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Vibration of FG viscoelastic nanobeams due to a periodic heat flux via fractional derivative model

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ABSTRACT

In this work, the vibrations of viscoelastic functionally graded Euler–Bernoulli nanostructure beams are investigated using the fractional-order calculus. It is assumed that the functionally graded nanobeam (FGN) is due to a periodic heat flux. FGN can be considered as nonhomogenous composite structures; with continuous structural changes along the thick- ness of the nanobeam usually, it changes from ceramic at the bottom of the metal at the top. Based on the nonlocal model of Eringen, the complete analytical solution to the problem is established using the Laplace transform method. The effects of different parameters are illustrated graphically and discussed. The effects of fractional order, damping coefficient, and periodic frequency of the vibrational behavior of nanobeam was investigated and discussed. It also provides a conceptual idea of the FGN and its distinct advantages compared to other engineering materials. The results obtained in this work can be applied to identify of many nano-structures such as nano-electro mechanical systems (NEMS), nano-actuators, etc.

1. Introduction

A functionally graded material (FGM) is considered as a material whose properties vary from one surface to another according to a continuous function depending on the position through the thickness of the material. Most often, these materials are made of ceramic and metal components. The metal part of the material acts as a structural support, while ceramic provides thermal protection when exposed to extreme temperatures. The function that describes the variation of materials throughout the material and most importantly the diversity of material property makes it possible to modify the function to appropriate the needs of different applications. In view of the fact that the materials can be designed for specific uses, the FGM materials are widely used in various industrial applications. In comparison with the laminated and isotropic materials, the FGM materials have diminished stress concentrations and thermal stresses and have the capacity of withstanding high-temperature gradient without losing structural reliability. Because of the extensive applications of FGM in structures, many researchers have made an analysis of mechanical and thermal responses of structures made of FGM [1-13].

The nonlocal elasticity theory proposed by Eringen [14-16] is widely used. Nano-materials are the base material of various nanoscale materials. Nanoscale objects are referred to as nanostructures. Recently, many one-dimensional nano-structures have been recognized. They contain nanowires, nano-dots, nano-rods, nanotubes, nano-belts, nano-nails, nano-bridges, nano-helices, nano-walls, seamless nano-rings and nano-beams, etc. The nonlocal elasticity and thermoelasticity theories are used to analyze the mechanical response of nanostructures by many authors [17-24].

Thermal vibrations of micro/nano-beam resonators have pulled in extensive considerable attention recently because of their several significant technical applications in Micro/ Nano-Electro-Mechanical Systems. Many authors have studied the elastic vibrations and heat transfer process of micro/nanobeams [17-24]. The calculation of fractions has been widely used to improve many existing models of physical processes, especially for polymerization modeling. The presence of the fractional order operator in the differential equations affects system history, which means that the following permissions in the system will depend on the current state and all its previous states. The

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different definitions and uses of thermoelasticity with fractional derivatives have become the main objective of many studies [25-33].

The viscous response occurs in a group of materials that show some kinds of liquid-like elastic manners. These materials include acrylics, rubber, and glass. The normal linear elastic Hooke's law is not an accurate presentation of viscoelastic material manners such as stress relaxation and creeps over small time-scales. Further, the behavior of material depends on it's a unique time history. The viscous model, called Kelvin-Voigt model, consists of a dashpot and a spring in parallel. Each element suffers from the same expansion, but strains adds [34].

In the present work, a nonlocal thermos-viscoelastic fractional order model for a nanoscale beam resonator is constructed. The nanobeam is considered to be made of functionally graded material and due to a periodic heat flux. A numerical technique based on the Laplace transform is applied to calculate the thermo-viscoelastic vibration of the deflection and temperature. The effect of frequency of heat flux is studied. The size effect the nanobeam and the effect of fractional derivative are analyzed.

2. Fractional Kelvin-Voigt thermos-viscoelastic model

The viscoelastic constitutive relation to Kelvin-Voigt viscoelastic model [34-37] is a combination of thermoelasticity and viscoelasticity theory. In addition, to consider the viscoelastic property of the material, Young's modulus of the material should be amended as follows, according to Kelvin-Voigt model [34]

$$E \rightarrow E_0\left(1+\tau_d \frac{\partial}{\partial t}\right), \tag{1}$$

where τ_d is the internal damping coefficient of microbeam (the viscous damping coefficient). Unfortunately, the model (1) is inaccurate to describe a wide category of nearly elastic engineering materials and the factor of resulting loss is proportional to the excitation frequency [38]. As a result, the fractional viscoelastic Voigt material model can accurately describe the properties of material damping to a wider range of frequencies [39-42].

The motion equation of the microbeam is resulting under the assumption of Euler-Bernoulli beam theory, shear deformation and rotary inertia. Furthermore, the uniform mass density and bending stiffness are assumed. Also, it is assumed that the first derivative in time in the Kelvin-Voigt model (1) is replaced by a fractional-order derivative of order α. In this fractional model, Young modulus E of the microbeam material is defined by the following relationship [43, 44]:

$$E \rightarrow E_0\left(1+\tau_d^\alpha \frac{\partial^\alpha}{\partial t^\alpha}\right). \tag{2}$$

Among the different definitions of fractional derivatives, we have followed the definition in the sense of Riemann-Liouville because it is closely related to Laplace transform and, consequently, to the Fourier transform. The operator $\frac{\partial^\alpha}{\partial t^\alpha}$ of the Riemann-Liouville fractional derivative of order α applied to the function $f(t)$ is defined by the expression $(0<\alpha<1)$:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t)=\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(u)}{(t-u)^\alpha} \mathrm{d}u, \tag{3}$$

where $\Gamma(1-\alpha)$ denotes the Gamma function. The constitutive relation with $\alpha=1$ represents Kelvin-Voight material with internal linear dissipation of mechanical energy.

The strain-displacement relations:

$$2e_{ij}=u_{j,i}+u_{i,j}. \tag{4}$$

Constitutive equations:

$$\tau_{ij}=2\mu e_{ij}+\lambda e_{ij}-\gamma\theta\delta_{ij}. \tag{5}$$

In last two relations, τ_{ij} represents the local stress tensor, e_{ij} denotes the strain tensor, δ_{ij} is Kronecker delta function, $\gamma=(3\lambda+2\mu)\alpha_t=E\alpha_t/(1-2\nu)=\alpha_T E$ is the coupling parameters, in which, λ and μ are Lame's constants and α_t is the coefficient of linear thermal expansion, $\theta=T-T_0$ denotes the thermodynamic temperature, T_0 represents the reference temperature.

Lame's constant λ and shear modulus μ that can be defined in terms of Young's modulus and Poisson's ratio as:

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu=\frac{E}{2(1+\nu)}. \tag{6}$$

The nonlocal differential constitutive equations for a homogenous thermoelastic materials is [14-16]

$$(1-\xi\nabla^2)\sigma_{ij}=\tau_{ij}, \tag{7}$$

where σ_{ij} is the nonlocal stress tensors, respectively, ∇^2 is Laplacian operator and ξ is the nonlocal parameter.

Equations of motion:

$$\sigma_{ji,j}+F_i=\rho\ddot{u}_i. \tag{8}$$

The strain-displacement relations:

$$2e_{ij}=u_{j,i}+u_{i,j}. \tag{9}$$

Due to the thermoelastic coupling, the heat conduction equation (non- Fourier model) for FGMs materials, is governed by the equation [45]

$$\nabla(K\nabla\theta)=\left(1+\tau_0\frac{\partial}{\partial t}\right)\left[\rho C_E\frac{\partial\theta}{\partial t}+\gamma T_0\frac{\partial}{\partial t}(\nabla\cdot\vec{u})-Q\right]. \tag{10}$$

3. Formulation of the problem

Let us consider a thermoelastic FG nanobeam of length L, width b and thickness h and initially at temperature T_0 such that x-axis is drawn along the axial direction of the beam and y, z axes correspond to the width and thickness, respectively (see Figure I).

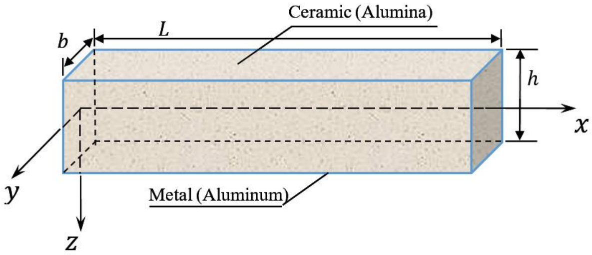


Figure 1: Schematic diagram for the FG nanobeam.

In the FGMs with high-temperature-environments, some material properties (elasticity modulus (E), thermal conductivity (K), mass density (ρ), coefficient of thermal expansion (α_t), and yield strength are of particular pertinence to this work) become temperature-thickness-dependent. Any material property $P(z)$ through the thickness of the present beam varies as a function of the volume fraction and is expressed as [3, 8, 13]

$$P(z) = P_m \mathrm{e}^{n_p(2z-h)/h}, \quad n_p = \ln\sqrt{P_m/P_c}\,, \tag{11}$$

where P_c represents the material property of the pure ceramic and P_m represents the material property of the pure metal. The material properties of the present beam are metal-rich (full metal) at the bottom surface $z = h/2$ and ceramic-rich (full ceramic) at the top surface $z = -h/2$ of the beam.

The axial and transverse displacements using the linear Euler-Bernoulli beam theory are given by

$$u = -z\frac{\partial w}{\partial x}, \quad v = 0, \quad w(x, y, z, t) = w(x, t), \tag{12}$$

where w is the lateral deflection.

In case of fractional model, Young modulus $\tilde{E}_m$ of the FGMs microbeam material can be expressed as [43,44]

$$\tilde{E}_m \to E_m\left(1 + \tau_d^\alpha \frac{\partial^\alpha}{\partial t^\alpha}\right). \tag{13}$$

With the aid of Eqs. (11)-(13), the thermal conduction equation, Eq. (10), for the nanobeam without the heat source ($Q = 0$), is expressed as

$$K_m \mathrm{e}^{n_K(2z-h)/h}\left[\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial z^2} + \frac{2n_K}{h}\frac{\partial\theta}{\partial z}\right] = \left(1 + \tau_0\frac{\partial}{\partial t}\right)\begin{bmatrix} \rho_m C_{Em} \mathrm{e}^{\frac{n_{\rho C_E}(2z-h)}{h}}\frac{\partial\theta}{\partial t} \\ -z\gamma_m \mathrm{e}^{n_\gamma(2z-h)/h} T_0\left(1 + \tau_d^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\right)\end{bmatrix} \tag{14}$$

It is to be noted that the parameters n_K, n_γ and $n_{\rho C_E}$ are given according to Eq. (12) in terms to the properties of ceramic and metal materials, and

$$\gamma_m = \frac{E_m\alpha_m}{1-2\nu_m}, \quad \rho_m C_{Em} = \frac{K_m}{\chi_m}. \tag{15}$$

Because there is no heat flow of heat across the upper and lower surfaces of the considered nanobeam, so, $\frac{\partial\theta}{\partial z} = 0$ at $z = \pm h/2$. Assuming that the temperature varies in a sinusoidal form along the direction of thickness. The solution that fulfills these conditions is given by

$$\theta(x, z, t) = \Theta(x, t)\sin\left(\frac{\pi z}{h}\right). \tag{16}$$

Substituting Eq. (16) into Eq. (14) and integrating with respect to z through the nanobeam thickness from $-h/2$ to $h/2$, one obtains

$$\frac{\partial^2\Theta}{\partial x^2} = \left(1 + \tau_0\frac{\partial}{\partial t}\right)\left[\bar{\mu}_{\rho C_E}\eta\frac{\partial\Theta}{\partial t} - \frac{\bar{\mu}_\gamma\gamma_m h T_0}{K_m}\left(1 + \tau_d^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\right)\right], \tag{17}$$

in which $\eta = \rho_m C_{Em}/K_m$, $\bar{\mu}_{\rho C_E} = \mu_{\rho C_E}/\mu_K$ and $\bar{\mu}_\gamma = \mu_\gamma/\mu_K$ and

$$\mu_{\rho C_E} = \frac{2n_{\rho C_E}\left(1+\mathrm{e}^{-2n_{\rho C_E}}\right)}{\pi^2 + 4\left(n_{\rho C_E}\right)^2}, \quad \mu_K = \frac{2n_K\left(1+\mathrm{e}^{-2n_K}\right)}{\pi^2 + 4(n_K)^2}, \quad \mu_\gamma = \frac{n_\gamma\left(1+\mathrm{e}^{-2n_\gamma}\right) + e^{-2n_\gamma} - 1}{4\left(n_\gamma\right)^2}. \tag{18}$$

According to the nonlocal theory of fractional viscoelastic materials, the nonlocal axial stress σ_x given in Eq. (7) at an arbitrary point in a nanobeam with the help of Eqs. (11), (13) becomes

$$\sigma_x - \xi\frac{\partial^2\sigma_x}{\partial x^2} = -E_m\left(1 + \tau_d^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\left[z\mathrm{e}^{\frac{n_E(2z-h)}{h}}\frac{\partial^2 w}{\partial x^2} + \alpha_m\theta\mathrm{e}^{\frac{n_{E\alpha}(2z-h)}{h}}\right], \tag{19}$$

in which $n_{E\alpha} = \ln\sqrt{E_m\alpha_m/E_c\alpha_c}$. To get the non-classical equations of motion for fractional thermo-viscoelastic nanobeams corresponding to the nonlocal thermoelasticity theory, it is essential to calculate the yield nonlocal bending moment. The expression for bending moment M with the help of Eq. (19) becomes

$$M(x, t) - \xi\frac{\partial^2 M}{\partial x^2} = -bh^2 E_m\left(1 + \tau_d^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\left[h\mu_E\frac{\partial^2 w}{\partial x^2} + \alpha_m\mu_{E\alpha}\Theta\right], \tag{20}$$

where

$$\mu_E = \frac{(n_E^2+2)(1-\mathrm{e}^{-2n_E}) - 2n_E(1-\mathrm{e}^{-2n_E})}{8n_E^3}, \quad \mu_K = \frac{2n_{E\alpha}(\pi^2+4n_{E\alpha}^2)(1-\mathrm{e}^{-2n_{E\alpha}}) + (\pi^2-4n_{E\alpha}^2)(1+\mathrm{e}^{-2n_{E\alpha}})}{\left(\pi^2+4n_{E\alpha}^2\right)^2}. \tag{21}$$

Based on Hamilton's principle, one can derive the governing transverse motion equation for FGM nanobeam as follows [8, 13]

$$\frac{\partial^2 M}{\partial x^2} = \frac{(1-\mathrm{e}^{-2n_\rho})\rho_m}{2n_\rho}A\frac{\partial^2 w}{\partial t^2}. \tag{22}$$

where, $A = bh$ is the area of cross section.

Introducing Eq. (20) into Eq. (22), the motion equation of the beam is given by

$$\left(1 + \tau_d^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial^4 w}{\partial x^4} + \frac{\rho_m(1-\mathrm{e}^{-2n_\rho})}{2E_m h^2 n_\rho\mu_E}\left(\frac{\partial^2 w}{\partial t^2} - \xi\frac{\partial^4 w}{\partial t^2\partial x^2}\right) + \frac{\alpha_m\mu_{E\alpha}}{\mu_E}\left(1 + \tau_d^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial^2\Theta}{\partial x^2} = 0. \tag{23}$$

Upon using expression (22), the flexure moment in Eq. (20) becomes

$$M(x, t) = \xi A\frac{(1-\mathrm{e}^{-2n_\rho})\rho_m}{2n_\rho}\frac{\partial^2 w}{\partial t^2} - bh^2 E_m\left(1 + \tau_d^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\left[h\mu_E\frac{\partial^2 w}{\partial x^2} + \alpha_m\mu_{E\alpha}\Theta\right]. \tag{24}$$

For convenience, we define the following non-dimensional quantities

$$\{x', z', u', w', L', h'\} = c_0\eta_0\{x, z, u, w, L, h\}, \quad \{t', \tau'_0, \xi'\} = c_0^2\eta_0\{t, \tau_0, \eta_0\xi\}, \quad \Theta' = \frac{\Theta}{T_0} \tag{25}$$

where

$$\eta_0 = \frac{\rho_m C_E}{K_m}, c_0^2 = \frac{E_m}{\rho_m}$$

Upon introducing variables in Eq. (25) into Eqs. (17), (23) and (24) after suppressing primes, we get

$$\left(1 + \tau_d^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial^4 w}{\partial x^4} + A_1\left(\frac{\partial^2 w}{\partial t^2} - \xi\frac{\partial^4 w}{\partial t^2\partial x^2}\right) = -A_2\left(1 + \tau_d^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial^2\Theta}{\partial x^2}, \tag{26}$$

$$\frac{\partial^2\Theta}{\partial x^2} = \left(1 + \tau_0\frac{\partial}{\partial t}\right)\left[A_3\frac{\partial\Theta}{\partial t} - A_4\frac{\partial}{\partial t}\left(1 + \tau_d^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\frac{\partial^2 w}{\partial x^2}\right)\right], \tag{27}$$

$$M(x, t) = A_1\left[\xi\frac{\partial^2 w}{\partial t^2} - \left(1 + \tau_d^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial^2 w}{\partial x^2}\right] - A_2\left(1 + \tau_d^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\Theta, \tag{28}$$

where

$$A_1 = \frac{1-\mathrm{e}^{-2n_\rho}}{2h^2 n_\rho \mu_E}, \quad A_2 = \frac{T_0 \alpha_m \bar{\mu}_{E\alpha}}{h}, \quad A_3 = \bar{\mu}_{\rho C_E}, \quad A_4 = \frac{\bar{\mu}_\gamma \gamma_m h}{\eta_0 K_m}. \tag{29}$$

Equations (26) and (27) characterizes the equations of motion and heat conduction in non-dimensional forms for the transverse vibrations in thermo-viscoelastic nanobeams, respectively. These equations are linear partial differential equations and may be solved by means of Laplace transform technique to find the fields w and Θ.

4. Laplace transform technique

Applying the Applying the Laplace transform technique under the initial conditions

$$\Theta(x,0) = \frac{\partial\Theta(x,0)}{\partial t} = 0 = w(x,0) = \frac{\partial w(x,0)}{\partial t}, \tag{30}$$

to the governing Eqs. (26)-(28), we obtain

$$\left(\frac{\mathrm{d}^4}{\mathrm{d}x^4} - \frac{\xi A_1 s^2}{1+\tau_d^\alpha s^\alpha}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A_1 s^2}{1+\tau_d^\alpha s^\alpha}\right)\bar{w} = -A_2 \frac{\mathrm{d}^2\bar{\Theta}}{\mathrm{d}x^2},$$
$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - qA_3\right)\bar{\Theta} = -qA_4 \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2}, \quad q = s(1+s^\alpha\tau_0^\alpha)(1+\tau_d^\alpha s^\alpha), \tag{31}$$

$$\bar{M}(x,t) = A_1\left[\xi s^2\bar{w} - (1+\tau_d^\alpha s^\alpha)\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2}\right] - A_2(1+\tau_d^\alpha s^\alpha)\bar{\Theta}. \tag{32}$$

From Eqs. (31), both the functions $\bar{\Theta}$ and $\bar{w}$, satisfies the differential equations:

$$(D^6 - AD^4 + BD^2 - C)\{\bar{\Theta},\bar{w}\}(x) = 0, \tag{33}$$

Where

$$A = \frac{\xi A_1 s^2}{1+\tau_d^\alpha s^\alpha} + qA_3 + qA_2A_4, \quad B = \frac{A_1 s^2 + q\xi A_1 A_3 s^2}{1+\tau_d^\alpha s^\alpha}, \quad C = \frac{qA_1A_3 s^2}{1+\tau_d^\alpha s^\alpha}, \quad D = \frac{\mathrm{d}}{\mathrm{d}x}. \tag{34}$$

Equation (33) can be rewritten as

$$(D^2 - m_1^2)(D^2 - m_2^2)(D^2 - m_3^2)\{\bar{\Theta},\bar{w}\}(x) = 0, \tag{35}$$

where m_n^2, $n = 1,2,3,4$ are roots of

$$m^6 - Am^4 + Bm^2 - C = 0. \tag{36}$$

The general solution of Eq. (30), can be expressed as

$$\{\bar{w},\bar{\Theta}\}(x) = \sum_{n=1}^{3}\{1, C_n'\}(C_n \mathrm{e}^{-m_n x} + C_{n+3}\mathrm{e}^{m_n x}). \tag{37}$$

Compatibility between Eq. (37) and Eqs. (31), gives

$$C_n' = -\frac{m_n^4 + A_1 s^2}{A_2 m_n^2} = \beta_n C_n. \tag{38}$$

From (37) and (12), axial displacement $\bar{u}$ is given by

$$\bar{u}(x) = -z\frac{\mathrm{d}\bar{w}}{\mathrm{d}x} = z\sum_{n=1}^{3} m_n(C_n\mathrm{e}^{-m_n x} - C_{n+3}\mathrm{e}^{m_n x}). \tag{39}$$

Substituting the values of $\bar{w}$ and $\bar{\Theta}$ from Eq. (37) in Eq. (32), we get the expressions of lateral deflection as

$$\bar{M}(x) = \sum_{n=1}^{3}[\xi s^2 - (m_n^2 + A_2\beta_n)(1+\tau_d^\alpha s^\alpha)](C_n\mathrm{e}^{-m_n x} + C_{n+3}\mathrm{e}^{m_n x}). \tag{40}$$

In addition, the strain will be

$$\bar{e}(x) = \frac{\mathrm{d}\bar{u}}{\mathrm{d}x} = -z\sum_{n=1}^{3} m_n^2(C_n\mathrm{e}^{-m_n x} + C_{n+1}\mathrm{e}^{m_n x}). \tag{41}$$

5. Applications

Introduce the following boundary conditions for the present application:

(1) Mechanical boundary conditions that the two ends of the nanobeam are simply-supported:

$$w(0,t) = w(L,t) = 0 = \frac{\partial^2 w(0,t)}{\partial x^2} = \frac{\partial^2 w(L,t)}{\partial x^2} \tag{42}$$

(2) Thermal boundary condition: We consider a dimensionless time dependent heat flux $q(t)$ of constant intensity q_0 is applied on the first end of the nanobeam $x = 0$ as

$$\frac{\partial\Theta}{\partial x} = q(t) \quad \text{on} \quad x = 0, \tag{43}$$

where $q(t)$ is varying periodically with time as:

$$q(t) = q_0\cos(\omega t), \quad \omega > 0 \quad \text{on} \quad x = 0, \tag{44}$$

where ω is the periodic frequency of heat flux. Not that, when constant heat flux is applied, we take $\omega = 0$. In addition, we assume that the second end $x = L$ is thermally insulated, yield

$$\frac{\partial\Theta}{\partial x} = 0 \quad \text{on} \quad x = L. \tag{45}$$

Applying Laplace transform to the boundary conditions (37) and (39), we can write

$$\begin{gathered}\bar{w}(0,s) = \bar{w}(L,s) = 0,\\ \frac{\partial^2\bar{w}(0,s)}{\partial t^2} = \frac{\partial^2\bar{w}(L,s)}{\partial t^2} = 0,\\ \frac{\partial\bar{\Theta}(0,s)}{\partial x} = \frac{q_0 s}{\omega^2 + s^2} = G(s),\\ \frac{\partial\bar{\Theta}(L,s)}{\partial x} = 0.\end{gathered} \tag{46}$$

Substituting Eqs. (32) into the above boundary conditions, one can get the following six linear equations:

$$\begin{gathered}\sum_{n=1}^{3}(C_n + C_{n+1}) = 0,\\ \sum_{n=1}^{3}(C_n\mathrm{e}^{-m_n L} + C_{n+1}\mathrm{e}^{m_n L}) = 0,\end{gathered} \tag{47}$$

$$\begin{gathered}\sum_{n=1}^{3} m_n^2(C_n + C_{n+1}) = 0,\\ \sum_{n=1}^{3} m_n^2(C_n\mathrm{e}^{-m_n L} + C_{n+1}\mathrm{e}^{m_n L}) = 0,\end{gathered} \tag{48}$$

$$\begin{gathered}\sum_{n=1}^{3} m_n(\beta_n C_n - \beta_{n+1}C_{n+1}) = -G(s),\\ \sum_{n=1}^{3} m_n(\beta_n C_n\mathrm{e}^{-m_n L} - \beta_{n+1}C_{n+1}\mathrm{e}^{m_n L}) = 0.\end{gathered} \tag{49}$$

The solution of the above system of linear equations gives the unknown parameters C_n, $(n = 1,2,\dots,6)$. To determine the studied fields in the physical domain, the Riemann-sum approximation method is used to obtain the numerical results. The details of these methods can be found in Honig and Hirdes [33].

6. Numerical results

Here, it is assumed that the metal and ceramic phases of the nanobeam are made of aluminum as lower metal surface and alumina as upper ceramic surface, respectively, with the following material properties [8]:

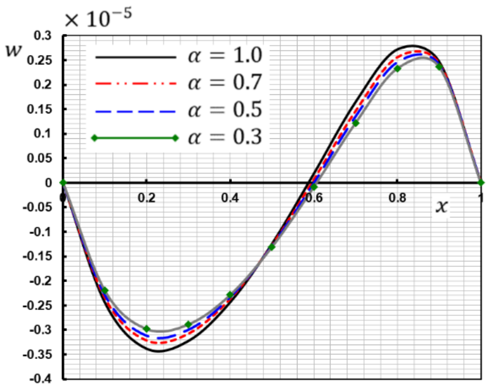
Table 1: Mechanical and thermoelastic properties parameter of the graded nanobeam

Material / Material properties	Metal (Aluminum)	Ceramic (Alumina)
Thermal conductivity ($\mathrm{W\,m^{-1}K^{-1}}$)	237	1.78
Young' modulus (GPa)	70	116
Density ($\mathrm{Kgm^{-3}}$)	2700	3000
Thermal expansion ($\mathrm{K^{-1}}$)	23.1×10^{-5}	8.7×10^{-6}
Thermal diffusivity ($\mathrm{m^2s^{-1}}$)	84.18×10^{-6}	1.06×10^{-6}
Poisson's ratio	0.35	0.33

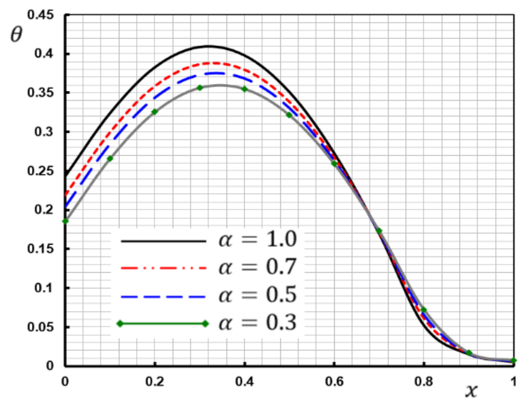
Unless otherwise stated, the values of the parameters utilized are as follows: $L/h = 10$, $L = 1$ and $z = h/3$ and $\bar{\xi} = \xi \times 10^{-5}$. Numerical calculations of the flexure moment M, temperature θ, displacement u, and lateral vibration w have been considered for various values of the nonlocal parameter $\bar{\xi} = \xi \times 10^{-6}$, fractional order α, thickness of nanobeam and periodic frequency ω. The results are investigated graphically in Figs. 2-5. Numerical calculations and graphs have been divided into four cases.

6.1 The effect of the fractional order parameter

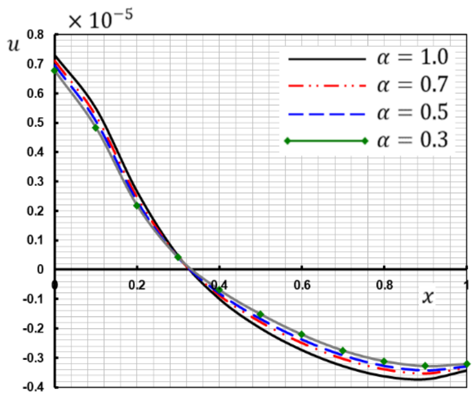
The effect of derivative with fractional order α on the behavior of the dimensionless studied variables of simply-supported viscoelastic nanobeams against axial distance x are investigated in Figs. 2. For a classical viscoelastic theory (integer derivative), one puts $\alpha = 1$ and for a fractional order viscoelastic theory, α may be 0.7, 0.5 or 0.3. As shown in all these figures, the fractional order parameter α has significant effects on flexure moment M, temperature θ, displacement u and lateral vibration w. Also, it is noted that the amplitudes of the studied variables increase with the increase of this coefficient, mainly due to increasing on viscous properties of the nanobeam material. It is observed that w vanishes at the boundaries of the nanobeam $x = 0, L$ in all cases, satisfying the boundary conditions of the problem. The variation of temperature θ is no longer increasing and has its maximum near the first edge of the beam. The variation of displacement u gradually decreases from positive to negative values with increasing x values. As discussed in [39, 40], we confirm that the fractional viscous models give physical motivation to analyze new models where memory and nonlinear effects are not ignored.



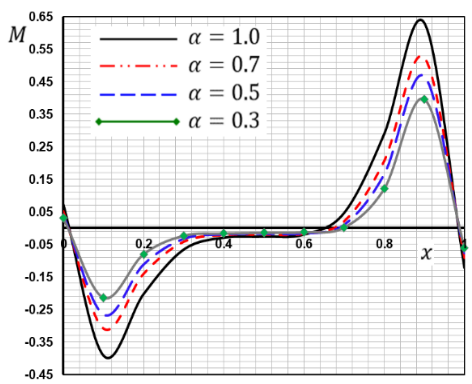
(a) Transverse deflection w versus distance x



(b) Temperature θ versus distance x



(c) Displacement u versus distance x

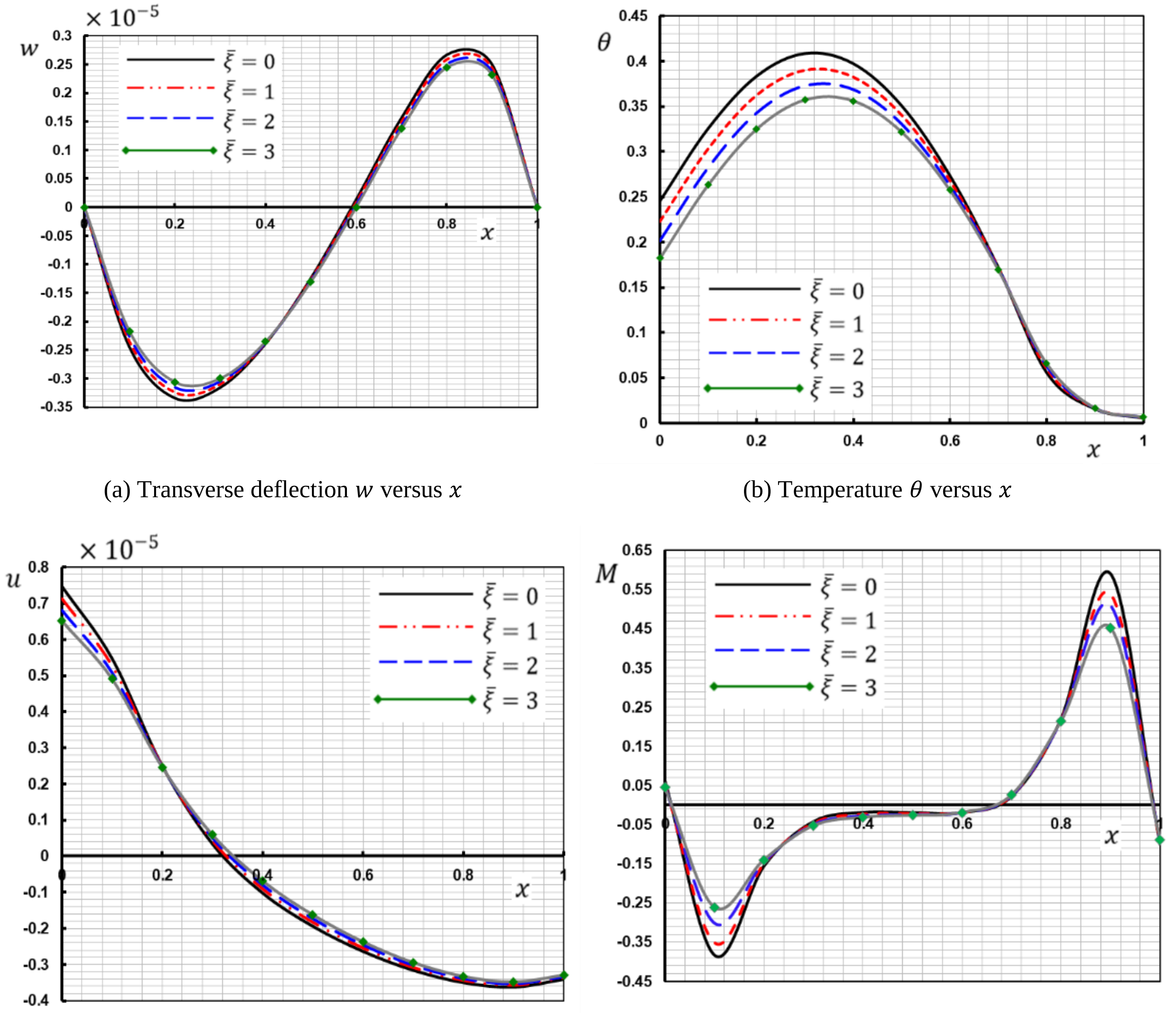


(d) Thermal stress σ_{xx} versus distance x

Figure 2: The transverse deflection, temperature, displacement and thermal stress distributions of the FG nanobeam for different values of the fractional order parameter α ($\bar{\xi} = 1, \omega = 5, \tau_0 = 0.02, z = h/4$)

6.2 Comparison with nonlocal parameter in the graded nanobeam

In the non-local theory of elasticity the stress σ_{ij} at a point x is a function of strains e_{ij} at all other points of the elastic body. Nonlocal theory proposed by Eringen [14-16] is introduced to include small-scale effects that appear at the nanoscale level. The effect of the nonlocal parameter $\bar{\xi}$ on the field variables response of viscoelastic nanobeams is illustrated in Figure 5. The case of $\bar{\xi} = 0$ indicates the old situation (local viscoelastic model) while the other values $\bar{\xi} = 1$ and $\bar{\xi} = 3$ indicate the nonlocal viscoelastic theory with fractional order. In this case the periodic frequency parameter of the applied heat flux remnants constant ($\omega = 5$) as well as $\alpha = 0.7, \tau_0 = 0.02$ and $z = h/6$. From these results, as stated in [20-22], the distributions of the temperature, deflection, displacement and bending moment are extremely sensitive to the variety of nonlocal parameter. It is noticed that the magnitude of the studied field variables increases in the domain $[0, L]$ with the increasing of nonlocal parameter $\bar{\xi}$. This investigation may be valuable for the future study of other single or various nanostructure based systems with the damping or in design techniques of Nano-devices.



(a) Transverse deflection w versus x (b) Temperature θ versus x

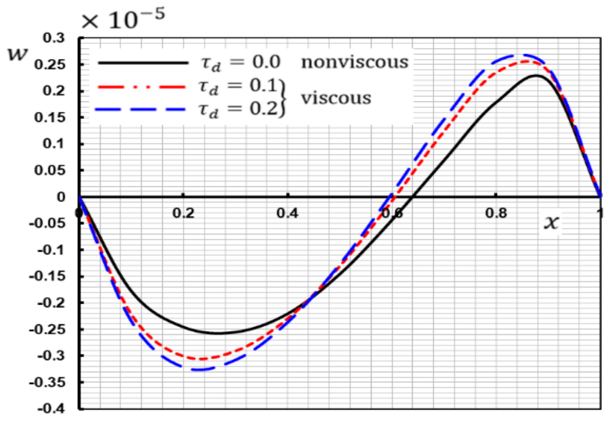
(c) Displacement u versus x (d) Thermal stress σ_{xx} versus x

Figure 3: The transverse deflection, temperature, displacement and thermal stress distributions of the FG nanobeam for different values of the nonlocal parameter $\bar{\xi}$ ($\alpha = 0.7, \omega = 5, \tau_d = 0.2, \tau_0 = 0.02, z = h/4$)

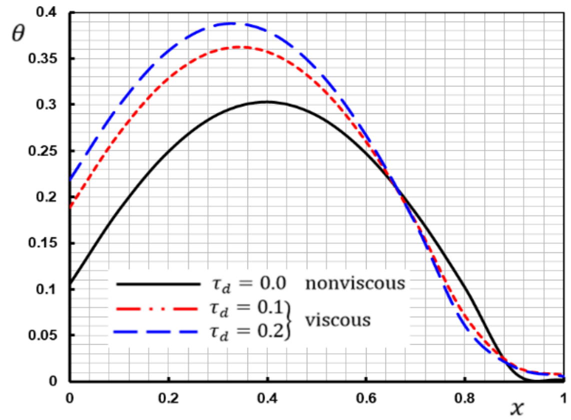
6.3 The effect of viscous damping coefficient

In the following case, the effects of the viscous damping coefficient τ_d (the viscous coefficient) on the vibrational behavior of viscoelastic Euler-Bernoulli nanobeam are investigated for constant values of ($\alpha = 0.7, \bar{\xi} = 1, \omega = 5, \tau_0 = 0.02, z = h/4$). It is of interest that when $\tau_d = 0$, the results of non-viscoelasticity theory are rendered. As it is seen in Figs. 2, by increasing the viscosity coefficient τ_d, the amplitude of vibration increases and the peak values of amplitudes are obtained.

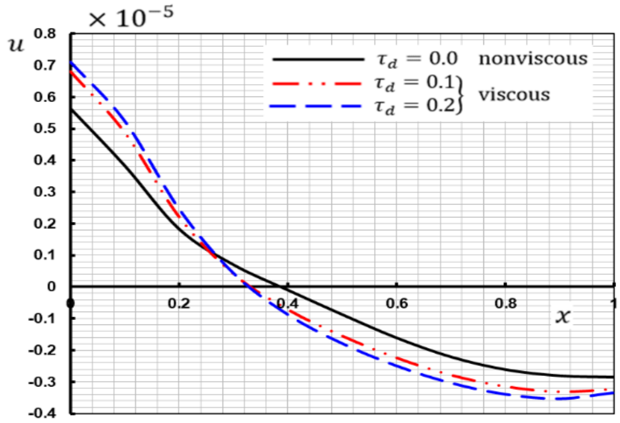
This fact that is because nanobeam tends to instability and hence, the jump phenomenon happens earlier. This phenomenon is consistent with that pronounced by Hosseini et al. [41] for viscoelastic piezoelectric cantilever beams.



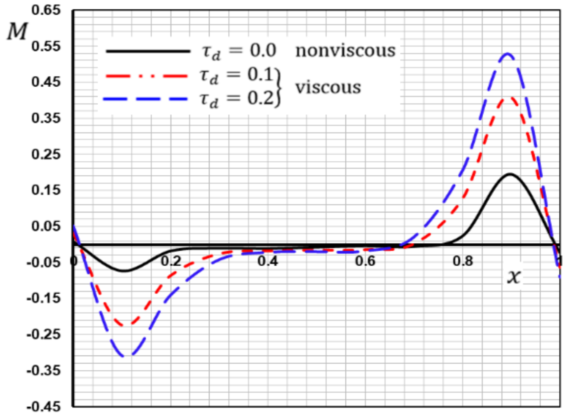
(a) Transverse deflection w versus x



(b) Temperature θ versus x



(c) Displacement u versus x



(d) Thermal stress σ_{xx} versus x

Figure 4: The transverse deflection, temperature, displacement and thermal stress distributions of the FG nanobeam for different values of the viscosity coefficient τ_d $(\alpha = 0.7, \omega = 5, \bar{\xi} = 1, \tau_0 = 0.02, z = h/4)$

6.4 The effect of the periodic frequency of the heat flux

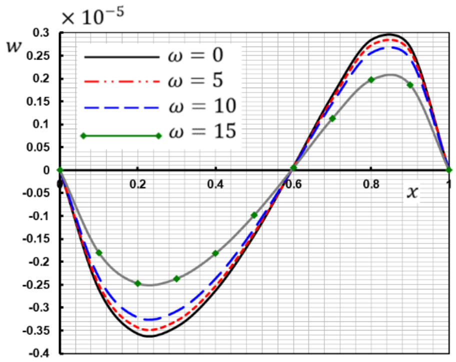
For constant values of $\alpha = 0.7$, $\tau_0 = 0.02$, $\bar{\xi} = 1$ and $z = h/6$, variations of the amplitude of field variables of viscoelastic nanobeams with distance corresponding to different values of the periodic frequency of the heat flux ω are displayed in Figs. 5. We take the values $\omega = 5,10,15$ for time dependent heat flux and $\omega = 0$ for constant heat flux. Based on these Figs., it is found that the viscoelastic vibration of the studied fields decreases, as the periodic frequency ω increases. As depicted, the influence of the frequency ω of the applied heat flux q on all studied areas is extremely significant.

7. Conclusions

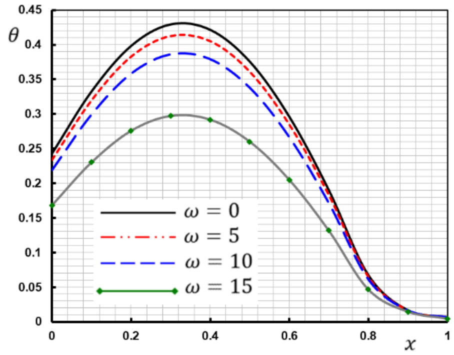
A fractional viscoelastic model of a simply supported nonlocal Euler-Bernoulli nanobeam is constructed in this paper. It is assumed that the properties of the nanobeam vary gradually through the thickness, e.g., from a pure ceramic to a pure metal. It is assumed that the FG nanobeam is due to a periodic heat flux. In this research, the effects of several parameters such as fractional order, damping coefficient, and periodic frequency of the vibrational behavior of nanobeam was investigated and discussed.

From this investigation, the following outlines could be highlighted:

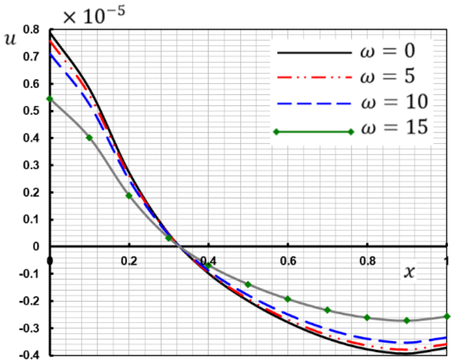
- The Nanobeam length (nanoscale) effect plays an important role in damping behavior of all the studied fields. Consequently, nonlocal influences should be considered in analyzing the mechanical behavior of nanoscale structures.
- The vibrational behavior system strongly depends on the periodic frequency of the applied heat flux.
- The technique and the introduced model used in this work can be applied to many solid mechanic and thermodynamics fields.
- It is further found that fractional order effects can make the nanobeam softer or stiffer, depending on the values of fractional order and viscous damping coefficient.
- Elasticity and heat propagation problems generally include both memory and nonlinear effects and it looks an ideal structure for considering new complex mathematical models, including partial fractional derivatives.
- Finally, the technique and analytical solution obtained here opens the scope of various additional studies in mathematics, science and engineering as well as the microstructure industry.



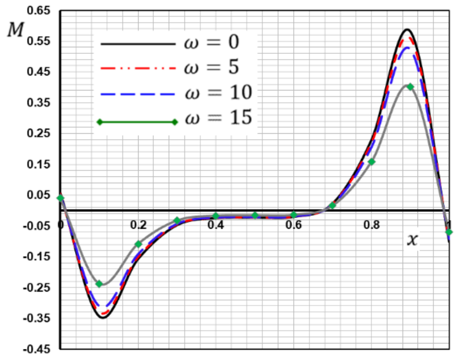
(a) Transverse deflection w versus x



(b) Temperature θ versus x



(c) Displacement u versus x



(d) Thermal stress σ_{xx} versus x

Figure 5: The transverse deflection, temperature, displacement and thermal stress distributions of the FG nanobeam for different values of the periodic frequency of the heat flux ω $(\alpha = 0.7, \tau_d = 0.2, \bar{\xi} = 1, \tau_0 = 0.02, z = h/4)$

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