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Solving single phase fluid flow instability equations using Chebyshev Tau- QZ polynomial

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ABSTRACT

In this article the instability of single phase flow in a circular pipe from laminar to turbulence regime has been investigated. To this end, after finding boundary conditions and equation related to instability of flow in cylindrical coordination system, which is called eigenvalue Orr Sommerfeld equation, the solution method for these equation has been investigated. In this article Chebyshev polynomial Tau-QZ algorithm has been selected for the solution technique to solve the Orr Sommerfeld equation because in this method some of complex terms in the instability equation in cylindrical coordination will be appeared. After finding Orr Sommerfeld parameters related to Chebyshev polynomial Tau-QZ algorithm the solution have been done for Re=5000 and Re=1000, then the results had been compared with the results of valid references where other methods had been used in them. It have been observed that the use of Chebyshev Tau-QZ algorithm has higher accuracy concerning the results and it also has a higher accurate technique to solve the Orr Sommerfeld instability equations in cylindrical coordination system.

1. Introduction

Pipes are used for transmission of liquid, gas fluids or multiphase fluids in all industries. The investigations of flow instability inside pipes is one of the challenging issues in fluid mechanics. Pipe poiseuille flow at the first has been studied experimentally by Osborne Reynolds at the end of the 19th century [1]. He found that the sensitivity to disturbances could be characterized by one non-dimensional number Re (Reynolds number). For Reynolds numbers lower than about 2000, and more than 2300 it is observable that the flow is stable to all disturbances. Laminar flow could be maintained at higher Reynolds numbers by carefully controlling the external disturbances of the flow. The laminar flow will be unstable by increasing the Reynolds number in transition zone because of flow disturbances [2]. Mechanism of flow instability and transition to turbulence and physics of flow instability and turbulent transition in shear flows had been studied by Hua-Shu Dou [3, 4]. Experimental studies of transition to turbulence in a pipe had been done by Mullin [5] and background information as well as more details may be found in the recent reviews Kerswell [6] and Eckhardt [7]. In this study after obtaining the equations related to instability of single phase fluid flow inside horizontal pipe, these equations will be solved. To solve these equations, the Eigenvalue equations related to instability, several methods have been used by the researchers with special strength and weakness. In order to simplify the numerical solving, we imagine that the fluid flow inside the pipe is symmetrically axial. After obtaining the equations related to instability of flow and boundary conditions governing it, the equations will be solved using Chebyshev Tau-QZ algorithm and the obtained results will be validated.

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2. Governing equations and numerical method

2.1. Obtaining equations governing the stability of flow inside the pipe

In order to obtain the governing equations for the instability of single-phase flow in the horizontal pipe with fully developed unidirectional flow regime and axial symmetric flow ($v_\theta = 0$), the continuity and momentum equations in cylindrical coordination system (r, θ, x) has been illustrated in Fig.1 as follow [8]:

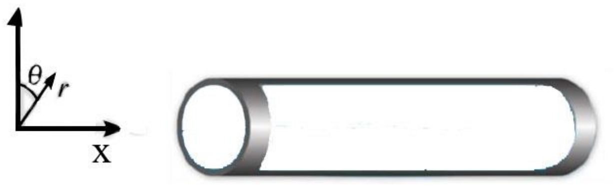

Figure 1. The components in cylindrical coordination system

Continuity equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right) + \frac{\partial u}{\partial x} = 0 \tag{1}$$

Momentum equations:

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + u\frac{\partial v_r}{\partial x} = -\frac{\partial P}{\partial r} + \upsilon\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right)\right) + \frac{\partial^2 v_r}{\partial x^2}\right) \tag{2}$$

and

$$\frac{\partial u}{\partial t} + v_r\frac{\partial v_r}{\partial r} + u\frac{\partial u}{\partial x} = -\frac{\partial P}{\partial x} + \upsilon\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial x^2}\right) \tag{3}$$

With consideration of disturbances are related to pressure and velocity according to Mellibovsky, Fernando, et al. [9], the terms related to these disturbances will be defined as follow:

$$\begin{aligned} v_r &= v_r' \\ u &= U + u' \\ P &= P(r) + p' \end{aligned} \tag{4}$$

Substituting equations (4) in (1), (2) and (3) and regardless of high order terms, we will have:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r'\right) + \frac{\partial u'}{\partial x} = 0 \tag{5}$$

and

$$\frac{\partial v_r'}{\partial t} + U\frac{\partial v_r'}{\partial x} = -\frac{\partial p'}{\partial r} + \upsilon\left(\frac{\partial^2 v_r'}{\partial r^2} + \frac{1}{r}\frac{\partial v_r'}{\partial r} - \frac{v_r'}{r} + \frac{\partial^2 v_r'}{\partial \theta^2}\right) \tag{6}$$

moreover

$$\frac{\partial u'}{\partial t} + v_r'\frac{\partial U}{\partial r} + U\frac{\partial u'}{\partial x} = -\frac{\partial p'}{\partial x} + \upsilon\left(\frac{1}{r}\frac{\partial u'}{\partial r} + \frac{\partial^2 u'}{\partial r^2} - \frac{\partial^2 u'}{\partial x^2}\right) \tag{7}$$

Now, using flow function, we will define the velocity fluctuations as below:

$$u' = \frac{1}{r}\frac{\partial \psi'}{\partial r} \tag{8}$$

$$v_r' = -\frac{1}{r}\frac{\partial \psi'}{\partial x} \tag{9}$$

Which ψ' is stream function. By substituting equations (8) and (9) in equations (6) and (7), we have:

$$\begin{aligned} &-\frac{1}{r}\frac{\partial^2\psi'}{\partial x\partial t} - U\frac{1}{r}\frac{\partial^2\psi'}{\partial x^2} = -\frac{1}{\rho}\frac{\partial p'}{\partial r} + \upsilon(-\frac{2}{r^3}\frac{\partial\psi'}{\partial x} + \frac{2}{r^2}\frac{\partial^2\psi'}{\partial x\partial r} \\ &-\frac{1}{r}\frac{\partial^3\psi'}{\partial x\partial r^2} + \frac{1}{r^3}\frac{\partial\psi'}{\partial x} - \frac{1}{r^2}\frac{\partial^2\psi'}{\partial x\partial r} + \frac{1}{r^2}\frac{\partial\psi'}{\partial x} - \frac{1}{r}\frac{\partial^3\psi'}{\partial x^3}) \end{aligned} \tag{10}$$

$$\begin{aligned} &\frac{1}{r}\frac{\partial^2\psi'}{\partial x\partial t} - \frac{1}{r}\frac{\partial\psi'}{\partial x}\frac{\partial U}{\partial r} + \frac{U}{r}\frac{\partial^2\psi'}{\partial x\partial r} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \upsilon(-\frac{1}{r^3}\frac{\partial\psi'}{\partial r} \\ &+\frac{1}{r^2}\frac{\partial^2\psi'}{\partial r^2} + \frac{2}{r^3}\frac{\partial\psi'}{\partial r} - \frac{2}{r^2}\frac{\partial^2\psi'}{\partial r^2} + \frac{1}{r}\frac{\partial^3\psi'}{\partial r^3} + \frac{1}{r}\frac{\partial^3\psi'}{\partial r\partial x^2}) \end{aligned} \tag{11}$$

In above equations, the terms related to pressure should be eliminated. Thus, one can write:

$$\begin{aligned} &\frac{1}{\rho}\frac{\partial p'}{\partial r} = \frac{1}{r}\frac{\partial^2\psi'}{\partial x\partial t} + \frac{U}{r}\frac{\partial^2\psi'}{\partial x^2} + \upsilon(-\frac{2}{r^3}\frac{\partial\psi'}{\partial x} \\ &+\frac{2}{r^2}\frac{\partial^2\psi'}{\partial x\partial r} - \frac{1}{r}\frac{\partial^3\psi'}{\partial x\partial r^2} + \frac{1}{r^3}\frac{\partial\psi'}{\partial x} - \frac{1}{r^2}\frac{\partial^2\psi'}{\partial x\partial r} \\ &+\frac{1}{r^2}\frac{\partial\psi'}{\partial x} - \frac{1}{r}\frac{\partial^3\psi'}{\partial x^3}) \end{aligned} \tag{12}$$

$$\begin{aligned} &\frac{1}{\rho}\frac{\partial p'}{\partial x} = -\frac{1}{r}\frac{\partial^2\psi'}{\partial r\partial t} + \frac{1}{r}\frac{\partial\psi'}{\partial x}\frac{\partial U}{\partial r} + \frac{U}{r}\frac{\partial^2\psi'}{\partial x\partial r} + (-\frac{1}{r^3}\frac{\partial\psi'}{\partial r^2} \\ &-\frac{1}{r^2}\frac{\partial^2\psi'}{\partial r^2} + \frac{2}{r^3}\frac{\partial\psi'}{\partial r} + \frac{1}{r}\frac{\partial^3\psi'}{\partial r^3} + \frac{1}{r}\frac{\partial^3\psi'}{\partial r\partial x^2}) \end{aligned} \tag{13}$$

Now, if we obtain derivation of equation (12) to x and derivation from equation (13) to r than put the right hand sides of equations in equal, the obtained an equation which is independent of pressure term. By defining stream function ψ' as disturb term [10]:

$$\psi'(x, r, t) = \varphi(r)\exp\left(i\alpha\left(x - ct\right)\right) \tag{14}$$

where α is a real wave number and c is the complex wave velocity $c = c_r + ic_i$. The real part of c gives the phase velocity of the wave, while the imaginary part of αc represents the growth rate of disturbances (for unstable flows $Im(\alpha c) > 0$). By subsiding equation (14) in the equation obtained from removing of pressure term in equations (12) and (13), it could be shown that $\varphi(r)$ function will be true in the following equation [11]:

$$(ia\,\mathrm{Re})^{-1}(L - a^2)^2\varphi = (U - c)(L - a^2)\varphi - r\left(\frac{U'}{x}\right)'\varphi \tag{15}$$

where:

$$L = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}$$

Equation (15) is similar to Orr-Somerfeld equation and this equation is instability equation for the flow in the pipe. This equation is valid for the fully developed flows. In the center of pipe ($r = 0$), equation (15) has a singular point. Thus, we should confining $\frac{\varphi}{r}$ and $\frac{\varphi'}{r}$ terms at $r \longrightarrow 0$. It is provable that velocity profile in the circular pipe is in the form of:

$$U(r) = 1 - r^2 \qquad (0 < r < 1) \tag{16}$$

Thus, the last term of equation (15) will be zero $(\frac{U'}{r})' = 0$, and the boundary conditions will be defined according to Sexl [10] as follow:

$$\varphi, \varphi' = 0 \qquad r = 0,1 \tag{17}$$

In this study, in order to solve equation (15) and obtain eigenvalue c, Chebyshev Tau – QZ algorithm has been used. The reason for using this method is because it is simple method for solving the problems in cylindrical or circular geometries where terms such as $(\frac{m}{r})\frac{d}{dr}$ exist [12].

2.2 Chebyshev Tau – QZ solution algorithm for solving O-S equation

According to Garder et al. [13] method, we consider:

$$\begin{aligned} L\varphi &= D^4\varphi - 2a^2D^2\varphi - ia\,\mathrm{Re}(U-c)(D^2-a^2)\varphi \\ &+ (a^4 - ia\,\mathrm{Re}U'')\varphi \end{aligned} \tag{18}$$

In order to solve O-S equation, concerning the existence of fourth order terms, we could write:

$$D^4\varphi = \sum_{i=0}^{N+4} \varphi_i^{(4)} T_i(r) \tag{19}$$

Where:

$$\begin{aligned} \varphi_i^{(4)} = \frac{1}{24C_i} \sum_{\substack{p=i+4 \\ p+i_{even}}}^{p=N+4} & p(p^2(p^2-4)^2 - 3p^4i^2 + 3p^2i^4 \\ & -i^2(i^2-4)^2)\varphi_p \end{aligned} \tag{20}$$

The above expression makes it possible for O-S equation to yield an equation for $\varphi_0, \dots, \varphi_{N+1}$ with multiplication by T_i to *N+1* system. To see this definition from differential operator, we have:

$$\begin{aligned} L\varphi &= D^4\varphi - 2a^2D^2\varphi - ia\,\mathrm{Re}(U-c)(D^2-a^2)\varphi \\ &+ (a^4 - ia\,\mathrm{Re}U'')\varphi \end{aligned} \tag{21}$$

In order to write the above equation for Chebyshev Tau – QZ algorithm, we write equation (21) as follow according to Fox [14]:

$$L\varphi = \tau_1 T_{N+1} + \tau_2 T_{N+2} + \tau_3 T_{N+3} + \tau_4 T_{N+4} \tag{22}$$

where τ_i denote tau coefficients. We take inner product of equation (22) by T_i for $i = 0, \dots, N$. Inner product with T_i for $i = N+1, \dots, N+4$ leads to four equations for Tau coefficients. Four remained condition are obtained from boundary conditions (17) and since $T_n'(\pm 1) = (\pm 1)^{n+1} n^2$, we have:

$$\sum_{i=0}^{N+4} (-1)^i \varphi_i = \sum_{i=0}^{N+4} \varphi_i = \sum_{i=0}^{N+4} (-1)^{i+1} i^2 \varphi_i = \sum_{i=0}^{N+4} i^i \varphi_i = 0 \tag{23}$$

Due to the way the terms split in the discretization of (15) for the condition where $U = 1 - r^2$, it is better the equations (23) will be written as follow:

$$\sum_{\substack{i=0 \\ i_{even}}}^{N+3} \varphi_i = 0, \quad \sum_{\substack{i=1 \\ i_{odd}}}^{N+4} \varphi_i = 0, \quad \sum_{\substack{i=1 \\ i_{odd}}}^{N+4} i^2 \varphi_i = 0, \quad \sum_{\substack{i=2 \\ i_{even}}}^{N+3} i^2 \varphi_i = 0 \tag{24}$$

In this condition, the obtained matrix can be divided into two sections including φ_i, i even and φ_i, i odd. Equation (24) for φ_{N+j}, $j = 1,2,3,4$ as linear combination of $\varphi_0, \dots, \varphi_N$ can be solved. The target is obtaining eigenvalues in form of $AX = \sigma BX$ which is a eigenvalue problem where $X = (\varphi_0, \dots, \varphi_N)$ and A is a D^2matrix and B is such that the results will be non-singular.

Since at D^2 condition in Chebyshev Tau – QZ algorithm for Orr-Somerfeld problems has high accurate results and no spurious eigenvalues, we could write O-S equation in two following:

$$\begin{aligned} L_1(\varphi, \chi) &= (D^2 - a^2)\varphi - \chi = 0 \\ L_2(\varphi, \chi) &= (D^2 - a^2)\chi - ia\,\mathrm{Re}(U-c)\chi + ia\,\mathrm{Re}U''\varphi = 0 \end{aligned} \tag{25}$$

By writing the equations as follow:

$$\begin{aligned} L_1(\varphi, \chi) &= \tau_1 T_{N+1} + \tau_2 T_{N+2} \\ L_2(\varphi, \chi) &= \tau_3 T_{N+1} + \tau_4 T_{N+2} \end{aligned} \tag{26}$$

where,

$$\begin{aligned} \varphi &= \sum_{i=0}^{N+2} \varphi_i T_i(r) \\ \chi &= \sum_{i=0}^{N+2} X_i T_i(z) \end{aligned} \tag{27}$$

and by multiplication of each of the above equations by T_i for $i = 0, \dots\dots, N$, equation (26) will be solved. The problem starts from the point where all boundary conditions are written based on φ_i and there are no constraints for χ_i so that rows of boundary conditions $\varphi_{N+1}, \varphi_{N+2}, \chi_{N+1}, \chi_{N+2}$obtained from the highly accurate results and no spurious eigenvalues result of the matrix D^2 could be eliminated.

The boundary conditions should be considered as a row inside the matrix which is done by Grander et al who obtained a system in form of four first order equations. A D^2 method for solving equations (15) and (17) is as follow:

$$X = (\varphi_0, \dots, \varphi_{N+2}, \chi_0, \dots, \chi_{N+2})^T \tag{28}$$

along with:

$$A_r = \begin{pmatrix} D^2 - a^2 I & -I \\ BC_1 & 0,\ldots,0 \\ BC_2 & 0,\ldots,0 \\ & \\ 0 & D^2 - a^2 I \\ BC_3 & 0,\ldots,0 \\ BC_4 & 0,\ldots,0 \end{pmatrix},$$

$$A_i = \begin{pmatrix} 0,\ldots,0 & 0,\ldots,0 \\ 0,\ldots,0 & 0,\ldots,0 \\ 0,\ldots,0 & 0,\ldots,0 \\ & \\ -2a\,\mathrm{Re}\, I & -2a\,\mathrm{Re}(P-I) \\ 0,\ldots,0 & 0,\ldots,0 \\ 0,\ldots,0 & 0,\ldots,0 \end{pmatrix}, \qquad (29)$$

$$B_r = 0,$$

$$B_i = \begin{pmatrix} 0 & 0 \\ & \\ 0 & -a\,\mathrm{Re}\, I \\ & 0 \\ & 0 \end{pmatrix}$$

where P is Chebyshev matrix shown by r^2 is $A = A_r + iA_i$ and $B = B_r + iB_i$ (P is a matrix that is obtained by writing $r^2 = \frac{1}{2}\left(1 + T^2(r)\right)$and then the inner product $(T_i, r^2\varphi)$ has been obtained, BC_1 and ….BC_4. Rows define boundary conditions on $\varphi_n$and for O-S equation. In this condition, O-S equation is written as four following equations:

$$\begin{aligned} &L_1 Y = D\varphi - \zeta = 0 \\ &L_2 Y = D\zeta - \eta = 0 \\ &L_3 Y = D\eta - \gamma = 0 \\ &L_4 Y = D\gamma - \left(2a^2 - ia\,\mathrm{Re}(U-c)\right)\eta \\ &+\left(a^4 + ia^3\,\mathrm{Re}(U-c) + ia\,\mathrm{Re}\,U''\right)\varphi = 0 \end{aligned} \qquad (30)$$

Where $L_i$specifies operators and is $Y = (\varphi, a, \beta, \gamma)$. In this case:

$$\begin{aligned} \varphi &= \sum_{i=0}^{N+1} \varphi_i T_i(r) \\ \zeta &= \sum_{i=0}^{N+1} \zeta_i T_i(r) \\ \eta &= \sum_{i=0}^{N+1} \eta_i T_i(r) \\ \gamma &= \sum_{i=0}^{N+1} \gamma_i T_i(r) \end{aligned} \qquad (31)$$

In following, the study of the obtained answers will be dealt with.

3. Numerical results

For the validation of numerical solution for single- phase fluid flow inside the axial symmetric fluid flow pipe with *Re=5000* and *Re=10000* with α=1 the results was compared by solution proposed by Davey-Darzin [15]. As it is observed, the obtained response for performed numerical solution on drawing the graph based on c_r, c_i is in full conformity with the above results proposed by Davey-Darzin.

As said before c is the complex wave velocity $c = c_r + ic_i$. The real part of c gives the phase velocity of the wave (which must be positive), while the imaginary part of αc represents the growth rate so if the Im(αc) is negative its means that the flow is stable.

Concerning the results obtained for numerical solution for *Re=5000* and α=1, it could be observed that in $-1 \leq c_i \leq -0.4$, the wave velocity $c_r$value will remain fixed. At $c_i < -1$ area, the $c_r$values in respect to c_i extend in right side with angle of 45° and in left side with angle of 30° and in $c_i > -0.3$, $c_r$values in respect to c_i grows in right side with angle of 60° and in right side with angle of 30°.

The obtained results of numerical solution for $Re = 10000$ and $\alpha = 1$, indicate that the obtained results from Davey and Darzin solution has dispersion of the results of solution; this is while the obtained solution in this problem conform to the real expected conditions. The same as the result obtained for *Re=5000*, it could be seen that in upper part of right side graph, it extends with angle of 45° and in left side with angle of 30°.

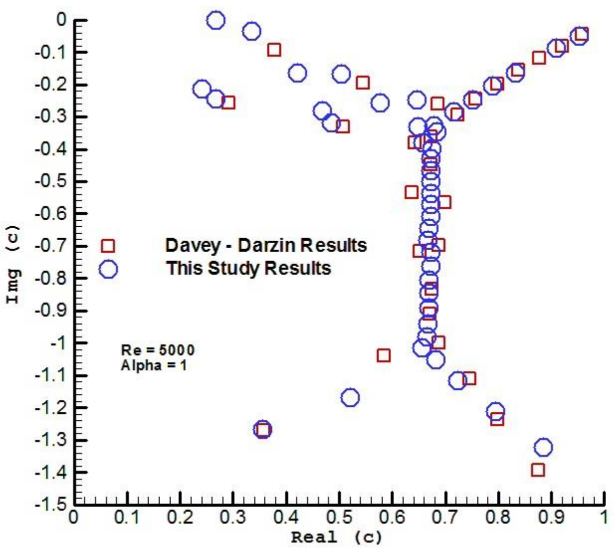

Fig 2. The comparison of numerical solution with Re=5000 and α= 1

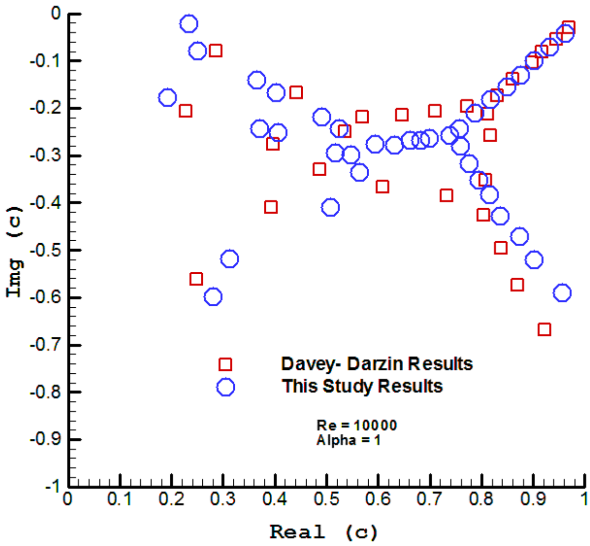

Fig 3. The comparison of numerical solution with Re=10000 and α= 1

4. Conclusion

The use of Chebyshev polynomial can be used as the appropriate solution for solving the Eigenvalue problem of stability equation of single-phase fluid flow instability inside the pipe. As could be seen in figures 2 and 3, the obtained results based on this solution have higher accuracy with compare to the results obtained in validation references.

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