

This fourth order ODE is solved using the method of trial functions or D operator methods.

Let $$\bar{\varphi}(k,\theta) = \exp m\theta \quad (52)$$

where $m(k)$ is a parameter we seek to find such that the assumed exponential form for $\bar{\varphi}(k,\theta)$ can be a solution.

Then, the Equation (50) becomes:

$$m^4 e^{m\theta} + (k^2 + (k+2)^2)m^2 e^{m\theta} + k^2(k+2)^2 e^{m\theta} = 0 \quad \ldots(53)$$

Simplifying,

$$(m^4 + (k^2 + (k+2)^2)m^2 + k^2(k+2)^2)e^{m\theta} = 0 \quad (54)$$

For non-trivial solutions,

$$e^{m\theta} \neq 0$$

The auxiliary (characteristic) polynomial is thus obtained as the fourth order polynomial:

$$m^4 + (k^2 + (k+2)^2)m^2 + k^2(k+2)^2 = (m^2 + k^2)(m^2 + (k+2)^2) = 0 \quad \ldots(55)$$

The roots of the characteristic polynomial yield the general solution to the Airy stress potential function in the Mellin transform space as:

$$\bar{\varphi}(k,\theta) = c_1 \sin k\theta + c_2 \cos k\theta + c_3 \sin(k+2)\theta + c_4 \cos(k+2)\theta \quad \ldots(56)$$

where c_1, c_2, c_3, and c_4 are four constants of integration which are determined from the boundary conditions of the specific 2D elasticity problem.

4.2 Stress fields in the Mellin transform space

The stress fields are obtained in the Mellin transform space by applying the Mellin transformation to the equations of the Airy stress potential functions – Equations (22) – (24).

Thus,

$$\int_0^{\infty}(r^2\sigma_{rr})r^{k-1}dr = \int_0^{\infty}\left(\frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right)r^2 r^{k-1}dr \quad (57)$$

$$\int_0^{\infty}(\sigma_{rr}r^2)r^{k-1}dr = \int_0^{\infty}\frac{1}{r}\frac{\partial\varphi}{\partial r}r^2 r^{k-1}dr + \int_0^{\infty}\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}r^2 r^{k-1}dr \quad \ldots(58)$$

Simplifying,

$$\int_0^{\infty}(\sigma_{rr}r^2)r^{k-1}dr = \int_0^{\infty}\frac{\partial\varphi}{\partial r}r r^{k-1}\,dr + \int_0^{\infty}\frac{\partial^2\varphi}{\partial\theta^2}r^{k-1}dr \quad (59)$$

$$\int_0^{\infty}\sigma_{rr}r^2 r^{k-1}dr = \int_0^{\infty}\frac{\partial\varphi}{\partial r}r r^{k-1}\,dr + \frac{\partial^2}{\partial\theta^2}\int_0^{\infty}\varphi r^{k-1}dr \quad (60)$$

$$\int_0^{\infty}\sigma_{rr}r^2 r^{k-1}dr = \frac{d^2\bar{\varphi}}{d\theta^2} - k\bar{\varphi} \quad (61)$$

Similarly, from Equation (9),

$$\int_0^{\infty}\sigma_{\theta\theta}r^2 r^{k-1}dr = \int_0^{\infty}\frac{\partial^2\varphi}{\partial r^2}r^2 r^{k-1}dr \quad (62)$$

$$= \frac{(-1)^2\Gamma(k+2)\bar{\varphi}}{\Gamma(k)} \quad (63)$$

$$= \frac{(-1)^2(k+1)!}{(k-1)!}\overline{\varphi} = k(k+1)\overline{\varphi} \quad (64)$$

Using Equation (23),

$$\int_0^\infty \sigma_{r\theta} r^2 r^{k-1} dr = \int_0^\infty -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) r^2 r^{k-1} dr \quad (65)$$

$$\int_0^\infty \sigma_{r\theta} r^2 r^{k-1} dr = \int_0^\infty \left(\frac{\partial\varphi}{\partial\theta} - r\frac{\partial^2\varphi}{\partial r\partial\theta}\right) r^{k-1} dr \quad (66)$$

$$\int_0^\infty \sigma_{r\theta} r^2 r^{k-1} dr = \int_0^\infty \frac{\partial\varphi}{\partial\theta} r^{k-1} dr - \frac{\partial}{\partial\theta}\int_0^\infty r\frac{\partial\varphi}{\partial r} r^{k-1} dr \quad (67)$$

$$\int_0^\infty \sigma_{r\theta} r^2 r^{k-1} dr = \frac{\partial}{\partial\theta}\int_0^\infty \varphi r^{k-1} dr - \frac{\partial}{\partial\theta}\int_0^\infty r\frac{\partial\varphi}{\partial r} r^{k-1} dr \quad (68)$$

$$\int_0^\infty \sigma_{r\theta} r^2 r^{k-1} dr = (k+1)\frac{d\overline{\varphi}}{d\theta} \quad (69)$$

4.3 Stress fields in the physical domain variables

The stress fields in the physical domain space variables (r, θ) of the problem are obtained by application of the inverse Mellin transformation to the Equations (61), (64) and (69). Thus, by inversion, the stresses are given by the line integrals:

$$\sigma_{rr}(r,\theta) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(-k\overline{\varphi} + \frac{d^2\overline{\varphi}}{d\theta^2}\right) r^{-k-2} dk \quad (70)$$

$$\sigma_{\theta\theta}(r,\theta) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} k(k+1)\overline{\varphi}(k,\theta) r^{-k-2} dk \quad (71)$$

$$\sigma_{r\theta}(r,\theta) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} (k+1)\frac{d\overline{\varphi}}{d\theta}(k,\theta) r^{-k-2} dk \quad (72)$$

where i is the imaginary number in complex analysis, $i = \sqrt{-1}$, a is a constant.

5. Specific applications of the Mellin transform method to the Flamant problem

5.1 The Flamant problem

The Flamant problem which is encountered in elasticity is the 2D elasticity problem of finding the stress fields in a semi-infinite body due to line loads of magnitudes (intensities) Q_1 and Q_2 and of infinite extent applied at the origin O (on the xz plane) on the boundary of an elastic half plane $(-\infty \le x \le \infty,\ 0 \le z \le \infty)$ as shown in Figure 1.

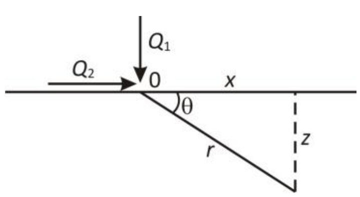

Figure 1: Flamant problem (line loads Q_1 and Q_2 of infinite extent applied at the origin O of an elastic half plane)

Airy stress formulation of the Flamant problem is given by Equations (22 – 24) and the biharmonic stress compatibility equation – Equation (26). The boundary conditions are:

$$\tau_{r\theta}(r,\theta=0) = 0 = \tau_{r\theta}(r,\theta=\pi) \quad (73)$$

$$\sigma_{\theta\theta}(r,\theta=0) = \sigma_{\theta\theta}(r,\theta=\pi) = 0 \quad (74)$$

The Mellin transformation of the biharmonic stress compatibility equation was performed to transform the governing biharmonic equation to a fourth order ODE which was solved to obtain the Airy stress function in the Mellin transform space as Equation (56).

By inversion, the Airy stress function is obtained in the physical domain space variables as:

$$\varphi(r,\theta) = \overline{c}_1 r\theta\sin\theta + \overline{c}_2 r\ln r\cos\theta + \overline{c}_3 r\theta\cos\theta + \overline{c}_4 r\ln r\sin\theta \quad (75)$$

where $\overline{c}_1, \overline{c}_2, \overline{c}_3, \overline{c}_4$ are constants that are obtained by the enforcement of stress boundary conditions.

Stress fields

The stresses are obtained from Equations (22 – 24) as follows:

$$\sigma_{rr}(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}(\overline{c}_1 r\theta\sin\theta + \overline{c}_2 r\ln r\cos\theta + \overline{c}_3 r\theta\cos\theta + \overline{c}_4 r\ln r\sin\theta) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(\overline{c}_1 r\theta\sin\theta + \overline{c}_2 r\ln r\cos\theta + \overline{c}_3 r\theta\cos\theta + \overline{c}_4 r\ln r\sin\theta) \quad (76)$$

$$\sigma_{rr}(r,\theta) = \overline{c}_1\frac{2\cos\theta}{r} + \overline{c}_2\frac{\cos\theta}{r} + \overline{c}_3\frac{2\sin\theta}{r} + \overline{c}_4\frac{\sin\theta}{r} \quad \ldots(77)$$

$$\tau_{r\theta}(r,\theta) = \frac{-\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}(\overline{c}_1 r\theta\sin\theta + \overline{c}_2 r\ln r\cos\theta + \overline{c}_3 r\theta\cos\theta + \overline{c}_4 r\ln r\sin\theta)\right) \quad (78)$$

$$\tau_{r\theta}(r,\theta) = \overline{c}_2\frac{\sin\theta}{r} + \overline{c}_4\left(\frac{-\cos\theta}{r}\right) \quad (79)$$

$$\sigma_{\theta\theta}(r,\theta) = \frac{\partial^2}{\partial r^2}(\overline{c}_1 r\theta\sin\theta + \overline{c}_2 r\ln r\cos\theta + \overline{c}_3 r\theta\cos\theta + \overline{c}_4 r\ln r\sin\theta) \quad (80)$$

$$\sigma_{\theta\theta}(r,\theta) = \overline{c}_2\left(\frac{\cos\theta}{r}\right) + \overline{c}_4\left(\frac{\sin\theta}{r}\right) \quad (81)$$

Enforcement of boundary conditions

Using the boundary conditions Equations (73) and (74) we have:

$$\tau_{r\theta}(r,\theta=0) = -\frac{\overline{c}_4}{r} = 0 \quad (82)$$

Hence, $\overline{c}_4 = 0$ (83)

$$\sigma_{\theta\theta}(r,\theta=0) = \frac{\overline{c}_2}{r} = 0 \quad (84)$$

$$\overline{c}_2 = 0 \quad (85)$$

Then, we have

$$\sigma_{\theta\theta}(r,\theta) = 0 \quad (86)$$

$$\tau_{r\theta}(r,\theta) = 0 \quad (87)$$

$$\sigma_{rr}(r,\theta) = \frac{2\bar{c}_1}{r}\cos\theta + \frac{2\bar{c}_3}{r}\sin\theta \quad (88)$$

Equilibrium equations

The equilibrium equations are:

$$\sum F_v = Q_1 + \int_0^{\pi} (\sigma_{rr}(r,\theta)\sin\theta + \tau_{r\theta}(r,\theta)\cos\theta) r d\theta = 0 \quad \ldots(89)$$

for the vertical direction.

$$\sum F_h = Q_2 + \int_0^{\pi} (\sigma_{rr}(r,\theta)\cos\theta - \tau_{r\theta}(r,\theta)\sin\theta) r d\theta = 0 \quad \ldots(90)$$

for the horizontal direction.

$$\sum M = \int_0^{\pi} r\tau_{r\theta}(r,\theta) r d\theta = 0 \quad (91)$$

Since $\tau_{r\theta}(r,\theta) = 0$ everywhere in the elastic half plane the equations of equilibrium become:

$$Q_1 + \int_0^{\pi} \sigma_{rr}(r,\theta)\sin\theta\, r d\theta = 0 \quad (92)$$

$$Q_2 + \int_0^{\pi} \sigma_{rr}(r,\theta)\cos\theta\, r d\theta = 0 \quad (93)$$

Hence,

$$Q_1 + \int_0^{\pi} \left(\frac{2\bar{c}_1\cos\theta}{r} + \frac{2\bar{c}_3\sin\theta}{r}\right)\sin\theta\, r d\theta = 0 \quad (94)$$

$$Q_2 + \int_0^{\pi} \left(\frac{2\bar{c}_1\cos\theta}{r} + \frac{2\bar{c}_3\sin\theta}{r}\right)\cos\theta\, r d\theta = 0 \quad (95)$$

$$Q_1 + 2\int_0^{\pi} (\bar{c}_1\cos\theta + \bar{c}_3\sin\theta)\sin\theta\, d\theta = 0 \quad (96)$$

$$Q_2 + 2\int_0^{\pi} (\bar{c}_1\cos\theta + \bar{c}_3\sin\theta)\cos\theta\, d\theta = 0 \quad (97)$$

$$Q_1 + \bar{c}_3\pi = 0 \quad (98)$$

$$Q_2 + \bar{c}_1\pi = 0 \quad (99)$$

Solving, $c_1 = -\dfrac{Q_2}{\pi}$ (100)

$$\bar{c}_3 = -\frac{Q_1}{\pi} \quad (101)$$

Hence,

$$\varphi(r,\theta) = -\frac{Q_2}{\pi} r\theta\sin\theta - \frac{Q_1}{\pi} r\theta\cos\theta \quad (102)$$

$$\sigma_{rr}(r,\theta) = -\frac{Q_2}{\pi}\frac{2\cos\theta}{r} - \frac{Q_1}{\pi}\frac{2\sin\theta}{r} \quad (103)$$

$$\sigma_{rr}(r,\theta) = -\frac{-2}{\pi r}(Q_2\cos\theta + Q_1\sin\theta) \quad (104)$$

The solutions can be decomposed into two: namely solution for Q_1 acting alone, and solution for Q_2 acting alone. The problem of Q_1 acting alone on the surface of a half plane is more frequently encountered.

For Q_2 applied alone,

$$\varphi(r.\theta) = -\frac{Q_2}{\pi} r\theta\sin\theta \quad (105)$$

$$\sigma_{rr}(r,\theta) = \frac{-2}{\pi r} Q_2\cos\theta \quad (106)$$

$$\tau_{r\theta}(r,\theta) = \sigma_{\theta\theta}(r,\theta) = 0 \quad (107)$$

For Q_1 applied alone,

$$\varphi(r.\theta) = -\frac{Q_1}{\pi} r\theta\cos\theta \quad (108)$$

$$\sigma_{rr}(r,\theta) = \frac{-2}{\pi r} Q_1\sin\theta \quad (109)$$

$$\tau_{r\theta}(r,\theta) = \sigma_{\theta\theta}(r,\theta) = 0 \quad (110)$$

Stress fields in Cartesian coordinates

The stress fields are obtained in Cartesian coordinates for the Flamant problem where the vertical line load Q_1 of infinite extent is applied alone.

Then,

$$\sigma_{zz}(x,z) = \sigma_{rr}\sin^2\theta = -\frac{2Q_1}{\pi r}\sin^3\theta \quad (111)$$

$$\sigma_{zz}(x,z) = \frac{-2Q_1}{\pi r}\left(\frac{z}{r}\right)^3 = \frac{-2Q_1 z^3}{\pi r^4} \quad (112)$$

$$\sigma_{zz}(x,z) = \frac{-2Q_1}{\pi z}\frac{z^4}{r^4} = \frac{-Q_1}{z}\frac{2}{\pi}\left(\frac{z}{r}\right)^4 \quad (113)$$

$$\sigma_{zz}(x,z) = \frac{-Q_1}{z}\frac{2}{\pi}\frac{z^4}{(x^2+z^2)^2} = \frac{-Q_1}{z}\frac{2}{\pi}\left(\frac{z^2}{x^2+z^2}\right)^2 \quad (114)$$

$$\sigma_{zz}(x,z) = \frac{-Q_1}{z}\frac{2}{\pi}\left(1+\left(\frac{x}{z}\right)^2\right)^{-2} = \frac{-Q_1}{z} I(x,z) \quad (115)$$

$$\sigma_{xx} = \frac{-2Q_1}{\pi z}\cos^2\theta\sin^2\theta \quad (116)$$

$$\sigma_{xx} = \frac{-2Q_1 x^2 z}{\pi(x^2+z^2)^2} = \frac{-2Q_1}{\pi z}\left(\frac{x^2}{x^2+z^2}\right)\left(\frac{z^2}{x^2+z^2}\right) \quad (177)$$

$$\sigma_{xx} = \frac{-2Q_1}{\pi z}\left(1+\frac{z^2}{x^2}\right)^{-1}\left(1+\frac{x^2}{z^2}\right)^{-1} \quad (118)$$

$$\sigma_{xx}(x,z) = \frac{Q}{z}\frac{2}{\pi}\left(1+\left(\frac{z}{x}\right)^2\right)^{-1}\left(1+\frac{x^2}{z^2}\right)^{-1} \quad (119)$$

$$\tau_{xz} = \frac{2Q_1 xz^2}{\pi(x^2+z^2)^2} \quad (120)$$

$$\sigma_{yy} = \mu(\sigma_{xx} + \sigma_{zz}) \quad (121)$$

6. Discussion

The Mellin transformation method which is an integral transform method has been used in this study to obtain general solutions for the stress fields in 2D elasticity problems. Stress based formulation of the 2D elasticity problem was adopted. This allowed the use of Airy stress functions as the stress potential functions that solved the differential equations of equilibrium. The Mellin transformation was applied to the stress compatibility equation expressed in terms of the Airy stress function, and the boundary value problem of 2D elasticity simplified to a fourth order ordinary differential equation (ODE) – (Equation (50)) – in terms of the Mellin transformed Airy stress potential function. The fourth order ODE was solved using the method of trial functions to obtain the conditions for non-trivial solutions as the characteristic (auxiliary) fourth degree polynomial-Equation (55). The Airy stress potential function was thus obtained from the four roots of the fourth degree polynomial as Equation (56) which had four unknown constants of integration corresponding to the fourth order ODE obtained as Equation (50).

The four constants of integration can be obtained for particular cases where the load is proscribed using the boundary conditions. The Mellin transformations of the stress field components were done using the Airy stress potential functions expressions for the stress fields to obtain the Mellin – transformed stress field components as Equations (61), (64) and (69). The Mellin transform inverse was then applied to Equations (61), (64) and (69) to obtain the general expressions for the stress field components in the physical space domain of the problem.

Flamant problem which is the problem of finding stresses in an elastic half plane due to line loads at the origin was used to illustrate the application of the Mellin transform to 2D elasticity problems. The Flamant problem is governed by biharmonic stress compatibility equation in terms of the Airy stress function. The solution Equation (75) – was obtained for the Airy stress function for the Flamant problem and was found by inversion of the Airy stress function in the Mellin transform space. The Airy stress function obtained is expressed in terms of four integration constants. The stresses are obtained using Equations (22 – 24) as Equations (77), (79) and (81).

Enforcement of boundary conditions yielded two constants of integration $\overline{c}_4$ and $\overline{c}_2$ as Equation (86) and (85), and the stresses $\sigma_{\theta\theta}$, $\tau_{r\theta}$ as Equations (84) and (87). The radial stress field is then found as Equation (88) which has two integration constants. The two constants are found from the requirement of equilibrium of internal and applied forces as Equations (100) and (101) and σ_{rr} is found as Equation (104).

The solution for applied vertical line load is found in terms of Cartesian coordinates as Equations (115), (119), (120) and (121). Vertical stress influence coefficients obtained for applied vertical line load on the half plane using Mellin transform in the present work are compared with solutions obtained by Das [15] and Onah et al [16] and found to be in exact agreement as shown in Table 1.

Table 1: Variation of vertical stress influence coefficients $I(x,z)$ in the Flamant problem

x/z	Das [15] $I(x,z)$	$I(x,z)$ Present work
0	0.637	0.637
0.1	0.624	0.624
0.2	0.589	0.589
0.3	0.536	0.536
0.4	0.473	0.473
0.5	0.407	0.407
0.6	0.344	0.344
0.7	0.287	0.287
0.8	0.237	0.237
0.9	0.194	0.194
1.0	0.159	0.159
1.5	0.060	0.060
2.0	0.025	0.025
3.0	0.006	0.006

7. Conclusions

The following conclusions are made:

(i) 2D elasticity problems that are formulated in terms of stresses can be solved using the Mellin transform method.

(ii) The Mellin transformation of the biharmonic stress compatibility equation in terms of the Airy stress function $\varphi(r,\theta)$ transforms the BVP to a fourth order ODE in terms of the Mellin transformed Airy stress potential function $\overline{\varphi}(k,\theta)$, and ultimately, a fourth degree auxiliary (characteristic) polynomial in $\overline{\varphi}(k,\theta)$.

(iii) Ultimately, the Mellin transformation transforms the biharmonic stress compatibility equation to an algebraic equation.

(iv) The general solution for the Airy stress potential function in the Mellin transform space variable is given in general as Equation (56) where the integration constants are determined from the boundary conditions of particular/specific 2D problems.

(v) The general solutions for the stress field components are obtained as the line integrals given by Equations (64) – (69).

(vi) The path integrals given as the general solutions contain unknown constants of integration, c_1, c_2, c_3 and c_4, which are evaluated for particular problems of 2D elasticity from the use of the appropriate boundary conditions.

(vii) The specific illustrations presented in the study show that by the Mellin transformation, the biharmonic equation in two dimensions (polar coordinates) is simplified to a fourth order ordinary differential equation in the Mellin transform space.

(viii) The solutions for the Flamant problem using the Mellin transform method are found to be identical with solutions in the literature obtained using other methods.

Nomenclature/Notations

r, θ	radial coordinate and angular coordinate in 2D polar coordinate system
$\left.\begin{matrix} r,\phi,z \\ r,q,z \end{matrix}\right\}$	cylindrical coordinate variables
σ_{rr}	radial normal stress
$\sigma_{\theta\theta}$	circumferential normal stress
$\sigma_{r\theta}(\tau_{r\theta})$	shear stress on the $r\theta$ plane
f_r, f_θ, f_z	body forces in the r, θ, and z coordinate directions
$\tau_{r\theta}$, $\tau_{z\theta}$, τ_{rz}	shear stresses in the 3D cylindrical polar coordinates
σ_{rr}, $\sigma_{\theta\theta}$, σ_{rz}	normal stresses in the 3D cylindrical polar coordinates
ε_{rr}, $\varepsilon_{\theta\theta}$, ε_{zz}	normal strains in the 3D cylindrical polar coordinates
E	Young's modulus of elasticity
G	shear modulus or modulus of rigidity
μ	Poisson's ratio
u_r, u_θ	displacements in the $r\theta(2D)$ polar coordinate system
$\varphi(r, \theta)$	Airy stress potential function
$\bar{\varphi}(r,\theta)$	Mellin transform of the Airy stress potential function
k	Mellin transform parameter
∇^2	Laplacian operator
∇^4	Biharmonic operator
$\Gamma(x)$	gamma function
!	factorial function
$\frac{\partial}{\partial\theta}, \frac{\partial}{\partial r}$	partial differential operator
$\int$	integration (integral)
i	imaginary number
∞	infinity
x,z	2D Cartesian coordinates
3D	three dimensional
2D	two dimensional
ODE	ordinary difernetial equation

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