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On Maxwell's Stress Functions for Solving three Dimensional Elasticity Problems in the Theory of Elasticity

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ABSTRACT

The governing equations of three dimensional elasticity problems include the six Beltrami–Michell stress compatibility equations, the three differential equations of equilibrium, and the six material constitutive relations; and these are usually solved subject to boundary conditions. The system of fifteen differential equations is usually difficult to solve, and simplified methods are usually used to achieve a solution. Stress-based formulation methods, and displacement-based formulation methods are two common simplified methods for solving elasticity problems. This work adopts a stress-based formulation for a three dimensional elasticity problem. In this work, the Maxwell's stress functions for solving three dimensional problems of elasticity theory are derived from fundamental principles. It is shown that the three Maxwell stress functions identically satisfy all the three differential equations of static equilibrium when body force components are ignored. It is further shown that the three Maxwell stress functions are solutions to the six Beltrami-Michell stress compatibility equations if the Maxwell stress functions are potential functions. It is also shown that the Airy's stress functions for two dimensional elasticity problems are special cases of the Maxwell stress functions.

1. Introduction

1.1 Background

The conventional strength of materials method of structural analysis assumes the structures behave like rigid bodies; and the deformation (elasticity) is neglected. The theory of elasticity method is an advanced method of structural analysis which accounts for the deformations of the structures. The objective of the theory of elasticity method, is the determination of stress, strain and displacement fields in structures due to applied loads or temperature changes. Theory of elasticity method of structural analysis has extensive applications in the areas of structural, mechanical, naval, marine, aeronautic and spacecraft engineering. The applications cover the fields of stress analysis, deformation analysis and the determination of internal force fields in the members of structural systems due to applied loads and/or temperature changes.

The solution of elasticity problems in three dimensional (3D) space variables entails solving a system of fifteen differential equations, made up of three differential equations of equilibrium, six strain-displacement relations (kinematic relations) and six equations relating stresses and strains [1-10]. The solution is often very tedious. Simplifications have been offered by formulations of the governing equations using displacement based methods and stress based methods. In displacement formulations, the governing equations are obtained (expressed) in terms of displacement field components as the primary unknown variables by combining the stress equilibrium equations, the stress-strain laws and the kinematic relations. The governing equations of 3D elasticity in a displacement formulation reduce to a system of three partial differential equations in terms of three unknown displacement field components, and thus can be more easily solved [11-17]. In stress formulation, the governing equations are expressed in terms of the six unknown stress components as the primary unknowns . The compatibility of strains are used to generate additional equations. Stress compatibility equations are obtained from the strain compatibility equations by using the stress-strain relations.

1.2 Literature review

A review of literature shows there are two basic methods of formulating and solving problems of the theory of elasticity. They are the displacement potential function method and the stress potential function method. A third method called the mixed or hybrid method is not common; but involves a formulation of the problem

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such that some displacement components and some stress components are the primary unknown variables.

1.2.1 Displacement potential functions

Displacement potential functions which are scalar fields (functions) of the space variables have been derived as solutions to the displacement equations of elasticity theory by Boussinesq, Green and Zerna [18]. The displacement functions have no obvious physical meaning other than their use in defining displacement components in terms of their derivatives [1].

Nwoji et al [16] used the Green and Zerna displacement potential function method to determine the stress, strain and displacement fields in an elastic half-space due to a point load at the origin. They obtained solutions which were identical to the solutions obtained by Boussinesq who used Boussinesq's potentials to solve the problem.

1.2.2 Stress potential functions

Stress potential functions are scalar fields (functions) of the space variables that are solutions of the stress formulation of elasticity problems, from which the stresses could be derived [1]. Airy [19] presented the first stress function solution of the differential equations of equilibrium for two dimensional elasticity problems. Airy's stress function is a single harmonic function of the two dimensional (2D) Cartesian coordinate variables for the problem from which equilibrating stress fields could be derived. Three dimensional generalizations of the stress functions of 3D elasticity were studied by Maxwell [20], Morera [21] and Beltrami [22].

Onah et al [23] used the Fourier transform method in an Airy's stress based formulation to determine the Cartesian stress field components for an elastic half plane problem in the *xz* coordinate plane due to infinitely long line load and uniformly distributed strip load on the surface.

Ike [24] used the exponential Fourier transform method, in an Airy's stress function formulation to find stress fields in elastic half-plane due to load acting on the boundary.

Ike [12] used an axisymmetric stress function to solve the Boussinesq problem in the theory of elasticity applied to geotechnical problems.

1.3 Research aim and objectives

The general aim of this study is to present the Maxwell stress functions for solving three dimensional elasticity problems. The specific objectives are:

(i) to derive the Maxwell's stress functions from fundamental principles and show that they identically satisfy the differential equations of equilibrium when body forces are disregarded.
(ii) to prove that the Maxwell's stress functions satisfy the six Beltrami-Michell stress compatibility equations only if the functions are harmonic functions.
(iii) to establish the relationship between the Maxwell stress potential functions of 3D elasticity and the Airy's stress potential functions of 2D elasticity, and hence prove that the Airy's stress functions are special cases of the Maxwell stress functions.
(iv) to present some illustrative applications of the use of the 2D specializations of the Maxwell stress functions in the determination of stress fields in cantilever beams under point load at the free end and simply supported beam under uniformly distributed load.

2. Theoretical framework

Beltrami-Michell compatibility equations in terms of stress are:

$$\nabla^2\sigma_{xx} + \frac{1}{1+\mu}\frac{\partial^2}{\partial x^2}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = \frac{-\mu}{1-\mu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right) - 2\frac{\partial F_x}{\partial x} \quad (1)$$

$$\nabla^2\sigma_{yy} + \frac{1}{1+\mu}\frac{\partial^2}{\partial y^2}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = \frac{-\mu}{1-\mu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right) - 2\frac{\partial F_y}{\partial y} \quad (2)$$

$$\nabla^2\sigma_{zz} + \frac{1}{1+\mu}\frac{\partial^2}{\partial z^2}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = \frac{-\mu}{1-\mu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right) - 2\frac{\partial F_z}{\partial z} \quad (3)$$

$$\nabla^2\tau_{xy} + \frac{1}{1+\mu}\frac{\partial^2}{\partial x\partial y}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = -\left(\frac{\partial F_x}{\partial y}+\frac{\partial F_y}{\partial x}\right) \quad (4)$$

$$\nabla^2\tau_{xz} + \frac{1}{1+\mu}\frac{\partial^2}{\partial x\partial z}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = -\left(\frac{\partial F_x}{\partial z}+\frac{\partial F_z}{\partial x}\right) \quad (5)$$

$$\nabla^2\tau_{yz} + \frac{1}{1+\mu}\frac{\partial^2}{\partial y\partial z}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = -\left(\frac{\partial F_y}{\partial z}+\frac{\partial F_z}{\partial y}\right) \quad (6)$$

where $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ are nomal stresses, $\tau_{xy}, \tau_{xz}, \tau_{yz}$ are shear stresses; μ is the Poisson's ratio, and F_x, F_y, and F_z are components of body forces in the *x*, *y*, and *z* Cartesian coordinate directions.

When body forces are disregarded, $F_x = F_y = F_z = 0$, the Beltrami-Michell compatibility equations are expressed in terms of stress by the following six scalar equations:

$$(1+\mu)\nabla^2\sigma_{xx} + \frac{\partial^2}{\partial x^2}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = 0 \quad (7)$$

$$(1+\mu)\nabla^2\sigma_{yy} + \frac{\partial^2}{\partial y^2}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = 0 \quad (8)$$

$$(1+\mu)\nabla^2\sigma_{zz} + \frac{\partial^2}{\partial z^2}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = 0 \quad (9)$$

$$(1+\mu)\nabla^2\tau_{xy} + \frac{\partial^2}{\partial x\partial y}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = 0 \quad (10)$$

$$(1+\mu)\nabla^2\tau_{yz} + \frac{\partial^2}{\partial y\partial z}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = 0 \quad (11)$$

$$(1+\mu)\nabla^2\tau_{zx} + \frac{\partial^2}{\partial z\partial x}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = 0 \quad (12)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \quad (13)$$

∇^2 is the Laplacian.

Differential equations of equilibrium

The differential equations of equilibrium for static cases are given by:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0 \tag{14}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0 \tag{15}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0 \tag{16}$$

when body forces are disregarded, the differential equations are simplified as follows:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{17}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \tag{18}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{19}$$

3. Methodology

Stress formulation of problems of the theory of elasticity is suitable for use with elasticity problems given with traction boundary conditions. Once the stresses have been determined by solving the system of governing equations of the elasticity problem presented in terms of stresses, the corresponding strains can be found from the stress-strain laws (generalized Hooke's law); and the displacement fields obtained from the strains using the strain-displacement equations. The system of governing differential equations presented in the stress formulation of elasticity problems is very complicated and complex; and solutions by integration of the equations are often impossible. Mathematically closed form solutions to the governing equations of elasticity presented in term of stress have been obtained by defining stress functions.

Six stress functions $\phi_{11}(x,y,z)$, $\phi_{22}(x,y,z)$, $\phi_{33}(x,y,z)$, $\phi_{12}(x,y,z)$, $\phi_{23}(x,y,z)$, $\phi_{31}(x,y,z)$ where $\phi_{ij}(x,y,z)$ and $\phi_{ij} = \phi_{ji}$ that satisfy the differential equations of equilibrium and the Beltrami-Michell stress compatibility equations are given by Equations (20) – (25) as follows:

$$\sigma_{xx} = \frac{\partial^2 \phi_{22}}{\partial z^2} + \frac{\partial^2 \phi_{33}}{\partial y^2} - 2\frac{\partial^2 \phi_{23}}{\partial y \partial z} \tag{20}$$

$$\sigma_{yy} = \frac{\partial^2 \phi_{33}}{\partial x^2} + \frac{\partial^2 \phi_{11}}{\partial z^2} - 2\frac{\partial^2 \phi_{31}}{\partial z \partial x} \tag{21}$$

$$\sigma_{zz} = \frac{\partial^2 \phi_{11}}{\partial y^2} + \frac{\partial^2 \phi_{22}}{\partial x^2} - 2\frac{\partial^2 \phi_{12}}{\partial x \partial y} \tag{22}$$

$$\tau_{yz} = \sigma_{yz} = \frac{\partial^2 \phi_{31}}{\partial x \partial y} + \frac{\partial^2 \phi_{12}}{\partial x \partial z} - \frac{\partial^2 \phi_{11}}{\partial y \partial z} - \frac{\partial^2 \phi_{23}}{\partial x^2} \tag{23}$$

$$\tau_{zx} = \sigma_{zx} = \frac{\partial^2 \phi_{12}}{\partial y \partial z} + \frac{\partial^2 \phi_{23}}{\partial y \partial x} - \frac{\partial^2 \phi_{22}}{\partial z \partial x} - \frac{\partial^2 \phi_{31}}{\partial y^2} \tag{24}$$

$$\tau_{xy} = \sigma_{xy} = \frac{\partial^2 \phi_{23}}{\partial z \partial x} + \frac{\partial^2 \phi_{31}}{\partial z \partial y} - \frac{\partial^2 \phi_{33}}{\partial x \partial y} - \frac{\partial^2 \phi_{12}}{\partial z^2} \tag{25}$$

4.0 Results

4.1 First principles derivation of Maxwell's stress functions $\phi_{11}(x,y,z)$, $\phi_{22}(x,y,z)$, $\phi_{33}(x,y,z)$,

Let the shear stress fields τ_{yz}, τ_{zx} and τ_{xy} be derivable from the three Maxwell's stress functions $\phi_{11}(x,y,z)$, $\phi_{22}(x,y,z)$, $\phi_{33}(x,y,z)$ as follows:

$$\tau_{yz} = -\frac{\partial^2 \phi_{11}}{\partial y \partial z}(x,y,z) = \tau_{zy} \tag{26}$$

$$\tau_{zx} = \tau_{xz} = -\frac{\partial^2 \phi_{22}}{\partial z \partial x}(x,y,z) \tag{27}$$

$$\tau_{xy} = \tau_{yx} = -\frac{\partial^2 \phi_{33}}{\partial x \partial y}(x,y,z) \tag{28}$$

Then for the Maxwell stress functions ϕ_{11}, ϕ_{22}, ϕ_{33} to satisfy the differential equations of equilibrium in the absence of body forces, the normal stresses σ_{xx}, σ_{yy}, σ_{zz} can be found. Thus, by substitution of Equations (26 – 28) into the differential equations of equilibrium, Equations (17 – 19), we obtain:

$$\frac{\partial \sigma_{xx}}{\partial x} = -\frac{\partial}{\partial y}\left(-\frac{\partial^2 \phi_{33}}{\partial x \partial y}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial^2 \phi_{22}}{\partial z \partial x}\right) \tag{29}$$

Simplifying Equation (29) becomes:

$$\frac{\partial \sigma_{xx}}{\partial x} = \frac{\partial^3 \phi_{33}}{\partial x \partial y^2} + \frac{\partial^3 \phi_{22}}{\partial x \partial z^2} \tag{30}$$

Further simplification yields:

$$\frac{\partial \sigma_{xx}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \phi_{33}}{\partial y^2}\right) + \frac{\partial}{\partial x}\left(\frac{\partial^2 \phi_{22}}{\partial z^2}\right) \tag{31}$$

Using linearity properties of the partial differential operator, we obtain:

$$\frac{\partial \sigma_{xx}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \phi_{33}}{\partial y^2} + \frac{\partial^2 \phi_{22}}{\partial z^2}\right) \tag{32}$$

Integration of Equation (32) with respect to *x* yields:

$$\sigma_{xx} = \int \frac{\partial}{\partial x}\left(\frac{\partial^2 \phi_{33}}{\partial y^2} + \frac{\partial^2 \phi_{22}}{\partial z^2}\right) dx \tag{33}$$

By integration, Equation (33) is simplified to obtain:

$$\sigma_{xx} = \frac{\partial^2 \phi_{33}}{\partial y^2} + \frac{\partial^2 \phi_{22}}{\partial z^2} \tag{34}$$

Similarly, substitution of Equations (26) and (28) into Equation (18) yields

$$\frac{\partial \sigma_{yy}}{\partial y} = -\frac{\partial}{\partial x}\left(-\frac{\partial^2 \phi_{33}}{\partial x \partial y}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial^2 \phi_{11}}{\partial y \partial z}\right) \tag{35}$$

Simplifying,

$$\frac{\partial \sigma_{yy}}{\partial y} = \frac{\partial^3 \phi_{33}}{\partial y \partial x^2} + \frac{\partial^3 \phi_{11}}{\partial y \partial z^2} \tag{36}$$

Further simplification yields:

$$\frac{\partial \sigma_{yy}}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 \phi_{33}}{\partial x^2}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 \phi_{11}}{\partial z^2}\right) \tag{37}$$

Use of linearity property of the partial differential operator gives:

$$\frac{\partial \sigma_{yy}}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 \phi_{33}}{\partial x^2} + \frac{\partial^2 \phi_{11}}{\partial z^2}\right) \quad (38)$$

Integrating with respect to y, we have:

$$\sigma_{yy} = \int \frac{\partial}{\partial y}\left(\frac{\partial^2 \phi_{33}}{\partial x^2} + \frac{\partial^2 \phi_{11}}{\partial z^2}\right) dy \quad (39)$$

By integration, Equation (39) becomes:

$$\sigma_{yy} = \frac{\partial^2 \phi_{33}}{\partial x^2} + \frac{\partial^2 \phi_{11}}{\partial z^2} \quad (40)$$

Substitution of Equations (26) and (27) into Equation (19) yields:

$$\frac{\partial \sigma_{zz}}{\partial z} = -\frac{\partial}{\partial x}\left(-\frac{\partial^2 \phi_{22}}{\partial z \partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial^2 \phi_{11}}{\partial y \partial z}\right) \quad (41)$$

Simplifying Equation (41) we obtain:

$$\frac{\partial \sigma_{zz}}{\partial z} = \frac{\partial^3 \phi_{22}}{\partial z \partial x^2} + \frac{\partial^3 \phi_{11}}{\partial z \partial y^2} \quad (42)$$

Further simplification of Equation (42) yields:

$$\frac{\partial \sigma_{zz}}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial^2 \phi_{22}}{\partial x^2}\right) + \frac{\partial}{\partial z}\left(\frac{\partial^2 \phi_{11}}{\partial y^2}\right) \quad (43)$$

Use of the linearity property of the partial differential operator gives:

$$\frac{\partial \sigma_{zz}}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial^2 \phi_{22}}{\partial x^2} + \frac{\partial^2 \phi_{11}}{\partial y^2}\right) \quad (44)$$

Integrating with respect to z, yields:

$$\sigma_{zz} = \int \frac{\partial}{\partial z}\left(\frac{\partial^2 \phi_{22}}{\partial x^2} + \frac{\partial^2 \phi_{11}}{\partial y^2}\right) dz \quad (45)$$

By integration, Equation (45) becomes:

$$\sigma_{zz} = \frac{\partial^2 \phi_{22}}{\partial x^2} + \frac{\partial^2 \phi_{11}}{\partial y^2} \quad (46)$$

When the stress functions $\phi_{12} = \phi_{23} = \phi_{31} = 0$, in Equations (20-25), we obtain the Maxwell's stress functions $\phi_{11}(x, y, z)$, $\phi_{22}(x, y, z)$, $\phi_{33}(x, y, z)$ defined in terms of the stress fields by Equations (26-28), or alternatively, by Equations (34), (40) and (46).

4.2 Proof that Maxwell's stress functions are equilibrating stresses

We prove that the Maxwell's stress functions, defined in Equations (26) – (28), or alternatively as Equations (34), (40) and (46) identically satisfy the differential equations of equilibrium when body forces are disregarded. By integration of Equations (26) – (28), we obtain:

$$\phi_{11}(x, y, z) = -\iint \tau_{yz}(x, y, z) dydz \quad (47)$$

$$\phi_{22}(x, y, z) = -\iint \tau_{zx}(x, y, z) dzdx \quad (48)$$

$$\phi_{33}(x, y, z) = -\iint \tau_{xy}(x, y, z) dxdy \quad (49)$$

By substitution of Equations (47) – (49) into Equations (34), (40) and (46), we obtain as follows:

From Equation (34),

$$\sigma_{xx}(x, y, z) = \frac{\partial^2}{\partial z^2}\left(-\iint \tau_{zx}(x, y, z) dzdx\right) + \frac{\partial^2}{\partial y^2}\left(-\iint \tau_{xy}(x, y, z) dxdy\right) \quad (50)$$

Simplifying,

$$\sigma_{xx}(x, y, z) = -\int \frac{\partial}{\partial z} \tau_{zx}(x, y, z) dx - \int \frac{\partial}{\partial y} \tau_{xy}(x, y, z) dx \quad (51)$$

Using the linearity property of the integral operator we have:

$$\sigma_{xx}(x, y, z) = -\int \left(\frac{\partial \tau_{zx}}{\partial z} + \frac{\partial \tau_{xy}}{\partial y}\right) dx \quad (52)$$

From Equation (40), we have:

$$\sigma_{yy}(x, y, z) = \frac{\partial^2}{\partial x^2} \iint -\tau_{xy} dxdy + \frac{\partial^2}{\partial z^2} \iint -\tau_{yz} dydz \quad (53)$$

Simplifying,

$$\sigma_{yy}(x, y, z) = -\frac{\partial}{\partial x} \int \tau_{xy} dy - \frac{\partial}{\partial z} \int \tau_{yz} dy \quad (54)$$

Using the linearity property of the integral operator we have:

$$\sigma_{yy}(x, y, z) = -\int \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z}\right) dy \quad (55)$$

From Equation (46), we have:

$$\sigma_{zz}(x, y, z) = -\frac{\partial^2}{\partial y^2} \iint \tau_{yz} dydz - \frac{\partial^2}{\partial x^2} \iint \tau_{zx} dzdx \quad (56)$$

Simplifying,

$$\sigma_{zz}(x, y, z) = -\frac{\partial}{\partial y} \int \tau_{yz} dz - \frac{\partial}{\partial x} \int \tau_{zx} dz \quad (57)$$

Using the linearity property of the integral operator,

$$\sigma_{zz}(x, y, z) = -\int \left(\frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zx}}{\partial x}\right) dz \quad (58)$$

4.3 Proof that Beltrami-Michell's stress compatibility equations are satisfied by Maxwell's stress functions if the stress functions are harmonic (potential) functions

By substitution of the Maxwell's stress functions – Equations (34), (40) and (46) – into the Beltrami-Michell stress compatibility equations – Equations (7 – 13) – for the case where body forces are disregarded, we have:

$$(1+\mu)\nabla^2\left(\frac{\partial^2 \phi_{22}}{\partial z^2} + \frac{\partial^2 \phi_{33}}{\partial y^2}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \phi_{22}}{\partial z^2} + \frac{\partial^2 \phi_{33}}{\partial y^2} + \frac{\partial^2 \phi_{33}}{\partial x^2} + \frac{\partial^2 \phi_{11}}{\partial z^2} + \frac{\partial^2 \phi_{11}}{\partial y^2} + \frac{\partial^2 \phi_{22}}{\partial x^2}\right) = 0 \quad (59)$$

Simplifying,

$$(1+\mu)\nabla^2\left(\frac{\partial^2 \phi_{33}}{\partial y^2} + \frac{\partial^2 \phi_{22}}{\partial z^2}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \phi_{22}}{\partial z^2} + \frac{\partial^2 \phi_{33}}{\partial y^2} + \frac{\partial^2 \phi_{33}}{\partial x^2} + \frac{\partial^2 \phi_{11}}{\partial z^2} + \frac{\partial^2 \phi_{11}}{\partial y^2} + \frac{\partial^2 \phi_{22}}{\partial x^2}\right) = 0 \quad (60)$$

Simplifying,

$$(1+\mu)\nabla^2\left(\frac{\partial^2 \phi_{33}}{\partial y^2} + \frac{\partial^2 \phi_{22}}{\partial z^2}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \phi_{11}}{\partial y^2} + \frac{\partial^2 \phi_{11}}{\partial z^2}\right.$$

$$+\frac{\partial^2\phi_{22}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial z^2}+\frac{\partial^2\phi_{33}}{\partial x^2}+\frac{\partial^2\phi_{33}}{\partial y^2}\Bigg)=0 \quad (61)$$

Simplifying,

$$(1+\mu)\nabla^2\left(\frac{\partial^2\phi_{33}}{\partial y^2}+\frac{\partial^2\phi_{22}}{\partial z^2}\right)+\frac{\partial^2}{\partial x^2}\Bigg(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{11}}{\partial y^2}+\frac{\partial^2\phi_{11}}{\partial z^2}+\frac{\partial^2\phi_{22}}{\partial x^2}$$

$$+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{22}}{\partial z^2}+\frac{\partial^2\phi_{33}}{\partial x^2}+\frac{\partial^2\phi_{33}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}$$

$$-\frac{\partial^2\phi_{11}}{\partial x^2}-\frac{\partial^2\phi_{22}}{\partial y^2}-\frac{\partial^2\phi_{33}}{\partial z^2}\Bigg)=0 \quad (62)$$

Simplifying,

$$(1+\mu)\nabla^2\left(\frac{\partial^2\phi_{33}}{\partial y^2}+\frac{\partial^2\phi_{22}}{\partial z^2}\right)+\frac{\partial^2}{\partial x^2}\Big(\nabla^2\phi_{11}+\nabla^2\phi_{22}+\nabla^2\phi_{33}$$

$$-\frac{\partial^2\phi_{11}}{\partial x^2}-\frac{\partial^2\phi_{22}}{\partial y^2}-\frac{\partial^2\phi_{33}}{\partial z^2}\Big)=0 \quad (63)$$

Simplifying,

$$(1+\mu)\nabla^2\left(\frac{\partial^2\phi_{33}}{\partial y^2}+\frac{\partial^2\phi_{22}}{\partial z^2}\right)+\frac{\partial^2}{\partial x^2}\Big(\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$-\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Big)=0 \quad (64)$$

Simplifying,

$$(1+\mu)\left(\frac{\partial^2}{\partial y^2}\nabla^2\phi_{33}+\frac{\partial^2}{\partial z^2}\nabla^2\phi_{22}\right)+\frac{\partial^2}{\partial x^2}\Big(\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$-\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Big)=0 \quad (65)$$

This Equation (65) is satisfied only if:

$$\nabla^2\phi_{11}=0 \quad (66)$$

$$\nabla^2\phi_{22}=0 \quad (67)$$

$$\nabla^2\phi_{33}=0 \quad (68)$$

Similarly, Equation (8), upon substitution of the Maxwell stress functions becomes:

$$(1+\mu)\nabla^2\left(\frac{\partial^2\phi_{11}}{\partial z^2}+\frac{\partial^2\phi_{33}}{\partial x^2}\right)+\frac{\partial^2}{\partial y^2}\Big(\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$-\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Big)=0 \quad (69)$$

Equation (9), after substitution of the Maxwell stress functions becomes:

$$(1+\mu)\nabla^2\left(\frac{\partial^2\phi_{22}}{\partial x^2}+\frac{\partial^2\phi_{11}}{\partial y^2}\right)+\frac{\partial^2}{\partial z^2}\Big(\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$-\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Big)=0 \quad (70)$$

So, the equations are satisfied if:

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)$$

$$+\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)=0 \quad (71)$$

Expanding,

$$\frac{\partial^4\phi_{11}}{\partial x^4}+\frac{\partial^4\phi_{22}}{\partial x^2\partial y^2}+\frac{\partial^4\phi_{33}}{\partial x^2\partial z^2}+\frac{\partial^4\phi_{11}}{\partial y^2\partial x^2}+\frac{\partial^4\phi_{22}}{\partial y^4}+\frac{\partial^4\phi_{33}}{\partial y^2\partial z^2}$$

$$+\frac{\partial^4\phi_{11}}{\partial z^2\partial x^2}+\frac{\partial^4\phi_{22}}{\partial z^2\partial y^2}+\frac{\partial^4\phi_{33}}{\partial z^4}=0 \quad (72)$$

From Equation (72), we obtain Equations (73) – (75) as follows:

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{11}}{\partial y^2}+\frac{\partial^2\phi_{11}}{\partial z^2}\right)=0 \quad (73)$$

$$\frac{\partial^2}{\partial y^2}\left(\nabla^2\phi_{22}\right)=0 \quad (74)$$

$$\frac{\partial^2}{\partial z^2}\left(\nabla^2\phi_{33}\right)=0 \quad (75)$$

Equation (10) is expressed in terms of the Maxwell stress functions as:

$$(1+\mu)\nabla^2\left(-\frac{\partial^2\phi_{33}}{\partial x\partial y}\right)+\frac{\partial^2}{\partial x\partial y}\Big(\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$-\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Big)=0 \quad (76)$$

Simplifying,

$$\frac{\partial^2}{\partial x\partial y}\Big(-(1+\mu)\nabla^2\phi_{33}+\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$-\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Big)=0 \quad (77)$$

Alternatively,

$$\frac{\partial^2}{\partial x\partial y}\Big((1+\mu)\nabla^2\phi_{33}-\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$+\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Big)=0 \quad (78)$$

Equation (11) is given in terms of the Maxwell stress functions as:

$$(1+\mu)\nabla^2\left(-\frac{\partial^2\phi_{11}}{\partial y\partial z}\right)+\frac{\partial^2}{\partial y\partial z}\Big(\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$-\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Big)=0 \quad (79)$$

Simplifying,

$$\frac{\partial^2}{\partial y\partial z}\Big(-(1+\mu)\nabla^2\phi_{11}+\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$-\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Big)=0 \quad (80)$$

or

$$\frac{\partial^2}{\partial y\partial z}\Big((1+\mu)\nabla^2\phi_{11}-\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$+\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Bigg)=0 \quad (81)$$

Equation (12) is given in terms of the Maxwell stress functions as:

$$(1+\mu)\nabla^2\left(-\frac{\partial^2\phi_{22}}{\partial z\partial x}\right)+\frac{\partial^2}{\partial z\partial x}\Bigg(\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$-\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Bigg)=0 \quad (82)$$

Simplifying,

$$\frac{\partial^2}{\partial z\partial x}\Bigg(-(1+\mu)\nabla^2\phi_{22}+\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$-\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Bigg)=0 \quad (83)$$

or

$$\frac{\partial^2}{\partial z\partial x}\Bigg((1+\mu)\nabla^2\phi_{22}-\nabla^2(\phi_{11}+\phi_{22}+\phi_{33})$$

$$+\left(\frac{\partial^2\phi_{11}}{\partial x^2}+\frac{\partial^2\phi_{22}}{\partial y^2}+\frac{\partial^2\phi_{33}}{\partial z^2}\right)\Bigg)=0 \quad (84)$$

The conditions for the Beltrami-Michell stress compatibility equations to be satisfied is that the Maxwell stress functions are harmonic.

4.4 Airy's stress functions as a special case of Maxwell's stress functions

Let the Maxwell's stress functions, $\phi_{11}=0$, $\phi_{22}=0$, and $\phi_{33}(x,y)$ where ϕ_{33} does not depend on the z Cartesian coordinate variable, then the Maxwell's stress functions Equations (26-31) reduce to:

$$\sigma_{xx}=\frac{\partial^2}{\partial y^2}\phi_{33}(x,y) \quad (85)$$

$$\sigma_{yy}=\frac{\partial^2}{\partial x^2}\phi_{33}(x,y) \quad (86)$$

$$\sigma_{zz}=0 \quad (87)$$

$$\tau_{yz}=0 \quad (88)$$

$$\tau_{zx}=0 \quad (89)$$

$$\tau_{xy}=-\frac{\partial^2\phi_{33}}{\partial x\partial y} \quad (90)$$

The resulting stress function $\phi_{33}(x,y)$ is identified as the Airy's stress potential function in the two dimensional (plane) elasticity problem. We thus observe that the Airy's stress harmonic functions are particular (special) cases of Maxwell's stress functions when $\phi_{11}=\phi_{22}=0$, and $\phi_{33}(x,y)$ for Airy's stress functions of 2D elasticity problems on the x, y Cartesian coordinate plane.

4.5 Applications of the 2D Maxwell's stress potential functions in solving 2D elasticity problems (Airy's stress potential functions)

4.5.1 Stress analysis for a cantilever beam having a rectangular cross-section of unit width and depth 2*h* subject to a point load *Q* at the free end

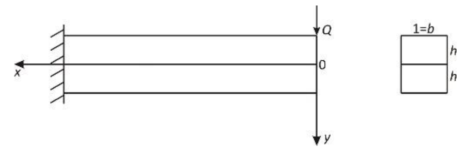

Figure 4.5.1: Cantilever beam of rectangular cross-section ($b\times 2h$) subject to a point load Q at the free end

The origin of coordinates is chosen as shown in Figure 4.5.1. The boundary conditions are:

(i) $\tau_{xy}(xy=\pm h)=0$ (91)

(ii) The resultant shear force at a section is $-Q$ or

$$\int_{-h}^{h}\tau_{xy}dy+Q=0 \quad (92)$$

or,

$$\int_{-h}^{h}\tau_{xy}dy=-Q \quad (93)$$

(iii) $\sigma_{yy}(x,y)=0$ for all x (94)

(iv) τ_{xy} will not vary with x.

The problem is solved by finding a suitable 2D Maxwell stress potential function (Airy's stress potential function) $\phi(x,y)$ that satisfies all the boundary conditions.

A suitable biharmonic stress potential function is chosen in polynomial form as:

$$\phi(x,y)=c_1xy+c_2xy^3 \quad (95)$$

where c_1 and c_2 are constants to be determined.

Then, the stresses are found as:

$$\sigma_{xx}=\frac{\partial^2\phi}{\partial y^2}=\frac{\partial^2}{\partial y^2}(c_1xy+c_2xy^3)=6c_2xy \quad (96)$$

$$\sigma_{yy}=\frac{\partial^2\phi}{\partial x^2}=\frac{\partial^2}{\partial x^2}(c_1xy+c_2xy^3)=0 \quad (97)$$

$$\tau_{xy}=\frac{-\partial^2\phi}{\partial x\partial y}=\frac{-\partial^2}{\partial x\partial y}(c_1xy+c_2xy^3)=-(c_1+3c_2y^2) \quad (98)$$

Using the boundary condition Equation (91) in Equation (98), we obtain:

$$-(c_1+3c_2h^2)=0 \quad (99)$$

$$c_2=-\left(\frac{c_1}{3h^2}\right) \quad (100)$$

Hence, the stress function is expressed in terms of one unknown constant as:

$$\phi(x,y)=c_1xy-\frac{c_1}{3h^2}xy^3=c_1\left(xy-\frac{xy^3}{3h^2}\right) \quad (101)$$

τ_{xy} then becomes

$$\tau_{xy}=-\left(c_1+3\left(\frac{-c_1}{3h^2}\right)y^2\right)=c_1\left(\frac{y^2}{h^2}-1\right) \quad (102)$$

Using the boundary condition Equation (93), we obtain:

$$\int_{-h}^{h}c_1\left(\frac{y^2}{h^2}-1\right)dy=-Q \quad (103)$$

$$\left[\frac{c_1}{h^2}\frac{y^3}{3}-c_1 y\right]_{-h}^{h}=-Q \quad (104)$$

$$c_1\left(-\frac{4h}{3}\right)=-Q \quad (105)$$

$$c_1=\frac{3Q}{4h} \quad (106)$$

Hence, we obtain the stress function as:

$$\phi(x,y)=\frac{3Q}{4h}xy+\frac{3Q}{12h^3}xy^3 \quad (107)$$

The stress field components become:

$$\sigma_{xx}(x,y)=\frac{\partial^2\phi}{\partial y^2}=\frac{-3Qxy}{2h^3}=\frac{-Qxy}{I} \quad (108)$$

$$=\frac{My}{I} \quad (109)$$

where

$$I=\frac{2bh^3}{3}=\frac{2h^3}{3} \quad (110)$$

$$\sigma_{yy}(x,y)=\frac{\partial^2\phi}{\partial x^2}(x,y)=\frac{\partial^2}{\partial x^2}\left(\frac{3Qxy}{4h}+\frac{3Qxy^3}{12h^3}\right)=0 \quad (111)$$

$$\tau_{xy}(x,y)=\frac{-\partial^2\phi(x,y)}{\partial x\partial y}=\frac{-\partial^2}{\partial x\partial y}\left(\frac{3Qxy}{4h}+\frac{3Qxy^3}{12h^3}\right)$$

$$=\frac{-3Q}{4h}\left(1-\frac{y^2}{h^2}\right) \quad (112)$$

$$\tau_{xy}(x,y)=\frac{-3Q}{4h}\left(\frac{h^2-y^2}{h^2}\right)=\frac{-3Q}{4h^3}(h^2-y^2)$$

$$\tau_{xy}=\frac{-Q}{2I}(h^2-y^2) \quad (113)$$

4.5.2 Stress analysis for a simply supported beam with a rectangular cross-section subject to a uniformly distributed load of intensity q(kN/m^2)

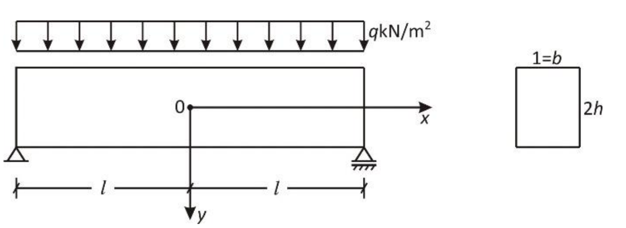

Figure 4.5.2: Simply supported beam with a rectangular cross-section subject to a uniformly distributed load of intensity q(kN/m^2)

A simply supported beam as shown in Figure 4.5.2, of span $2l$ with a rectangular cross-section of unit width end depth $2h$ carrying a uniformly distributed load of intensity q kN/m^2 is considered. The origin of coordinates is chosen at the centre of the beam as shown in Figure 4.5.2. The boundary conditions are as follows:

The loading conditions at the lower and upper edges of the beam are:

$$\tau_{xy}(x,y=\pm h)=0 \quad (114)$$

$$\sigma_{yy}(x,y=h)=0 \quad (115)$$

$$\sigma_{yy}(x,y=-h)=-q \quad (116)$$

at the ends, $x=\pm l$,

$$\int_{-h}^{h}\tau_{xy}(x=\pm l,y)dy=\mp q,l \quad (117)$$

$$\int_{-h}^{h}\sigma_{xx}(x=\pm l,y)dy=0 \quad (118)$$

$$M_{xx}(x=\pm l)=\int_{-h}^{h}\sigma_{xx}(x=\pm l,y)ydy=0 \quad (119)$$

A suitable biharmonic stress function can be considered in polynomial form as follows:

$$\phi(x,y)=a_1\frac{x^2}{2}+\frac{a_2}{2}x^2y+\frac{a_3}{6}y^3+\frac{a_4}{6}x^2y^3-\frac{a_4}{30}y^5 \quad (120)$$

where a_1, a_2, a_3 and a_4 are the four unknown constants of the stress function which are found using the boundary conditions.

The stresses are found from $\phi(x,y)$ as follows:

$$\sigma_{xx}(x,y)=\frac{\partial^2\phi}{\partial y^2}=a_3y+a_4\left(x^2y-\frac{2}{3}y^3\right) \quad (121)$$

$$\sigma_{yy}(x,y)=\frac{\partial^2\phi}{\partial x^2}=a_1+a_2y+\frac{a_4}{3}y^3 \quad (122)$$

$$\tau_{xy}=\frac{-\partial^2\phi}{\partial x\partial y}=-a_2x-a_4xy^2 \quad (123)$$

Using the boundary conditions, we obtain: using Equation (114),

$$\tau_{xy}(x,y=\pm h)=-a_2x-a_4xh^2=0 \quad (124)$$

Solving, we have:

$$a_2+a_4h^2=0 \quad (125)$$

$$\sigma_{yy}(x,y=+h)=a_1+a_2h+\frac{a_4h^3}{3}=0 \quad (126)$$

$$\sigma_{yy}(x,y=-h)=a_1-a_2h-\frac{a_4h^3}{3}=-q$$

(127)

From Equations (126) and (127) we obtain:

$$2a_1=-q \quad (128)$$

$$a_1=-\frac{q}{2} \quad (129)$$

Then, Equations (126) and (127) become:

$$a_2h+\frac{a_4h^3}{3}=-a_1=\frac{q}{2} \quad (130)$$

$$-a_2h-\frac{a_4h^3}{3}=-q-a_1=-q+\frac{q}{2}=\frac{-q}{2} \quad (131)$$

From Equation (125)

$$a_2=-a_4h^2 \quad (132)$$

Then from Equation (130), we have:

$$-a_4h^2h+\frac{a_4}{3}h^3=\frac{q}{2} \quad (133)$$

$$-a_4h^3 + \frac{a_4h^3}{3} = \frac{-2a_4h^3}{3} = \frac{q}{2} \quad (134)$$

$$a_4 = \frac{-3q}{2(2h^3)} = \frac{-3q}{4h^3} \quad (135)$$

$$a_2 = -a_4h^2 = \frac{3q}{4h} \quad (136)$$

Then the stresses become:

$$\sigma_{yy}(x, y) = \frac{-q}{2} + \frac{3q}{4h}y - \frac{q}{4h^3}y^3 \quad (137)$$

$$\tau_{xy}(x, y) = \frac{-3q}{4h}x + \frac{3q}{4h^3}xy^2 \quad (138)$$

$$\sigma_{xx}(x, y) = a_3y - \frac{3q}{4h^3}\left(x^2y - \frac{2}{3}y^3\right) \quad (139)$$

Substitution of Equation (139) into the boundary condition Equation (119) yields:

$$\int_{-h}^{h} \sigma_{xx}(x = \pm l, y)y\,dy = \int_{-h}^{h}\left(a_3y + a_4\left(l^2y - \frac{2}{3}y^3\right)\right)y\,dy = 0 \quad (140)$$

$$\int_{-h}^{h} a_3y^2 + a_4\left(l^2y^2 - \frac{2}{3}y^4\right)dy = 0 \quad (141)$$

$$\left[\frac{a_3y^3}{3} + a_4\left(\frac{l^2y^3}{3} - \frac{2}{3}\frac{y^5}{5}\right)\right]_{-h}^{h} = 0 \quad (142)$$

$$\left(\frac{a_3h^3}{3}+a_4\left(\frac{l^2h^3}{3}-\frac{2h^5}{15}\right)\right)-\left(\frac{-a_3h^3}{3}+a_4\left(\frac{-l^2h^3}{3}+\frac{2h^5}{15}\right)\right)=0 \quad (143)$$

$$\frac{a_3h^3}{3} + a_4\left(\frac{l^2h^3}{3} - \frac{2h^5}{15}\right) + \frac{a_3h^3}{3} + a_4\left(\frac{l^2h^3}{3} - \frac{2h^5}{15}\right) = 0 \quad (144)$$

$$\frac{2a_3h^3}{3} + 2a_4\left(\frac{l^2h^3}{3} - \frac{2h^5}{15}\right) = 0 \quad (145)$$

$$a_3 = -2a_4\left(\frac{l^2h^3}{3} - \frac{2h^5}{15}\right)\cdot\frac{3}{2h^3} \quad (146)$$

$$a_3 = -a_4\left(\frac{l^2h^3}{3} - \frac{2h^5}{15}\right)\frac{3}{h^3} = \frac{3q}{4h^3}\left(l^2 - \frac{2h^2}{5}\right)$$

$$a_3 = \frac{3q}{4h}\left(\frac{l^2}{h^2} - \frac{2}{5}\right) \quad (147)$$

Hence,

$$\sigma_{xx}(x, y) = \frac{3q}{4h}\left(\frac{l^2}{h^2} - \frac{2}{5}\right)y - \frac{3q}{4h^3}\left(x^2y - \frac{2}{3}y^3\right)$$

$$= \frac{3q}{4h^3}l^2y - \frac{3q}{10h}y - \frac{3q}{4h^3}x^2y + \frac{q}{2h^3}y^3 \quad (148)$$

$$\sigma_{xx}(x, y) = \frac{q}{2I}(l^2 - x^2)y + \frac{q}{I}\left(\frac{y^3}{3} - \frac{h^2}{5}y\right) \quad (149)$$

where

$$I = \frac{2}{3}h^3 \quad (150)$$

5 Discussion

In this study, Maxwell's stress functions have been derived from first principles, such that they satisfy all the differential equations of equilibrium when body forces are ignored, and such that the stress fields could be derived from them. It is observed that the Maxwell stress functions are special cases of Beltrami stress functions when $\phi_{12}(x, y, z)=\phi_{23}(x, y, z)=\phi_{31}(x, y, z)=0$. It was also proved that by integration of Equations (26) – (28) and (34), (40) and (46), that the Maxwell stress functions all satisfy the differential equations of equilibrium when body forces are disregarded. Finally it is proved that the conditions for the Maxwell stress functions to satisfy the stress compatibility equations are that all the Maxwell stress functions are harmonic functions. It is further shown that the Maxwell stress functions reduce to the Airy stress potential functions when the problem reduces to two dimensions.

6 Conclusions

The following conclusions are drawn:

(i) Maxwell's stress functions satisfy all the differential equations of equilibrium in three dimensions in the absence of body force components.

(ii) The normal and shear stress fields are derivable from the three Maxwell's stress functions.

(iii) The Maxwell stress functions satisfy the Beltrami-Michell stress compatibility equations if they are harmonic (potential) functions.

(iv) The Maxwell stress functions are particular cases of the Beltrami-Michell stress potential functions.

(v) The Maxwell stress functions simplify to the Airy stress function when the elasticity problem reduces from a 3D problem to a 2D problem

(vi) The Airy stress potential function is thus a special case of the Maxwell stress potential function for applications to 2D problems of the theory of elasticity.

Nomenclature

σ_{xx}, σ_{yy}, σ_{zz}	normal stress field components
τ_{xy}, τ_{yz}, τ_{xz}	shear stress field components
F_x, F_y, F_z	body force components in the x, y, and z Cartesian coordinate directions
μ	Poisson's ratio
ϕ_{11}, ϕ_{12}, ϕ_{13}, ϕ_{22}, ϕ_{23}, ϕ_{31}, ϕ_{32}, ϕ_{33}	Beltrami-Michell stress functions
ϕ_{11}, ϕ_{22}, ϕ_{33}	Maxwell stress functions
$2h$	depth of beam considered in the work
Q	point load acting at the free end of a cantilever beam
x, y, z	Cartesian coordinates
b	width of a beam's cross-section
c_1, c_2	constants of integration or
a_1, a_2, a_3, a_4	unknown parameters of the stress function
I	moment of inertia
q	intensity of uniformly distributed load on a simply supported beam
$2l$	span of simply supported beam considered in the study
2D	two dimensional
3D	three dimensional
∇^2	Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$\frac{\partial}{\partial x}$ partial differential operator, partial derivative with respect to x

$\int$ integral operator, integration sign

$\iint$ double integral

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