JCAMECH

Vol. 50, No. 1, June 2019, pp 191-196
DOI: 10.22059/jcamech.2018.250335.233

Size dependent on vibration and flexural sensitivity of atomic force microscope cantilevers

Reza Javidi [a], Hamid Haghshenas Gorgani [b], Mohsen Mahdavi Adeli [c,]*

[a]School of Mechanical Engineering, College of Engineering, University of Tehran,Iran
[b]Engineering Graphics Center, Sharif University of Technology, Tehran, Iran
[c]Department of Mechanical Engineering, Sousangerd Branch, Islamic Azad University, Sousangerd, Iran

ARTICLE INFO

Article history:
Received: 15 January 2018
Accepted: 16 February 2018

Keywords:
Consistent couple stress theory
Atomic force microscope (AFM)
Euler–Bernoulli beam
Hamilton's principle
Sensitivity

ABSTRACT

In this paper, the free vibration behaviors and flexural sensitivity of atomic force microscope cantilevers with small-scale effects are investigated. To study the small-scale effects on natural frequencies and flexural sensitivity, the consistent couple stress theory is applied. In this theory, the couple stress is assumed skew-symmetric. Unlike the classical beam theory, the new model contains a material-length-scale parameter and can capture the size effect. For this purpose, the Euler–Bernoulli beam theory is used to develop the AFM cantilever. The tip interacts with the sample that is modeled by a spring with constant of. The equation of motion is obtained through a variational formulation based on Hamilton's principle. In addition, the analytical expressions for the natural frequency and sensitivity are also derived. At the end, some numerical results are selected to study the effects of material-length-scale parameter and dimensionless thickness on the natural frequency and flexural sensitivity.

1. INTRODUCTION

Micro and nano technologies include a wide range of advanced techniques used to fabricate and study artificial systems with dimensions ranging from several micrometers to a few nanometers [1, 2]. At nano and micro scales, size effects often become important parameters [3]. The results of both experimental and Molecular dynamics simulation have shown that the small-scale effects cannot be neglected in analyzing the mechanical properties of nano and microstructures, so the classical continuum theories become unusable in these scales [4-7]. Molecular dynamics simulation is a convenient method for simulating the mechanical behavior of small size structures; however, it is computationally expensive for structures with large number of atoms [8-16]. Thus, researchers were stimulated to develop higher-order continuum theories such as nonlocal theory [17-24], strain gradient theory [25] and etc. with the capability of predicting size effect by considering material length scale parameters. In 1960s, the couple stress theory introduced by Toupin [26], Mindlin and Tiersten [27], and Koiter [28], became a popular non-classical theory for analyzing micro and nano scale structures due to the higher predicted value of the stiffness of micro and nano scale structures compared to the classical theory. Free vibration of a C-CST Euler-Bernoulli nano-beams made of arbitrary bidirectional functionally graded materials have illustrated by Nejad et al. [11]. Farajpour et al. [29] have investigated the nonlinear buckling of magnetoelectro-elastic (MEE) hybrid nanoshells in thermal environment using a size-dependent continuum model. The stability analysis and vibration of rotating nanobeams under the effect of the compressive loading and a non-uniform magnetic field have investigated by Beghani et al. [30]. The influence of thermo-electro-mechanical loads on the free and forced vibration of a piezoelectric nanowire consideration of the framework of Timoshenko beam theory

[*]Corresponding author. E-mail address: mahdavi_mech_eng@yahoo.com

within the nonlocal elasticity theory is illustrated with a numerical solution procedure by Kordani et al. [31]. In this paper effect of surface piezoelectricity, surface elasticity and residual surface stress is investigated. An analytical approach to study buckling of bundles of MTs considering surface effects provided by Farajpour et al. [32]. Yang, Chong, Lam, & Tong [33] proposed the modified version of the couple stress theory and considered the couple stress tensor to be symmetric. In their theory, two higher-order material-length-scale parameters are introduced in addition to the two Lame constants. One of the superiority of this theory compared to other theories was that the four additional parameters in the micro polar theory and five additional parameters in the strain gradient theory were reduced to only two additional parameters in this theory. In recent years, this property has attracted researchers to derive formulations for the mechanical analysis of the micro-beams and micro-plates and investigate their mechanical behavior based on this theory. Investigations about the formulations and mechanical behavior have been presented in recent years based on the modified couple stress theory for homogeneous linear micro-beams [34, 35], homogenous nonlinear micro-beams [36-38], functionally graded linear micro-beams [39], functionally graded nonlinear micro-beams [40-42], linear micro-plates [43-45], nonlinear micro-plates [46, 47], composite laminated beams [48, 49]. Recently, by considering true continuum kinematical displacement and rotation, Hajesfandiari & Dargush [50] have demonstrated that the couple tensor is skew-symmetric and present the consistent couple stress theory by adopting the skew-symmetric part of the rotation gradients as the curvature tensor.

In this article, free vibration behaviors and flexural sensitivity of atomic force microscope cantilevers with small-scale effects are investigated. To study the small-scale effects on natural frequencies and flexural sensitivity, the consistent couple stress theory is applied.

2. ANALYSIS

A schematic diagram of an atomic force microscope probe cantilevered at one end is shown in Fig. 1. The cantilever has length L, thickness h and width b. A mass M is attached at the free end of the cantilever, which interacts with the sample by a spring constant k. Cartesian coordinates (x,y,z) are considered as presented in the figure.

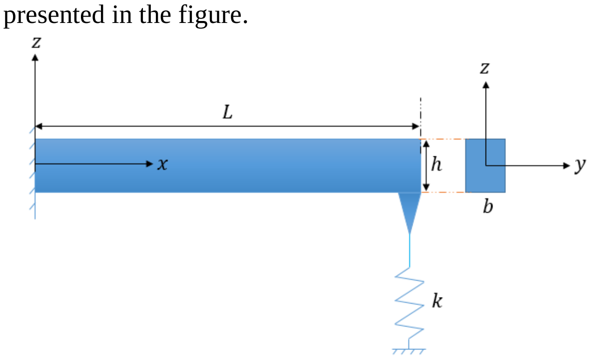

Fig. 1. Euler–Bernoulli micro-cantilever of an AFM probe with the end spring.

According to the consistent couple stress developed by Hadjesfandiari [50], first variation of the strain energy for an isotropic linear elastic material with volume Ω experiencing an infinitesimal displacement is defined as:

$$\delta U = \int_{\Omega} \left(\sigma_{ij} \delta e_{ij} + m_{ij} \delta \kappa_{ji} \right) dv \tag{1}$$

Where, σ_{ij}, m_{ij}, e_{ij} and κ_{ij} represent the stress, couple stress, strain and skew-symmetric curvature tensors, respectively. These tensors are defined by

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} \tag{2}$$

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} \right) \tag{3}$$

$$m_{ij} = -8\mu l^2 \kappa_{ij} \tag{4}$$

$$\kappa_{ij} = \frac{1}{2}\left(\omega_{i,j} - \omega_{j,i} \right) \tag{5}$$

Where, λ and μ are Lame's constants, and u_i and ω_j are the components of the displacement and the rotation vectors, respectively. The size-dependent parameter, l, is dependent on the material and scale and varies from one material to another or from one scale to another scale. This parameter should be obtained via conducting experiments for various dimensions in different working conditions. Also, it can be approximated by more accurate techniques such as molecular dynamics simulation. Where the ω_j, rotation vector is defined as:

$$\omega_i = \frac{1}{2} \varepsilon_{ijk} u_{k,j} \tag{6}$$

In which, ε_{ijk}, denotes the permutation or Levi-Civita symbol. Components of displacement vector (u_1, u_2 and u_3) for micro beams can be expressed based on Euler–Bernoulli beam theories as follows:

$$u_1 = -z\frac{dw}{dx} \tag{7}$$

$$u_2 = 0 \tag{8}$$

$$u_3 = w(x) \tag{9}$$

Substitution of Eqs (6-9) into the Eq. (5) yields to the expression for the skew-symmetric curvature tensor as

$$\kappa = \frac{1}{2}\frac{\partial^2 w}{\partial x^2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

From Eq. (4), the couple stress tensor is defined as follows

$$m = 4\mu l^2 \frac{\partial^2 w}{\partial x^2}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

From the displacement field, the strain components can be calculated by substituting Eqs. (7-9) into equation, Eq. (3).

$$e = -z\frac{\partial^2 w}{\partial x^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

For a slender beam with a large aspect ratio, the Poisson effect is secondary and can be disregarded for simplifying the beam theory formulation. Hence, the stress component is presented as

$$\sigma = -Ez\frac{\partial^2 w}{\partial x^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13}$$

Where, E is the modulus of elasticity. Considering the first variation of the kinetic energy on time interval [0,t] the following relation can be obtained:

$$\int_0^t \delta K dt = \int_0^t \left[\int_\Omega \rho \left(\dot{u}_1 \delta \dot{u}_1 + \dot{u}_2 \delta \dot{u}_2 + \dot{u}_3 \delta \dot{u}_3 \right) dV \right] dt$$

Where, ρ is the volume density. In order to obtain governing equations and corresponding boundary conditions for the AFM cantilever flexural vibration, the Hamilton's principle can be used as follows:

$$\delta \int_0^t \left(U - K \right) dt = 0$$

In which U and K are strain energy and kinematic energy, respectively. The governing equation of the AFM cantilever flexural vibration is:

$$\left(EI + 4\mu l^2 A \right) \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \tag{16}$$

Where, A is the cross-sectional area of the AFM cantilever. The corresponding boundary conditions are:

$$\begin{cases} w\left(0,t\right) = 0 \\ \dfrac{\partial w\left(0,t\right)}{\partial x} = 0 \\ \dfrac{\partial^2 w\left(L,t\right)}{\partial x^2} = 0 \\ \left(EI + 4\mu l^2 A \right) \dfrac{\partial^3 w\left(L,t\right)}{\partial x^3} = kw\left(L,t\right) + M\dfrac{\partial^2 w\left(L,t\right)}{\partial t^2} \end{cases} \tag{17}$$

For free vibration, we assume that the w(x,t) varies harmonically with respect to the time variable t as follows:

$$w\left(x,t\right) = f\left(x\right) e^{i\omega t} \tag{18}$$

For convenience and generality, the following non-dimensional variables are introduced:

$$\bar{x} = \frac{x}{L} \qquad \bar{f} = \frac{f}{L} \qquad \bar{k} = \frac{kL^3}{EI}$$

$$\eta = \frac{4\mu A l^2}{EI} \qquad \gamma^2 = \frac{\rho L^4 \omega^2 A}{EI} \qquad \bar{M} = \frac{M}{\rho A L}$$

Use of the above non-dimensional parameters in the ordinary differential equation and boundary conditions results in:

$$\frac{d^4 \bar{f}}{d\bar{x}^4} - \frac{\gamma^2}{1+\eta} \bar{f} = 0 \tag{19}$$

$$\begin{cases} \bar{f}\left(0\right) = 0 \\ \left.\dfrac{d\bar{f}}{d\bar{x}}\right|_{\bar{x}=0} = 0 \\ \left.\dfrac{d^2 \bar{f}}{d\bar{x}^2}\right|_{\bar{x}=1} = 0 \\ \left(1+\eta\right) \left.\dfrac{d^3 \bar{f}}{d\bar{x}^3}\right|_{\bar{x}=1} = \left(\bar{k} - \bar{M}\gamma^2 \right) \bar{f}\left(1\right) \end{cases} \tag{20}$$

(14)

From equation (19), the general solution for $\bar{f}$ is given below:

$$\bar{f} = C_1 \sin a\bar{x} + C_2 \cos a\bar{x} + C_3 \sinh a\bar{x} + C_4 \cosh a\bar{x} \tag{21}$$

Where

$$a = \frac{\gamma^{\frac{1}{2}}}{\left(1+\eta\right)^{\frac{1}{4}}} \tag{22}$$

(15)

In which C_i are some constants. By applying the introduced boundary conditions, the characteristic equation can be obtained as:

$$C = a^3 \left(1+\eta\right)\left(1+\cos a \cosh a\right) + \left(\bar{k} - \bar{M} a^4 \left(1+\eta\right) \right) \left(\cosh a \sin a - \cos a \,\mathrm{si} \right. \tag{23}$$

And the natural frequency can be determined from the characteristic equation. The flexural sensitivity of the cantilever to the surface stiffness variations can be calculated from the following equation:

$$S = \frac{\partial \gamma}{\partial \bar{k}} = \frac{\partial \gamma}{\partial a}\frac{\partial a}{\partial \bar{k}} \tag{24}$$

$\dfrac{\partial \gamma}{\partial \bar{a}}$ can be calculated from equation (22) as

$$\frac{\partial \gamma}{\partial a} = 2a\left(1+\eta\right)^{\frac{1}{2}} \tag{25}$$

And from equation (23) we have:

$$\frac{\partial a}{\partial \bar{k}} = -\left(\cosh a \sin a - \cos a \sinh a \right) \times \begin{bmatrix} 3a^2 \left(1+\eta\right)\left(1+\cos a \cosh a\right) \\ +a^3 \left(1+\eta\right)\left(\cos a \sinh a - \cosh a \sin a\right) \\ +2\left(\bar{k} - \bar{M}\gamma^2\right) \sin a \sinh a \end{bmatrix}^{-1} \tag{26}$$

Substituting Eqs. (25) and (26) in Eq.(24), the flexural sensitivity of the AFM cantilever is expressed as

$$S = -2a\left(1+\eta\right)^{\frac{1}{2}} \left(\cosh a \sin a - \cos a \sinh a \right) \begin{bmatrix} 3a^2 \left(1+\eta\right)\left(1+\cos a \cosh a\right) + \\ +a^3 \left(1+\eta\right)\left(\cos a \sinh a - \cosh a \sin a\right) \\ +2\left(\bar{k} - \bar{M}\gamma^2\right) \sin a \sinh a \end{bmatrix}^{-1} \tag{27}$$

3. Result and discussion

In this section, the free vibration and sensitivity analysis of AFM cantilever are investigated by numerical results based on a consistent couple stress theory.

In AFM micro cantilever, the variation of the natural frequency of the first modes versus to the parameter $\overline{k}$ for various values of h/l is shown in Figure 2. As can be seen, the natural frequency increases by increasing the contact stiffness. It should be noted that the $\frac{h}{l}=\infty$ case presents the classical beam theory results. It is also seen that for a constant value of $\overline{k}$, the natural frequency decreases as h/l increases. Therefore, the natural frequency obtained by the consistent couple stress theory is significantly greater than the frequency obtained by the classical beam theory. Moreover, in very high values of $\overline{k}$,changing $\overline{k}$ has little effect on the natural frequency.

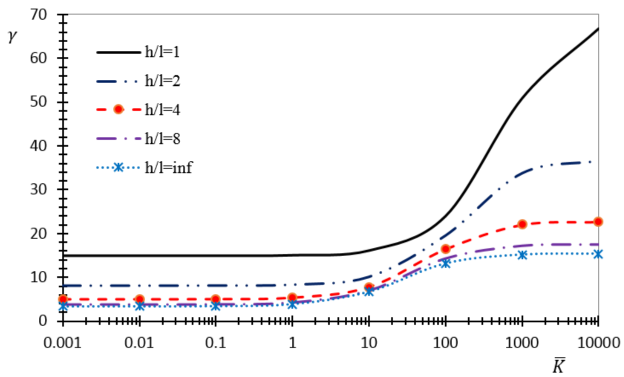

Fig. 2. The changes of the natural frequency of the first modes of an AFM micro-cantilever versus to the parameter $\overline{k}$ for various values h/l

Figure 3 shows the ratio of natural frequency in the case of considering consistent couple stress, γ_{CL}, to the classic case, γ_{CL}, in terms of dimensionless thickness,h/l for $\overline{k}=0.01$. As it can be seen, with increasing the dimensionless thickness, the natural frequency ratio tends to 1, which indicates the reduction in the couple stress effect by increasing the thickness against size scale parameter. If the dimensionless thickness equals to 1, relative natural frequency becomes 4.46, which shows the difference between the classic and consistent couple stress theory in small sizes.

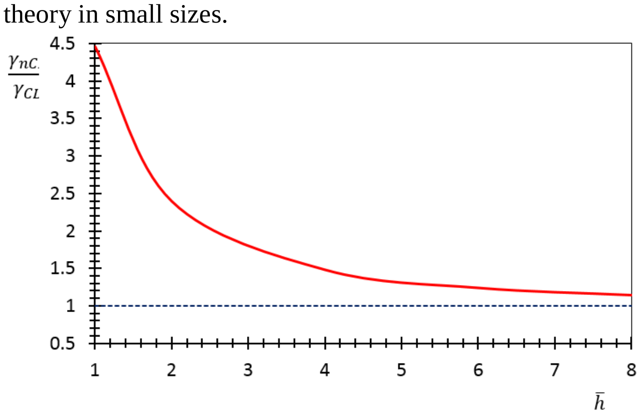

Fig. 3. Natural frequency ratio of the first modes of an AFM micro-cantilever versus to dimensionless thickness for $\overline{k}=0.01$.

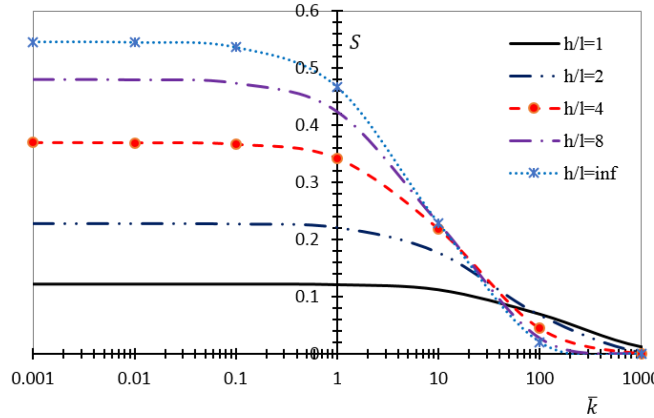

Fig. 4. The changes of the sensitivity of the first modes of an AFM micro-cantilever versus to the parameter $\overline{k}$ for various values h/l

The distribution of the first mode sensitivity of an AFM micro-cantilever versus to the parameter $\overline{k}$ for various values $\frac{h}{l}$ is shown in Figure 4. As it can be seen, with increasing $\overline{k}$, the sensitivity tends to zero. It may be important that for low values of contact stiffness, the decrease in the $\frac{h}{l}$ leads to the reduction of the sensitivity, while the opposite trend is observed for high values of contact stiffness. As an important result, it is observed that when the thickness of micro-cantilevers is close to the internal material-length-scale parameter, the difference between the sensitivities obtained from the coupled stress theory and the sensitivities predicted by the classical beam theory is relatively high.

4. Conclusion

In this paper, using an Euler–Bernoulli beam theory, the sensitivity of the flexural vibration modes for an atomic force microscope cantilever has been analyzed by the consistent couple stress theory. According to the conducted analysis, the sensitivity of the cantilever obtained by the consistent couple stress theory was smaller compared to the one obtained by classical beam theory. The presented results in this work may provide useful guidance for design and development of AFM cantilever-based micro devices.

References

[1] A. Hadi, A. Rastgoo, A. Bolhassani, N. Haghighipour, Effects of stretching on molecular transfer from cell membrane by forming pores, *Soft Materials*, pp. 1-9, 2019.

[2] H. H. Gorgani, M. M. Adeli, M. Hosseini, Pull-in behavior of functionally graded micro/nano-beams for MEMS and NEMS switches, *Microsystem Technologies*, pp. 1-9, 2018.

[3] M. Farajpour, A. Shahidi, A. Hadi, A. Farajpour, Influence of initial edge displacement on the nonlinear vibration, electrical and magnetic instabilities of magneto-electro-elastic nanofilms,

Mechanics of Advanced Materials and Structures, pp. 1-13, 2018.

[4] M. Shishesaz, M. Hosseini, K. N. Tahan, A. Hadi, Analysis of functionally graded nanodisks under thermoelastic loading based on the strain gradient theory, *Acta Mechanica,* Vol. 228, No. 12, pp. 4141-4168, 2017.

[5] A. Hadi, M. Z. Nejad, A. Rastgoo, M. Hosseini, Buckling analysis of FGM Euler-Bernoulli nano-beams with 3D-varying properties based on consistent couple-stress theory, *Steel and Composite Structures*, Vol. 26, No. 6, pp. 663-672, 2018.

[6] A. Hadi, A. Rastgoo, N. Haghighipour, A. Bolhassani, Numerical modelling of a spheroid living cell membrane under hydrostatic pressure, *Journal of Statistical Mechanics: Theory and Experiment*, Vol. 2018, No. 8, pp. 083501, 2018.

[7] M. Hosseini, M. Shishesaz, A. Hadi, Thermoelastic analysis of rotating functionally graded micro/nanodisks of variable thickness, *Thin-Walled Structures,* Vol. 134, pp. 508-523, 2019.

[8] S. Gopalakrishnan, S. Narendar, 2013, *Wave Propagation in Nanostructures: Nonlocal Continuum Mechanics Formulations,* Springer Science & Business Media,

[9] M. Hosseini, M. Shishesaz, K. N. Tahan, A. Hadi, Stress analysis of rotating nano-disks of variable thickness made of functionally graded materials, *International Journal of Engineering Science,* Vol. 109, pp. 29-53, 2016.

[10] A. Hadi, M. Z. Nejad, M. Hosseini, Vibrations of three-dimensionally graded nanobeams, *International Journal of Engineering Science,* Vol. 128, pp. 12-23, 2018.

[11] M. Z. Nejad, A. Hadi, A. Farajpour, Consistent couple-stress theory for free vibration analysis of Euler-Bernoulli nano-beams made of arbitrary bi-directional functionally graded materials, *Structural Engineering and Mechanics,* Vol. 63, No. 2, pp. 161-169, 2017.

[12] M. Hosseini, H. H. Gorgani, M. Shishesaz, A. Hadi, Size-dependent stress analysis of single-wall carbon nanotube based on strain gradient theory, *International Journal of Applied Mechanics,* Vol. 9, No. 06, pp. 1750087, 2017.

[13] M. M. Adeli, A. Hadi, M. Hosseini, H. H. Gorgani, Torsional vibration of nano-cone based on nonlocal strain gradient elasticity theory, *The European Physical Journal Plus,* Vol. 132, No. 9, pp. 393, 2017.

[14] A. Soleimani, K. Dastani, A. Hadi, M. H. Naei, Effect of out-of-plane defects on the postbuckling behavior of graphene sheets based on nonlocal elasticity theory, *Steel and Composite Structures,* Vol. 30, No. 6, pp. 517-+, 2019.

[15] M. Z. Nejad, A. Hadi, A. Omidvari, A. Rastgoo, Bending analysis of bi-directional functionally graded Euler-Bernoulli nano-beams using integral form of Eringen's non-local elasticity theory, *Structural Engineering and Mechanics,* Vol. 67, No. 4, pp. 417-425, 2018.

[16] M. Mohammadi, M. Hosseini, M. Shishesaz, A. Hadi, A. Rastgoo, Primary and secondary resonance analysis of porous functionally graded nanobeam resting on a nonlinear foundation subjected to mechanical and electrical loads, *European Journal of Mechanics-A/Solids,* 2019.

[17] A. C. Eringen, Theory of micromorphic materials with memory, *International Journal of Engineering Science,* Vol. 10, No. 7, pp. 623-641, 1972.

[18] A. C. Eringen, Nonlocal polar elastic continua, *International journal of engineering science,* Vol. 10, No. 1, pp. 1-16, 1972.

[19] A. C. Eringen, On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves, *Journal of applied physics,* Vol. 54, No. 9, pp. 4703-4710, 1983.

[20] A. C. Eringen, 2002, *Nonlocal continuum field theories*, Springer Science & Business Media,

[21] A. Daneshmehr, A. Rajabpoor, A. Hadi, Size dependent free vibration analysis of nanoplates made of functionally graded materials based on nonlocal elasticity theory with high order theories, *International Journal of Engineering Science,* Vol. 95, pp. 23-35, 2015.

[22] M. Z. Nejad, A. Hadi, A. Rastgoo, Buckling analysis of arbitrary two-directional functionally graded Euler–Bernoulli nano-beams based on nonlocal elasticity theory, *International Journal of Engineering Science,* Vol. 103, pp. 1-10, 2016.

[23] M. Z. Nejad, A. Hadi, Non-local analysis of free vibration of bi-directional functionally graded Euler–Bernoulli nano-beams, *International Journal of Engineering Science,* Vol. 105, pp. 1-11, 2016.

[24] M. Z. Nejad, A. Hadi, Eringen's non-local elasticity theory for bending analysis of bi-directional functionally graded Euler–Bernoulli nano-beams, *International Journal of Engineering Science,* Vol. 106, pp. 1-9, 2016.

[25] D. Lam, F. Yang, A. Chong, J. Wang, P. Tong, Experiments and theory in strain gradient elasticity, *Journal of the Mechanics and Physics of Solids,* Vol. 51, No. 8, pp. 1477-1508, 2003.

[26] R. A. Toupin, Elastic materials with couple-stresses, *Archive for Rational Mechanics and Analysis,* Vol. 11, No. 1, pp. 385-414, 1962.

[27] R. Mindlin, H. Tiersten, Effects of couple-stresses in linear elasticity, *Archive for Rational Mechanics and Analysis,* Vol. 11, No. 1, pp. 415-448, 1962.

[28] W. KOlTER, Couple stresses in the theory of elasticity, *Proc. Koninklijke Nederl. Akaad. van Wetensch,* Vol. 67, 1964.

[29] A. Farajpour, A. Rastgoo, M. Farajpour, Nonlinear buckling analysis of magneto-electro-elastic CNT-MT hybrid nanoshells based on the nonlocal

continuum mechanics, *Composite Structures,* Vol. 180, pp. 179-191, 2017.

[30] M. Baghani, M. Mohammadi, A. Farajpour, Dynamic and stability analysis of the rotating nanobeam in a nonuniform magnetic field considering the surface energy, *International Journal of Applied Mechanics*, Vol. 8, No. 04, pp. 1650048, 2016.

[31] N. Kordani, A. Fereidoon, M. Divsalar, A. Farajpour, Forced vibration of piezoelectric nanowires based on nonlocal elasticity theory, *Journal of Computational Applied Mechanics*, Vol. 47, No. 2, pp. 137-150, 2016.

[32] A. Farajpour, A. Rastgoo, Influence of carbon nanotubes on the buckling of microtubule bundles in viscoelastic cytoplasm using nonlocal strain gradient theory, *Results in physics*, Vol. 7, pp. 1367-1375, 2017.

[33] F. Yang, A. Chong, D. C. C. Lam, P. Tong, Couple stress based strain gradient theory for elasticity, *International Journal of Solids and Structures,* Vol. 39, No. 10, pp. 2731-2743, 2002.

[34] H. Ma, X.-L. Gao, J. Reddy, A microstructure-dependent Timoshenko beam model based on a modified couple stress theory, *Journal of the Mechanics and Physics of Solids*, Vol. 56, No. 12, pp. 3379-3391, 2008.

[35] S. Park, X. Gao, Bernoulli–Euler beam model based on a modified couple stress theory, *Journal of Micromechanics and Microengineering*, Vol. 16, No. 11, pp. 2355, 2006.

[36] M. Asghari, M. Kahrobaiyan, M. Ahmadian, A nonlinear Timoshenko beam formulation based on the modified couple stress theory, *International Journal of Engineering Science*, Vol. 48, No. 12, pp. 1749-1761, 2010.

[37] W. Xia, L. Wang, L. Yin, Nonlinear non-classical microscale beams: static bending, postbuckling and free vibration, *International Journal of Engineering Science,* Vol. 48, No. 12, pp. 2044-2053, 2010.

[38] M. Şimşek, Nonlinear static and free vibration analysis of microbeams based on the nonlinear elastic foundation using modified couple stress theory and He's variational method, *Composite Structures*, Vol. 112, pp. 264-272, 6//, 2014.

[39] M. Asghari, M. Rahaeifard, M. Kahrobaiyan, M. Ahmadian, The modified couple stress functionally graded Timoshenko beam formulation, *Materials & Design,* Vol. 32, No. 3, pp. 1435-1443, 2011.

[40] L.-L. Ke, Y.-S. Wang, J. Yang, S. Kitipornchai, Nonlinear free vibration of size-dependent functionally graded microbeams, *International Journal of Engineering Science,* Vol. 50, No. 1, pp. 256-267, 2012.

[41] N. Shafiei, S. S. Mirjavadi, B. M. Afshari, S. Rabby, A. Hamouda, Nonlinear thermal buckling of axially functionally graded micro and nanobeams, *Composite Structures,* Vol. 168, pp. 428-439, 2017.

[42] S. Srividhya, P. Raghu, A. Rajagopal, J. Reddy, Nonlocal nonlinear analysis of functionally graded plates using third-order shear deformation theory, *International Journal of Engineering Science,* Vol. 125, pp. 1-22, 2018.

[43] E. Jomehzadeh, H. Noori, A. Saidi, The size-dependent vibration analysis of micro-plates based on a modified couple stress theory, *Physica E: Low-dimensional Systems and Nanostructures,* Vol. 43, No. 4, pp. 877-883, 2011.

[44] L. Yin, Q. Qian, L. Wang, W. Xia, Vibration analysis of microscale plates based on modified couple stress theory, *Acta Mechanica Solida Sinica*, Vol. 23, No. 5, pp. 386-393, 2010.

[45] L. He, J. Lou, E. Zhang, Y. Wang, Y. Bai, A size-dependent four variable refined plate model for functionally graded microplates based on modified couple stress theory, *Composite Structures,* Vol. 130, pp. 107-115, 10/15/, 2015.

[46] M. Asghari, Geometrically nonlinear micro-plate formulation based on the modified couple stress theory, *International Journal of Engineering Science*, Vol. 51, pp. 292-309, 2012.

[47] J. Lou, L. He, Closed-form solutions for nonlinear bending and free vibration of functionally graded microplates based on the modified couple stress theory, *Composite Structures,* Vol. 131, pp. 810-820, 11/1/, 2015.

[48] M. Mohammad-Abadi, A. Daneshmehr, Modified couple stress theory applied to dynamic analysis of composite laminated beams by considering different beam theories, *International Journal of Engineering Science,* Vol. 87, pp. 83-102, 2015.

[49] D. Shao, S. Hu, Q. Wang, F. Pang, Free vibration of refined higher-order shear deformation composite laminated beams with general boundary conditions, *Composites Part B: Engineering,* Vol. 108, pp. 75-90, 2017.

[50] A. R. Hadjesfandiari, G. F. Dargush, Couple stress theory for solids, *International Journal of Solids and Structures,* Vol. 48, No. 18, pp. 2496-2510, 2011.