JCAMECH

Vol. 50, No. 2, December 2019, pp 256-262

DOI: 10.22059/jcamech.2018.263984.310

Stability analysis of stratified two-phase liquid-gas flow in a horizontal pipe

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ARTICLE INFO

Article history:
Received: 18 August 2018
Accepted: 04 September 2018

Keywords:
Two phase flow
Stratified
Instability equations
Eigenvalue equations
Chebyshev polynomial

ABSTRACT

This study aimed at linear stability analysis of the stratified two-phase liquid-gas flow in a horizontal pipe. First, equations governing the linear stability of flow in each phase and boundary conditions were obtained. The governing equations were eigenvalue Orr Sommerfeld equations which are difficult and stiff problems to solve. After obtaining the velocity profiles of the gas and liquid phases in the pipe, the instability equations for each phase with related boundary conditions were coupled and simultaneously solved by using the Chebyshev Tau-QZ polynomial method. The instability spectra for some points has been plotted and some curves about instability conditions the same as neutral stability curve which shown stable and unstable region respect to Reynolds number had been drown. According to the neutral stability curve for each phase, the liquid phase is more exposed to instability than the gas phase. The liquid phase was unstable in low Reynolds numbers and a large amplitude of the wave velocity α but gas was unstable in higher Reynolds number and small amplitude of α.

1. Introduction

Pipelines are used as a common method to transfer fluids in industry. In some cases, the flow in horizontal pipes is a two phase liquid-gas fluid with a stratified flow pattern. Due to viscous flow in the pipe, the pipe flow is susceptible to instability with increasing velocity [1]. Most of prior studies were about investigation of flow instability in the channels [2]. Hydrodynamic linear theories are used to analyze flow instability. Instability analysis of interface layer between two stratified inviscid flow has been studied by Kelvin- Helmholtz [3]. Yih studied instability of two phase viscous flow in the channel [1]. Thin film of two phase flow instability analysis in the channel has been studied by Boomkamp [2] and Miesen et al [4]. Hinch [5] investigated mechanism of the instability at the interface between two shearing fluids. Flow disturbances and their growth cause instability of the flow. Flow stability is described by Orr Sommerfeld equation. For shear flows with homogeneous boundary conditions, the Orr Sommerfeld equation is of eigenvalue type. There are several methods to solve these equations including shooting methods [2]. However, the Chebyshev Tau polynomial algorithm is used since these methods require a good initial prediction of an eigenvalue [2, 6-7].

2. Geometry and governing equations for flow stability

Flow stability in a horizontal pipe containing a two-phase liquid-gas fluid is investigated in this study. The stable and unstable flow patterns of the stratified flow are shown in Figs. 1a & 1b [8], respectively.

a)  b) 

Figure 1. a) The flow pattern of the stable stratified flow in the pipe, b) The flow pattern of the unstable stratified flow (wavy) in the pipe

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It is first necessary to obtain the velocity profiles of the liquid and gas phases in the pipe to derive the governing equations for flow stability.

2.1. The velocity profiles of the main phases

The three-dimensional velocity profiles of liquid and gas phases in the pipe are obtained for the case where half of the pipe is filled with liquid and the other half is filled with the gas phase. To this end, the simple two-dimensional relationships provided by Malik are expanded by defining $m = \mu_g / \mu_l = 0.05$ considering $U_{max} = 96\ m/s$ (for air and water [9]).

$$U_g(y) = \left(\frac{20}{96}\right) * \left[-y^2 + 0.9y + 0.1\right] * \left(1 - \frac{z^2}{1-y^2}\right) \qquad 0 \le y \le \sqrt{1^2 - Z^2} \quad , -1 \le Z \le 1 \tag{1}$$

The equation for the liquid velocity profile (lower layer) is written as follows:

$$U_l(y) = \left(\frac{1}{96}\right) * \left[-y^2 + 0.9y + 1.9\right] * \left(1 - \frac{z^2}{1-y^2}\right) \qquad -\sqrt{1^2 - Z^2} \le y \le 0 \quad , -1 \le Z \le 1 \tag{2}$$

The overall velocity profile of the pipe flow for both phases will be as follows:

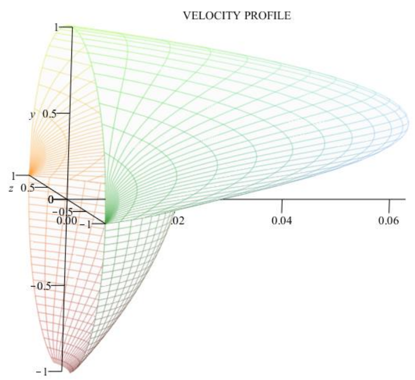


Figure 2. The velocity profiles of the liquid and gas phases in the pipe

3. Stability analysis of three-dimensional disturbances of viscous fluid flow

First, we assume a two-dimensional incompressible viscous flow which is moving along the x-axis. The boundaries are located at a height of y = ± 1 and z is considered perpendicular to the flow [10] [11]:

$$u = U(y), \quad v = 0, \quad w = 0 \tag{3}$$

where u,v,w are velocity components in the direction of three main axes. It is assumed that U(y) is a continuous and differentiable function in y. If a small disturbance is applied to the velocities u_1, v_1 and w_1 and the velocities are inserted in the continuity and momentum equations, neglecting the body forces and high-order terms as well as pressure terms, then we have:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \tag{4}$$

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \vartheta\nabla^2\right]\left[\frac{\partial w_1}{\partial y} - \frac{\partial v_1}{\partial z}\right] + U'\frac{\partial w_1}{\partial x} = 0 \tag{5}$$

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \vartheta\nabla^2\right]\left[\frac{\partial u_1}{\partial z} - \frac{\partial w_1}{\partial x}\right] + U'\frac{\partial v_1}{\partial z} = 0 \tag{6}$$

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \vartheta\nabla^2\right]\left[\frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y}\right] + U'\frac{\partial w_1}{\partial z} - U''v_1 = 0 \tag{7}$$

It is assumed that the disturbances are applied alternately and defined as below:

$$\begin{aligned} u_1 &= u(y).\exp i(jx + kz - \sigma t) \\ v_1 &= v(y).\exp i(jx + kz - \sigma t) \\ w_1 &= w(y).\exp i(jx + kz - \sigma t) \end{aligned} \tag{8}$$

where k and j are positive real number, but σ is a complex number in general. Inserting in equations (4) to (7), we can write:

$$iju + Dv + ikw = 0 \tag{9}$$

$$\left[\vartheta\left(D^2 - j^2 - K^2\right) + i\sigma - ijU\right]\left[Dw - ikv\right] - ijU'w = 0 \tag{10}$$

$$\left[\vartheta\left(D^2 - j^2 - K^2\right) + i\sigma - ijU\right]\left[ikv - ijw\right] - ikU'v = 0 \tag{11}$$

$$\left[\vartheta\left(D^2 - j^2 - K^2\right) + i\sigma - ijU\right]\left[ijv - Du\right] - ikU'w + U''v = 0 \tag{12}$$

where *R, V, cU₀, β, α, y* are non-dimensionalized as *y/b*, *jb*, *kb*, *σ/j*, *U/U₀* and *U₀b/ν*. Accordingly, the wave-lengths of disturbances along the x and z axes equals 2π/α and 2π/β, respectively. Therefore, the above equations can be rewritten as follows:

$$i\alpha u + Dv + i\beta w = 0 \tag{13}$$

$$\Delta\left[Dw - i\beta v\right] - iR\alpha.V'w = 0 \tag{14}$$

$$\Delta\left[i\beta u - i\alpha w\right] - iR\beta.V'v = 0 \tag{15}$$

$$\Delta\left[i\alpha v - Du\right] - iR\beta.V'w + RV''v = 0 \tag{16}$$

where the operator Δ is defined as $\Delta = D^2 - \left(\alpha^2 + \beta^2\right) - iR\alpha\left(V - c\right)$.

Due to the lack of disturbances on the boundaries, the boundary conditions on the walls are defined as follows:

$$u = v = w = 0$$

If *V (y)* is given for the main flow, equations (13) to (16) with the relevant boundary conditions will be sufficient to calculate c corresponding to the values of the independent quantities *R, α* and *β*. In general, this is a complex quantity with a negative imaginary part at low Reynolds numbers *R* indicating a stable flow. As the Reynolds number increases, *c* becomes a real number. In this case, *R* is called the critical Reynolds number. If *R* exceeds the critical Reynolds number, the flow will become unstable.

By eliminating *u* and *w* from equations (13) to (15) using the associative property of *D* and Δ operators, equation (16) is written as follows:

$$D.\Delta w - i\beta\Delta v = 0 \tag{17}$$

By eliminating *U* in equations (13) and (14), then we have:

$$i\left(\alpha^2+\beta^2\right)\Delta w+\beta\left[\Delta.D+iR\alpha.V'\right]v=0 \tag{18}$$

By subtracting equation (17) from equation (18), the term Δw is eliminated. Thus:

$$\left[D.\Delta.D+iR\alpha.D.V'\right]v=\left(\alpha^2+\beta^2\right)\Delta v \tag{19}$$

For v, we can write:

$$\left[\Delta.D^2-iR\alpha.V'.D+iR\alpha.D.V'\right]v=\left(\alpha^2+\beta^2\right)\Delta v \tag{20}$$

By replacing the operator Δ with that defined above, we can write:

$$\begin{aligned}&\left[D^4-2\left(\alpha^2+\beta^2\right)D^2+\left(\alpha^2+\beta^2\right)^2\right]v=\\&iR\alpha\left[\left(V-C\right)\left\{D^2-\left(\alpha^2+\beta^2\right)\right\}-V''\right]v\end{aligned} \tag{21}$$

Given the boundary conditions, u, v, and w will be eliminated on the boundaries. It is clear from equation (13) that Dv should also be eliminated on the boundaries for this purpose. Therefore, v will be obtained by solving the fourth order differential equation (21) with four boundary conditions. If v is obtained from equation (21), then w should be obtained from a second-order differential equation (equation (20)) and $y = \pm 1$ will be eliminated from the boundary conditions. If v and w are constant, u will be obtained from equation (13) and Dv and w will be zero on the walls.

The flow stability for disturbances of this type can be limited to examining equation (21) with boundary conditions that satisfy v.

Equation (21) is equivalent to the equations obtained for flow with two-dimensional disturbances. To complete this equivalence, we make the following replacements:

$$\begin{aligned}&\alpha^2+\beta^2=\overline{\alpha}^2\\&R\alpha=\bar{R}\bar{\alpha}=\bar{R}\left(\alpha^2+\beta^2\right)^{1/2}\end{aligned} \tag{22}$$

Therefore, equation (21) is written as follows:

$$\begin{aligned}&\left[D^4-2\overline{\alpha^2}.D^2+\overline{\alpha^4}\right]v=\\&i\bar{R}\bar{\alpha}\left[\left(V-C\right)\left\{D^2-\overline{\alpha^2}\right\}-V''\right]v\end{aligned} \tag{23}$$

The above equation is the same as the differential equation for two-dimensional disturbances with a wavelength of $2\pi\sqrt{\alpha}$ and $\bar{R}$ is the Reynolds number.

Based on the equivalence provided by Squire, it can be concluded that the three-dimensional disturbances defined in [8] are exactly similar to two-dimensional disturbances with a wavelength of $2\pi\sqrt{\alpha}$ and a Reynolds number of $\bar{R}$. In the case where $\bar{R}$ should be less than R, any three dimensional instability is displayed as a two-dimensional instability with a lower Reynolds number. Therefore, the study of two-dimensional disturbances is sufficient for flow stability analysis. The flow will be stable if the two-dimensional disturbances do not grow.

4. Governing equations for the two-phase parallel flow in the pipe

Based on the Yih [10] and Squire [11] theory outlined in Section 3, the stability analysis for three-dimensional flows can be performed two-dimensionally on a plane with the maximum Reynolds number. The two-phase flow in the pipe is one-directional and symmetrical relative to the plane perpendicular to the pipe along the main axis (as shown in Fig. 3). According to the velocity profiles of the liquid and gas phases in sections with the maximum velocity, the maximum Reynolds number is observed on this plane. Given the above discussion, instability in the pipe flow begins in the middle of the pipe perpendicular to the flow direction. A two-dimensional flow is considered for flow stability analysis [1].

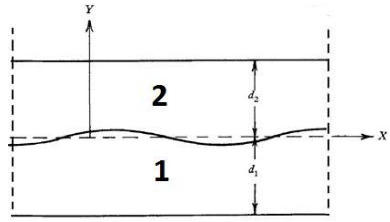


Figure 3. Two dimensional flow of the two-phase liquid-gas fluid on the plane

As shown in Fig. 3, the liquid film is considered in the bottom layer in the pipe with the gas phase on the upper layer. The gas flow exerts the shear stress, τ, on the surface of the liquid. The indices 1 and 2 respectively show the lower liquid phase and upper gas phase, d1 represents the thickness of the liquid layer and μ and ρ respectively show the viscosity and density.

As shown in Fig. 3, the coordinate components X and Y are in the direction of flow and perpendicular to the flow in the pipe respectively and the origin is considered at liquid-gas interface. Taking into account the velocity components u and v, the governing stability equations (Orr Sommerfeld) are obtained. The flow is assumed two-dimensional and incompressible. The continuity and momentum equations are written as follows after non-dimensionalization by $(\hat{u},\hat{v})=\frac{(u,v)}{V},(x,y)=\frac{(X,Y)}{d_1},\hat{P}=\frac{p}{\rho_1 U^2}$, and $\tau=\frac{tV}{d_1}$ it can be write [12] , [3], [13], [14], [15], [16]:

$$\begin{aligned}&\frac{D\hat{u}}{Dt}=-\frac{\partial\hat{P}}{\partial x}+\frac{1}{R}\nabla^2\hat{u}\\&\frac{D\hat{v}}{Dt}=-\frac{\partial\hat{P}}{\partial y}+\frac{1}{R}\nabla^2\hat{v}\end{aligned} \tag{24}$$

where R represents the Reynolds number and is defined as $\frac{\rho_1 V d_1}{\mu_1}$

By applying the disturbance terms to the fluid motion equations and given that there is no velocity component along the y axis, we can write:

$$\begin{aligned}&\hat{u}=U_1+u'\\&v=v'\\&\hat{p}=P+p'.\end{aligned} \tag{25}$$

Taking into account the flow function, the velocity disturbances are defined as follows:

$$u' = \psi_y \qquad (26)$$

$$v' = -\psi_x$$

Defining the Tollmien–Schlichting waves, the flow function can be written as follows:

$$\psi_j(x,y,t) = \phi_j(y)e^{i\alpha(x-c\tau)}$$
$$p'(x,y,t) = f(y)e^{i\alpha(x-c\tau)}. \qquad (27)$$

where i the imaginary part and α is the real wave number and c is the complex wave velocity which is defined as $c = c_r + ic_i$. The real part of c represents the fuzzy velocity of the wave while the imaginary part, αc, shows the growth rate of the wave (in other words, when $Im(\alpha c) > 0$, the wave is not damped and the flow will be unstable). If equations (25) to (27) are inserted in equation (24), neglecting the high-order disturbance terms, we can write after simplification:

$$i\alpha\left\{(U_1-c)\phi_1' - U_1'\phi_1\right\} = -i\alpha f + R^{-1}\left(\phi_1''' - \alpha^2\phi_1'\right) \qquad (28)$$

$$\alpha^2(c-U_1)\phi_1 = f' + \left(\frac{i\alpha}{R}\right)\left(\phi_1'' - \alpha^2\phi_1\right) \qquad (29)$$

In the above equations, the prime on φ and U indicates differentiation relative to y. By differentiating equation (28) and obtaining f' and inserting the result in equation (29), the Orr Sommerfeld equation for the function $\varphi_1(y)$ is obtained as follows:

$$\begin{aligned}&\phi_1^{iv} - 2\alpha^2\phi_1'' + \alpha^4\phi_1 = \\ &i\alpha R\left\{(U_1-c)\left(\phi_1'' - \alpha^2\phi_1\right) - U_1''\phi_1\right\}\end{aligned} \qquad (30)$$

The above equation will determine the flow stability based on boundary conditions. This equation is used for the lower layer (liquid phase). The index 2 is used to write the governing equation for the upper layer (gas phase). Applying disturbances, equations (25) will be written as follows after non-dimensionalization:

$$i\alpha R\left\{(U_2-c)\phi_2' - U_2'\phi_2\right\} = i\alpha f + \frac{m}{R}\left(\phi_2''' - \alpha^2\phi_2'\right) \qquad (31)$$

$$\alpha^2 r(c-U_2)\phi_2 = f' + \left(\frac{i\alpha m}{R}\right)\left(\phi_2'' - \alpha^2\phi_2\right) \qquad (32)$$

By eliminating the function f in equations (31) and (32), the Orr Sommerfeld equation for the gas phase is written as follow:

$$\phi_2^{iv} - 2\alpha^2\phi_2'' + \alpha^4\phi_2 = \left(\frac{i\alpha Rr}{m}\right)\left\{(U_2-c)\left(\phi_2'' - \alpha^2\phi_2\right) - U_2''\phi_2\right\} \qquad (34)$$

It should be noted that in this case, the liquid is located at the height (-1 <y <0) and gas phase at the height (0 <y <1). In equations (33) and (30), r is the density ratio of fluids and m is the viscosity ratio which are respectively defined as $r = \rho_2/\rho_1$ and $m = \mu_2/\mu_1$. Further, $R = \frac{\rho_1 U_\tau d_1}{\mu_1}$ and c is the complex wave velocity. Boundary conditions governing this problem for the liquid phase include wall impermeability and no-slip condition. So we have:

$$\phi_1(-1) = 0 \qquad (34)$$

$$\phi_1'(-1) = 0$$

Given the insignificant disturbances at large distances, the boundary condition for the gas phase is as follows:

$$\phi_2(1) = 0 \qquad (35)$$

$$\phi_2'(1) = 0$$

The boundary conditions on the interface include continuity of velocity components and equilibrium of stress components in tangential and vertical directions. From the continuity of v' and u', we will have:

$$\phi_1(0) = \phi_2(0) \qquad (36)$$

$$\phi_1'(0) + U_1'(0)\phi_1(0)/c = \phi_2'(0) + U_2'(0)\phi_2(0)/c \qquad (37)$$

Equations (36) and (37) show the continuity of velocities of two phases at interface. The equations for the tangential and vertical components of stress on the interface are as follow:

$$\begin{aligned}&\phi_2''(0) + \alpha^2\phi_2(0) + U_2''\phi_2(0)/c = \\ &m\left(\phi_1''(0) + \alpha^2\phi_1(0) + U_1''\phi_1(0)/c\right)\end{aligned} \qquad (38)$$

$$\begin{aligned}&\frac{m}{irR\alpha}\left(\phi_2'''(0) - 3\alpha^2\phi_2'(0)\right) + \left(c\phi_2'(0) + U_2'(0)\phi_2(0)\right) - \\ &\frac{1}{irR\alpha}\left(\phi_1'''(0) - 3\alpha^2\phi_1'(0)\right) - \frac{1}{r}\left(c\phi_1'(0) + U_1'(0)\phi_1(0)\right) \\ &+\frac{1}{r}\left(F + \alpha^2 S\right)\phi_1/c) = 0\end{aligned} \qquad (39)$$

where $S = \frac{\sigma}{\rho_1 U_1^2 d_1}$ is inverse of Weber number and $F = \frac{d_1 g(\rho_1 - \rho_2)}{\rho_1 U_1^2}$ is the inverse of Froude number which are related to surface tension, σ, and gravity acceleration, g.

The differential equations (30) and (33) along with conditions on the boundaries and the interface in equations (36) to (39) describe flow stability as an eigenvalue problem. The wave velocity, c, must have a certain value to obtain a non-zero solution for this system.

5. Chebyshev polynomial numerical solution

Spectral methods have had a significant impact on the accurate discretization of both initial value and eigenvalue problems. Especially in a bounded domain, the use of Chebyshev polynomials has been advantageous. Most of the stability calculations have been obtained by a Chebyshev discretization of the inhomogeneous coordinate direction. Below we will present the most useful relations for the discretization of derivatives and integrals. Chebyshev polynomials can be defined in many ways, for example, in terms of trigonometric functions [6], [17], [18], [19]:

$$T_n(y) = cos\left(n\, cos^{-1}(y)\right) \tag{40}$$

as solutions of the singular Sturm-Liouville problem

$$\frac{d}{dy}\left(\sqrt{1-y^2}\,\frac{d}{dy}T_n(y)\right) + \frac{n^2}{\sqrt{1-y^2}}T_n(y) = 0 \tag{41}$$

in terms of a recurrence relation

$$\begin{aligned} &T_0(y) = 1,\\ &T_1(y) = y,\\ &.\\ &.\\ &.\\ &T_{n+1}(y) = 2yT_n(y) - T_{n-1}(y) \end{aligned} \tag{42}$$

We will approximate the dependent variables by a Chebyshev expansion

$$f(y) = \sum_{n=0}^{N} a_n T_n(y) \tag{43}$$

and evaluate the Chebyshev polynomials at the extrema of the N -th Chebyshev polynomial given as

$$y_j = \cos\left(\frac{j\pi}{N}\right) \tag{44}$$

These locations are also known as the Gauss-Lobatto points. When discretizing ordinary or partial differential equations, derivatives of the solution are needed as well. These derivatives have to be expressed in terms of Chebyshev polynomials and the following recurrence relation between Chebyshev polynomials and their derivatives are used.

$$\begin{aligned} &T_0^{(k)}(y_j) = 0,\\ &T_1^{(k)}(y_j) = T_0^{(k-1)}(y_j),\\ &T_2^{(k)}(y_j) = 4T_1^{(k-1)}(y_j)\\ &.\\ &.\\ &. \end{aligned} \tag{45}$$

$$T_n^{(k)}(y_j) = 2nT_{n-1}^{(k-1)}(y_j) + \frac{n}{n-1}T_{n-1}^{(k)}(y_j) \qquad n = 3, 4, \ldots \tag{46}$$

with the superscript $k \ge 1$ denoting the order of differentiation.

Now Chebyshev Discretization presents as a spectral collocation method based and apply it to the Orr-Sommerfeld equation. This method has been used extensively to compute the stability characteristics of shear flows. The Orr-Sommerfeld equation for single phase flow presented as [7], [18], [19]:

$$\left(-Uk^2 - U'' - \frac{k^4}{i\alpha Re}\right)\phi + \left(U + \frac{2k^2}{i\alpha Re}\right)D^2\phi - \frac{1}{i\alpha Re}D^4\phi = c\left(D^2\phi - k^2\phi\right) \tag{47}$$

with the boundary conditions

$$\phi(\pm 1) = D\phi(\pm 1) = 0 \tag{48}$$

By expanding the eigen-functions in a Chebyshev series

$$\phi(y) = \sum_{n=0}^{N} a_n T_n(y) \tag{49}$$

The derivatives of the eigen-functions are obtained by differentiating the expansion above. We obtain for the second derivative, for example,

$$D^2\phi(y) = \sum_{n=0}^{N} a_n T_n''(y) \tag{50}$$

and similarly for the fourth derivative. Upon substitution into the Orr-Sommerfeld equation we get

$$\begin{aligned} &\left(U(y)k^2 - U''(y) - \frac{k^4}{i\alpha Re}\right)\sum_{n=0}^{N} a_n T_n(y) + \left(U(y) + \frac{2k^2}{i\alpha Re}\right)\\ &\sum_{n=0}^{N} a_n T_n''(y) - \frac{1}{i\alpha Re}\sum_{n=0}^{N} a_n T_n''''(y) =\\ &c\left(\sum_{n=0}^{N} a_n T_n''(y) - k^2\sum_{n=0}^{N} a_n T_n(y)\right) \end{aligned} \tag{51}$$

We then require this equation to be satisfied at the Gauss-Lobatto collocation points $y_j = \cos(j\pi/N)$ .This allows us to use the recurrence relations (45) & (46) to evaluate the derivatives of the Chebyshev polynomials. The discretized boundary conditions read

$$\begin{aligned} &\sum_{n=0}^{N} a_n T_n(1) = 0 \qquad \sum_{n=0}^{N} a_n T_n(-1) = 0\\ &\sum_{n=0}^{N} a_n T_n'(1) = 0 \qquad \sum_{n=0}^{N} a_n T_n'(-1) = 0 \end{aligned} \tag{52}$$

The final result is a generalized eigenvalue problem of the form

$$Aa = cBa \tag{53}$$

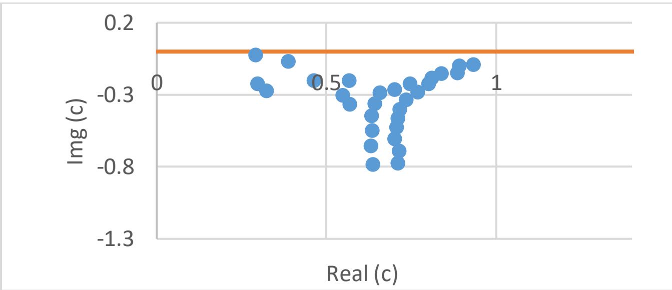
with the right-hand side

$$cBa = c\begin{bmatrix} T_0(1) & T_1(1) & \cdots \\ T_0'(1) & T_1'(1) & \cdots \\ T_0''(y_2) - k^2T_0(y_2) & T_1''(y_2) - k^2T_1(y_2) & \cdots \\ . & . & . \\ . & . & . \\ . & . & . \\ T_0''(y_{N-2}) - k^2T_0(y_{N-2}) & T_1''(y_{N-2}) - k^2T_1(y_{N-2}) & \cdots \\ T_0'(-1) & T_1'(-1) & \cdots \\ T_0(-1) & T_1(-1) & \cdots \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_{N-2} \\ a_{N-1} \\ a_N \end{bmatrix} \tag{54}$$

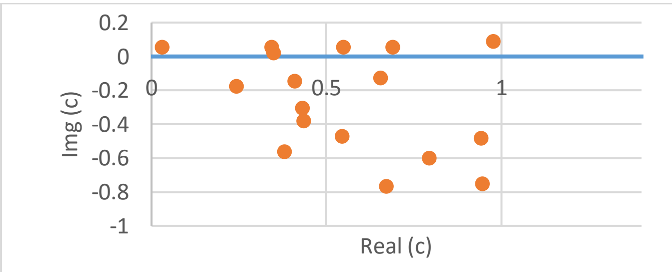
and similarly for the left-hand side Aa. We have chosen to use the first, second, last and next-to-last row of B to implement the four boundary conditions. The same rows in the matrix A can be chosen as a complex multiple of the corresponding rows in B. in this study two Orr-Sommerfeld equation (for two phases) has been solved simultaneously by Chebyshev polynomial. For validation of Chebyshev Tau- QZ polynomial numerical method for solving instability eigenvalue problems, this method had been used for solving and evaluating single phase instability equations in the pipe by authors [18].

6. Results

By simultaneous solution of equations with the boundary conditions of D^2 Chebyshev polynomial, the stability of the stratified two-phase flow in the pipe can be examined. Neglecting gravitational effects and assuming $F^{-1} \approx 0$ and taking into account *N=100, r=0.001*, α=1, *m=0.05, R=1500*, the eigenvalue spectra are obtained as follows:



(a)



(b)

Figure 4. Flow stability for (a) gas phase and (b) liquid phase in the case where N=100, r=0.001, α=1, m=0.05, R=1500

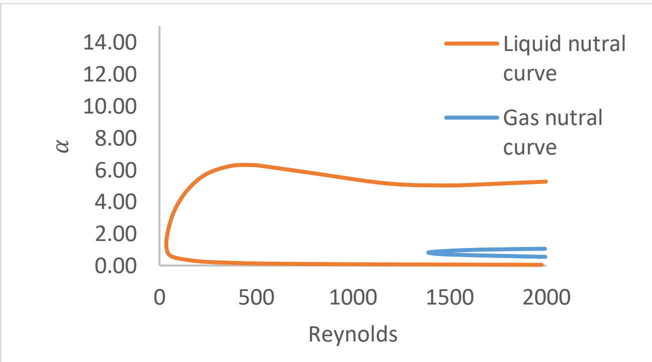


Figure 5. The stability plot for the two-phase flow (air-water)

If C_i *<0*, the flow is stable disturbances in the flow are damped. In contrast, if the imaginary part of the wave number is greater than zero (C_i*> 0*), the flow is unstable and flow disturbances grow. Figure 4a shows the flow stability for the gas phase. As seen, all C_i values are negative and thus the gas phase is stable. As seen in Fig.4b for the liquid phase, C_i values are positive showing the growth of disturbances in the liquid phase and thus an unstable liquid phase. If the stability curve is plotted for both gas and liquid phases for *r* = 0.001, *m*= 0.05 and *We=100*, Figure 5 is obtained:

As seen, the liquid phase flow is unstable at very low Reynolds numbers (approximately *R=50*). However, the gas phase becomes unstable at higher Reynolds numbers (approximately *R=1400*). Furthermore, α has a higher impact on instability of the liquid phase than the gas phase. The gas phase stable curve is located within the liquid phase curve. This means that if the gas phase is unstable, the liquid phase is already unstable.

If αc_i is plotted versus α for *m*=0.05, *r*=0.001, *R*=2000 and *We*=100, Figure 6 is obtained for the liquid phase.

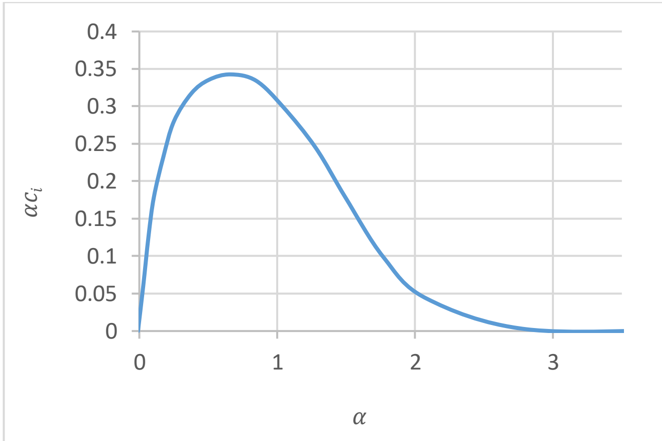


Figure 6. αc_i versus α for m=0.05, r=0.001, R=2000 and We=100

As seen, all αc_i values are positive. In other words, the liquid phase flow is unstable in this region. On the other hand αc_i for the liquid phase is maximum in the range of *0.4 <α <1* where $\alpha c_{i\ max}$ exists. The gas phase is unstable in this range. In other words, the instability of the gas phase leads to maximum instability in the liquid phase. Therefore, both phases are unstable in the range of *0.4 <α <1.*

7. Conclusion

It was observed that the Chebyshev Tau polynomial algorithm is a powerful tool capable of solving the eigenvalue equations of stability problems. In this study, the stability of the stratified two-phase liquid-phase pipe flow was investigated. According to the results, the liquid phase flow is unstable at very low Reynolds numbers (*Re* ≈ 50). However, the gas phase flow becomes unstable at Re ≈ 1400. Therefore, one can conclude that the two-phase pipe flow becomes unstable at *Re* ≈ 1400. At 50 <*R* <1400 where both liquid and gas phases are unstable, disturbances grow in the liquid phase, but are damped in the gas phase.

αc_i is maximum in the range of 0.4 <α<1 so the gas phase becomes more unstable in this region than any other region. In other words, when gas phase is unstable, the liquid phase is unstable too, but unstable flow of liquid phase is not a condition for gas phase instability.

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