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Strongly stable multi-time stepping method with the option of controlling amplitude decay in responses

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ABSTRACT

Recently, multi-time stepping methods have become very popular among scientist due to their high stability in problems with critical conditions. One important shortcoming of these methods backs to their high amount of uncontrolled amplitude decay. This study proposes a new multi-time stepping method in which the time step is split into two sub-steps. The first sub-step is solved using the well-known Newmark method and for the second sub-step an extended version of Newmark method is applied. In fact, similarity in basic formulas of the mentioned methods makes it available to control the amount of amplitude decay in responses obtained by the proposed method; in other words, the amplitude decay in the proposed method is controlled through constant parameters of the two methods applied on each sub-step. The precision assessment of the proposed method is performed using numerical approaches and revealed the minor period elongation error of the proposed method in comparison with other existing methods. In addition to this, the unconditional stability region of constant parameters is also determined through computation of spectral radius of the proposed method. Finally, practical assessment of the proposed method is performed through several numerical examples.

1. Introduction

Equation of motion is the well-known governing equation of dynamic systems. Analytical solutions of this equation is limited to special problems with special conditions [1, 2], also Modal analysis and Frequency domain analysis suffer from severe limitations such as not having the ability of solving nonlinear problems due use of superposition. It is common to use a direct time integration method to solve the motion equation in a dynamic system. After almost five decades, time integration methods are still playing the main role in analysis of dynamic systems. In these methods within each time interval, a specific type of variation of the displacement, velocity, and acceleration are assumed. By increasing the order of variation of these quantities, since more terms are kept in the Taylor series expansion, higher accuracy could be achieved. Several numerical integration algorithms are available depending on the type of variation assumed for those quantities within each time interval. This procedure is a form of finite difference solution for differential equations. Nowadays there are a wide variety of methods with different performances and arrangements. These time marching methods have been categorized according to their formulations and behavior [3-5].

One categorization is related to the equilibrium point on the time step. Explicit methods use the differential equation at time t to predict a solution at time $t+\Delta t$. For most real structures, a very small time step is required to obtain a stable solution using explicit methods. Of course recently unconditionally stable explicit methods have also been developed [6]. On the contrary to the explicit methods, implicit methods attempt to satisfy the differential equation at time $t+\Delta t$ after the solution at time t is found [7-9]. There is also another class in this categorization called predictor-corrector which utilizes both formulations of explicit and implicit methods [5].

In another point of view, time integration methods are classified as conditionally and unconditionally stable. This characterization is performed by defining conditions in which the given method is stable. An integration algorithm is said to be stable if the numerical solution, under any initial conditions, does not grow without bound; and is said to be unconditionally stable if the convergence of the solution is independent of the size of the time step Δt. The Newmark's family of methods, depending on the assumed values of the constant parameters, stand in this category. In the Newmark integration method, the acceleration varies linearly or remains constant within two instances of time. Wilson-θ is another example of such methods.

Another important factor in time integration algorithms is related to the previous steps' information needed for current time

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step to reach equilibrium. According to this feature, some of methods are single-step; that is, only the information obtained from last equilibrium point is needed for the solution of current time step and some are double-step.

It is noteworthy that through all developed methods, only some of them have the reliability to find their way into a commercial computer program. Since the problems being solved by commercial programs are usually consisted of huge numbers of degrees of freedom, the solution process usually takes a long time. Thus, the methods need to be robust in solving different problems with different specifications especially nonlinear problems. For instance, the method proposed by Bathe and Baig [10], is now available in ADINA computer program.

Multi-time stepping methods are the latest type of time integration methods. These methods attempt to march each time increment by multiple sub-steps. Usually for the first sub-step, it is common to apply a single-step method; so that the following sub-steps could be solved using methods which use the data obtained from multiple previous equation points [10-18]. Unlike the non-composite types of time marching algorithms, multi-time stepping methods are proven to be robustly stable even in highly nonlinear problems or in problems including ill conditioned matrices where even unconditionally stable methods such as Newmark's Trapezoidal rule sometimes lose their stability. This study proposes a new composite time integration method to increase the accuracy of responses and provide controllable numerical dissipation. The proposed method in this study uses two sub-steps in which the first sub-step is solved by Newmark method and a second order accurate double-step method, which is an extended version of Newmark's family of methods developed by Gholampour and Ghassemieh [19], is applied on the second sub-step.

2. The Proposed Method

Consider the nonlinear equation of motion in a single degree of freedom system, which is written in the following form:

$$\mathbf{M}\ddot{U}_{t+\Delta t} + \mathbf{C}\dot{U}_{t+\Delta t} + \mathbf{K}\Delta U_{t+\Delta t} + f_{s_t} = P_{t+\Delta t} \tag{1}$$

in which U is the displacement, $\dot{U}$ is the velocity, $\ddot{U}$ is the acceleration, $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the tangent stiffness, P is the exciting force, f_s is the internal force, and Δt is the time step duration. Imagine the solution is known up to time t and the solution of time $t+\Delta t$ is to be calculated. The time step is divided into two equal or unequal sub-steps; as presented in Fig. 1. According to this figure, the first sub-step is from t to $t+\alpha\Delta t$ and the second sub-step is from $t+\alpha\Delta t$ to $t+\Delta t$. In this study, the time increment is divided equally ($\alpha = 0.5$).

Fig. 1. Sub-steps in a time increment

As for the first sub-step, which is to be solved by Newmark method, the following equations are used:

$$\ddot{U}_{t+0.5\Delta t} = \frac{4}{\beta(\Delta t)^2}\left(\Delta U_{t+0.5\Delta t} - \frac{1}{2}\Delta t\dot{U}_t - \left(\frac{1}{2}-\beta\right)\left(\frac{1}{2}\Delta t\right)^2\ddot{U}_t\right) \tag{2}$$

$$\dot{U}_{t+0.5\Delta t} = \frac{2\gamma}{\beta(\Delta t)}\left(\Delta U_{t+0.5\Delta t}\right) + \left(1-\frac{\beta}{\gamma}\right)\dot{U}_t + \left(1-\frac{\gamma}{2\beta}\right)\left(\frac{1}{2}\Delta t\right)\ddot{U}_t \tag{3}$$

in which β and γ are the constants of Newmark family of methods in which the choice of $\beta = 1/4$ and $\gamma = 1/2$ leads to the Trapezoidal rule. Substituting Eqs. (2) and (3) into the equation of motion, the incremental equation is obtained; as follows:

$$\mathbf{K}_{e1}\Delta U_{t+0.5\Delta t} = R_1 \tag{4}$$

in which:

$$\mathbf{K}_{e1} = \frac{4\mathbf{M}}{\beta(\Delta t)^2} + \frac{2\gamma\mathbf{C}}{\beta\Delta t} + \mathbf{K} \tag{5}$$

$$\begin{aligned} R_1 = {} & P_{t+0.5\Delta t} - f_{s_t} + \left(\frac{2}{\beta(\Delta t)}\dot{U}_t + \left(\frac{1}{2\beta}-1\right)\ddot{U}_t\right)\mathbf{M} + \\ & \left(\left(\frac{\gamma}{\beta}-1\right)\dot{U}_t + \left(\frac{\gamma}{2\beta}-1\right)\left(\frac{1}{2}\Delta t\right)\ddot{U}_t\right)\mathbf{C} \end{aligned} \tag{6}$$

$$\Delta U_{t+0.5\Delta t} = U_{t+0.5\Delta t} - U_t \tag{7}$$

It is notable that in nonlinear problems the incremental equation, presented in Eq. (4), can be solved using any equilibrium path tracing algorithm such as Newton-Raphson method.

The second sub-step is solved using a double-step time integration method with a quadratic variation of acceleration over time step [19]. The following equations are related to the second sub-step:

$$\begin{aligned} \ddot{U}_{t+\Delta t} = {} & \frac{4}{(\Delta t)^2(\upsilon+1/12)}\left(\Delta U_{t+\Delta t}\right) - \frac{2}{\Delta t(\upsilon+1/12)}\dot{U}_{t+0.5\Delta t} - \\ & \frac{1/2-2\upsilon}{\upsilon+1/12}\ddot{U}_{t+0.5\Delta t} - \frac{\upsilon-1/12}{\upsilon+1/12}\ddot{U}_t \end{aligned} \tag{8}$$

$$\begin{aligned} \dot{U}_{t+\Delta t} = {} & \frac{(2\mu+1/2)}{(\Delta t)(\upsilon+1/12)}\left(\Delta U_{t+\Delta t}\right) + \left(1-\frac{(\mu+1/4)}{(\upsilon+1/12)}\right)\dot{U}_{t+0.5\Delta t} + \\ & \frac{1}{2}(\Delta t)\left(1-2\mu-\frac{(\mu+1/4)}{(\upsilon+1/12)}(1/2-2\upsilon)\right)\ddot{U}_{t+0.5\Delta t} + \\ & \frac{1}{2}(\Delta t)\left(\mu-1/4-\frac{(\mu+1/4)}{(\upsilon+1/12)}(\upsilon-1/12)\right)\ddot{U}_t \end{aligned} \tag{9}$$

Substituting Eqs. (8) and (9) into the equation of motion, the incremental equation is obtained; as follows:

$$\mathbf{K}_{e2}\Delta U_{t+\Delta t} = R_2 \tag{10}$$

in which:

$$\mathbf{K}_{e2} = \frac{4}{(\Delta t)^2(\upsilon+1/12)}\mathbf{M} + \frac{2\mu+1/2}{(\Delta t)(\upsilon+1/12)}\mathbf{C} + \mathbf{K}_{t+0.5\Delta t} \tag{11}$$

$$\begin{aligned} R_2 = {} & P_{t+\Delta t} - f_{s_{t+0.5\Delta t}} + \left(\frac{2}{(\Delta t)(\upsilon+1/12)}\mathbf{M} - \left(1-\frac{\mu+1/4}{\upsilon+1/12}\right)\mathbf{C}\right)\dot{U}_{t+0.5\Delta t} \\ & + \left(\frac{1/2-2\upsilon}{\upsilon+1/12}\mathbf{M} - \left(\frac{1}{2}\Delta t\right)\left(1-2\mu-\frac{\mu+1/4}{\upsilon+1/12}(1/2-2\upsilon)\right)\mathbf{C}\right)\ddot{U}_{t+0.5\Delta t} \\ & + \left(\frac{\upsilon-1/12}{\upsilon+1/12}\mathbf{M} - \left(\frac{1}{2}\Delta t\right)\left(\mu-1/4-\frac{\mu+1/4}{\upsilon+1/12}(\upsilon-1/12)\right)\mathbf{C}\right)\ddot{U}_t \end{aligned} \tag{12}$$

$$\Delta U_{t+\Delta t} = U_{t+\Delta t} - U_{t+0.5\Delta t} \tag{13}$$

It must be noted that μ and υ are the constant parameters of the quadratic acceleration method designed to make it available for the operator to induce numerical damping into the analysis; and for unconditional stability these constants are found; as follows:

$$\mu \geq 1/3 \quad \& \quad \mu/2 \leq \upsilon \leq \mu - 1/6 \tag{14}$$

It is also notable that the choice of $\mu = 1/3$ and $\upsilon = 1/6$ produces zero numerical damping in this method. As formulated, the constant parameters of both methods, being Newmark method and quadratic acceleration method, are kept into account in the proposed method; so that the method could provide reasonable amount of numerical damping while keeping its stability.

3. Numerical Properties

Numerical approaches in solving differential equations have specific properties due accepting a reasonable amount of error in each time increment. There are several ways to assess the numerical properties of a time integration. The proposed method in this study is assessed with some of these processes.

3.1. Stability

Stability assessment of a time marching method is carried out considering the equation of motion for a single degree of freedom with arbitrary initial conditions. Thus, free vibration is considered at time step $t + \Delta t$ and the amplification matrix $[A]$ is calculated. The method is stable if the spectral radius, being maximum eigenvalue of the amplification matrix in modulus, is less than unit [5, 6, 11]. Eq. (15) shows the well-known recursive matrix form of the Newmark method in a free vibration problem, as follows:

$$\begin{Bmatrix} \ddot{U}_{t+0.5\Delta t} \\ \ddot{U}_{t} \\ \dot{U}_{t+0.5\Delta t} \\ U_{t+0.5\Delta t} \end{Bmatrix} = [A] \begin{Bmatrix} \ddot{U}_{t} \\ \ddot{U}_{t-\Delta t+0.5\Delta t} \\ \dot{U}_{t} \\ U_{t} \end{Bmatrix}, \quad [A] = \begin{bmatrix} a_{11} & 0 & a_{12} & a_{13} \\ 1 & 0 & 0 & 0 \\ a_{21} & 0 & a_{22} & a_{23} \\ a_{31} & 0 & a_{32} & a_{33} \end{bmatrix} \tag{15}$$

The constants of the amplification matrix are:

$$a_{11} = -(1/2 - \beta)\lambda - 2(1-\gamma)k \,;\, a_{12} = -\frac{\lambda + 2k}{0.5\Delta t} \,; \quad a_{13} = -\frac{\lambda}{(0.5\Delta t)^2} \,;$$

$$a_{21} = (0.5\Delta t)(1 - \gamma - (1/2 - \beta)\gamma\lambda - 2(1-\gamma)\gamma k) \,;\, a_{22} = 1 - \frac{\lambda}{2} - k \,;$$

$$a_{23} = -\frac{2\gamma\lambda}{\Delta t} \,;\, a_{31} = (0.5\Delta t)^2 (1/2 - \beta - (1/2 - \beta)\beta\lambda - 2(1-\gamma)\beta k);$$

$$a_{32} = (0.5\Delta t)(1 - \beta\lambda - 2\beta k) \,;\, a_{33} = 1 - \beta\lambda \tag{16}$$

in which:

$$\lambda = \left(\frac{1}{\omega_n^2 (0.5\Delta t)^2} + \frac{2\gamma\xi}{\omega_n (0.5\Delta t)} + \beta \right)^{-1} ; \quad k = \frac{\xi\lambda}{\omega_n (0.5\Delta t)} \tag{17}$$

Then according to the second sub-step the following amplification matrix is obtained:

$$\begin{Bmatrix} \ddot{U}_{t+\Delta t} \\ \ddot{U}_{t+0.5\Delta t} \\ \dot{U}_{t+\Delta t} \\ U_{t+\Delta t} \end{Bmatrix} = [B] \begin{Bmatrix} \ddot{U}_{t+0.5\Delta t} \\ \ddot{U}_{t} \\ \dot{U}_{t+0.5\Delta t} \\ U_{t+0.5\Delta t} \end{Bmatrix}; \quad [B] = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \tag{18}$$

and the constants of the matrix is obtained, as follows:

$$b_{11} = -\left(\frac{1}{2} - 2\upsilon\right) r - (2 - 4\mu)s \,;\, b_{12} = -\left(\upsilon - \frac{1}{12}\right) r - \left(2\mu - \frac{1}{2}\right) s \,;$$

$$b_{13} = \frac{2}{\Delta t}(-r - 2s) \,;\, b_{14} = \frac{4}{(\Delta t)^2}(-r) \,;\, b_{21} = 1 \,;$$

$$b_{22} = b_{23} = b_{24} = 0 \,;$$

$$b_{31} = \left(\frac{1}{2}\Delta t\right)\left(1 - 2\mu - \left(\frac{1}{2} - 2\upsilon\right)\left(\mu + \frac{1}{4}\right) r - (1 - 2\mu)\left(2\mu + \frac{1}{2}\right) s\right);$$

$$b_{32} = \left(\frac{1}{2}\Delta t\right)\left(\mu - \frac{1}{4} - \left(\mu + \frac{1}{4}\right)\left(\upsilon - \frac{1}{12}\right) r - \left(\mu - \frac{1}{4}\right)\left(2\mu + \frac{1}{2}\right) s\right);$$

$$b_{33} = 1 - \left(\mu + \frac{1}{4}\right)(r + 2s) \,;\; b_{34} = -\frac{2r}{\Delta t}\left(\mu + \frac{1}{4}\right)$$

$$b_{41} = \left(\frac{1}{2}\Delta t\right)^2 \left(\frac{1}{2} - 2\upsilon - \left(\frac{1}{2} - 2\upsilon\right)\left(\upsilon + \frac{1}{12}\right) r - (2 - 4\mu)\left(\upsilon + \frac{1}{12}\right) s\right);$$

$$b_{42} = \left(\frac{1}{2}\Delta t\right)^2 \left(\upsilon - \frac{1}{12} - \left(\upsilon - \frac{1}{12}\right)\left(\upsilon + \frac{1}{12}\right) r - \left(2\mu - \frac{1}{2}\right)\left(\upsilon + \frac{1}{12}\right) s\right);$$

$$b_{43} = \left(\frac{1}{2}\Delta t\right)\left(1 - \left(\upsilon + \frac{1}{12}\right)(r + 2s)\right);\; b_{44} = 1 - \left(\upsilon + \frac{1}{12}\right) r \tag{19}$$

in which:

$$r = \left(\frac{1}{\omega_n^2 (0.5\Delta t)^2} + \frac{2\xi(\mu + 1/4)}{\omega_n (0.5\Delta t)} + (\upsilon + 1/12) \right)^{-1} ; \; s = \frac{\xi r}{\omega_n (0.5\Delta t)} \tag{20}$$

In the above equations, ξ represents the damping ratio, and ω_n is the natural frequency of system. Finally, extending the amplification matrix for the multi-time stepping method is obtained; as follows:

$$\begin{Bmatrix} \ddot{U}_{t+\Delta t} \\ \ddot{U}_{t+0.5\Delta t} \\ \dot{U}_{t+\Delta t} \\ U_{t+\Delta t} \end{Bmatrix} = [B][A] \begin{Bmatrix} \ddot{U}_{t} \\ \ddot{U}_{t-\Delta t+0.5\Delta t} \\ \dot{U}_{t} \\ U_{t} \end{Bmatrix} \tag{21}$$

The critical state in the analysis of stability occurs when the value of the damping ratio is zero. Fig. 2 demonstrates the spectral radius as a function of $\Delta t/T$ for different choices of constant parameters. Please be noticed that for the sake of having clear figures in this study, the proposed method is referred to as P in all figures.

According to Fig. 2, unconditional stability is obtained with the same constant parameters of the quadratic acceleration method, mentioned in Eq. (14), provided that the Trapezoidal rule is applied on the first sub-step. Also, another choice of constant parameters chosen for the proposed method being ($\beta = 3/10$; $\gamma = 11/20$) and ($\mu > 0.4$; $\upsilon > \mu/2$) offers the method unconditional stability. It is interesting that with the choice of ($\beta = 0.25$; $\gamma = 0.49$) the spectral radius equals to 1.

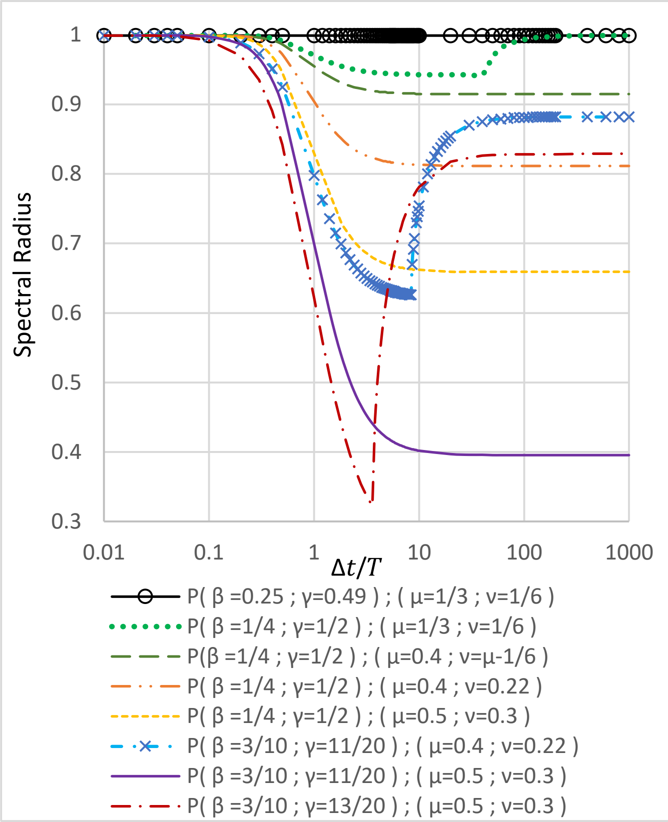

Fig. 2. Spectral Radius as a function of Δt/T

3.2. Precision

The decision as to which time-marching scheme should be used in analysis strongly depends on the cost of the solution, which in turn is determined by the number of time steps required in the integration. When an unconditional stable method is employed, the size of time step has to be chosen to yield an accurate and effective solution. The precision of a direct time integration method is studied using Eq. (22) instead of Eq. (1). This way, the variables considered in the stability and accuracy analyses are only Δt, ω, and ξ [5], as displayed in the following equation:

$$\ddot{U} + 2\xi\omega\dot{U} + \omega^2 U = r \qquad (22)$$

As a matter of fact, the obtained solutions to the Eq. (22) using various methods usually suffer from some distortions related to the magnitude of computed amplitudes and phase shifts called amplitude decay (AD), and period elongation (PE), respectively. The following sub-sections represent the AD and PE amount produced by the proposed method.

Amplitude decay, also called numerical damping, can hardly be considered as error. Scientists try to control it in a time marching algorithm in order to induce it in problems which demand numerical damping to yield better responses or even stable solutions like in some nonlinear problems. Fig. 3 shows the percentage numerical damping as a function of Δt/T. In order to reach a conclusion on the choices of constant parameters in the proposed method, the results of the proposed method with various choices of parameters are shown in this figure.

Fig. 3 reveals a huge advantage of the proposed method having the ability to induce any reasonable amount of numerical damping into the analysis in case needed; this is while other methods, like Bathe method, have an uncontrolled amount of numerical damping. Another fact in Fig. 3 about the proposed method, is that in order to induce a considerable amount of numerical damping into the analysis, the choice of ($\beta = 3/10$; $\gamma = 13/20$) needs to be adopted for the constant parameters of the Newmark method used in the first sub-step.

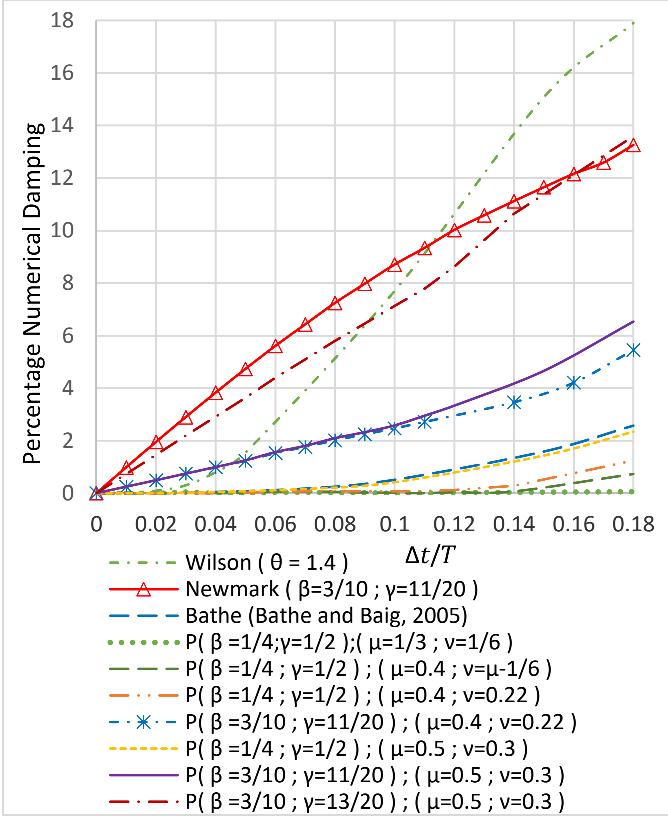

Fig. 3. Percentage numerical damping versus Δt/T

In addition to distortion in computed amplitudes, the complex exponentials in time-marching schemes introduce a systematic phase shift in each time step, which is known as the period elongation. This error is generally reported as the percentage difference between the true period and the period of computed solution of undamped free vibration of SDOF systems [5, 20]. Fig. 4 presents the computed percentage period elongation (PPE) for various methods.

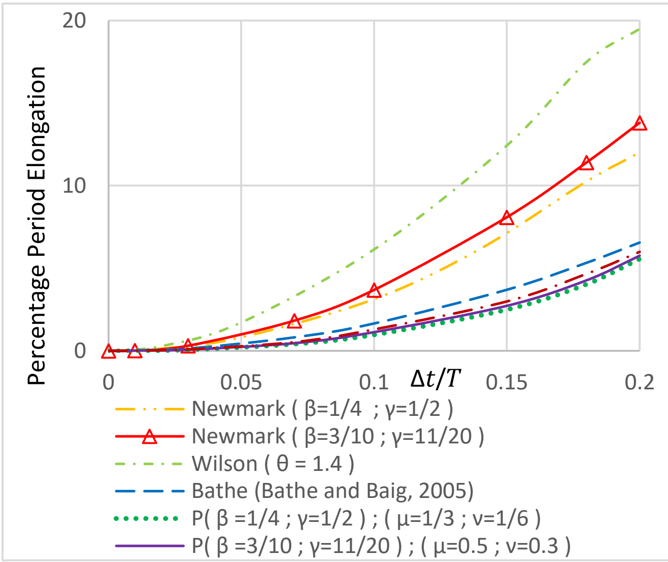

Fig. 4. Percentage period elongation as a function of Δt/T

As illustrated in Fig. 4, the proposed method has the lowest period elongation error. Please be noticed that for the sake of having a clear figure, other choices of constant parameters for the proposed method, which have very similar period elongation to the ones shown in Fig. 4, have been removed. An interesting point about Fig. 4 is that the proposed method even with the choice of ($\beta = 3/10$; $\gamma = 13/20$), which induces considerable amount of numerical damping to the analysis, produces less period elongation than other methods.

As a conclusion to the precision section and as a help of choosing between various constant parameters, it is suggested to consider 3 conditions; 1) if no numerical damping is needed in the problem, the choice of ($\beta = 1/4$; $\gamma = 1/2$) and ($\mu = 1/3$; $\upsilon = 1/6$) be utilized, 2) if a medium level of numerical damping is needed, the choice of ($\beta = 3/10$; $\gamma = 11/20$) and ($\mu = 0.5$; $\upsilon = 0.3$) be utilized, and 3) in case more numerical damping is required, ($\beta = 3/10$; $\gamma = 13/20$) and ($\mu = 0.5$; $\upsilon = 0.3$) be used in order to yield the best responses.

So as to see the differences between these three choices, consider the equation of motion with the following initial conditions:

$$\ddot{U} + \omega^2 U = 0 \,;\, U_0 = 1.0 \,;\, \dot{U}_0 = 0.0 \,;\, \ddot{U}_0 = -\omega^2 \,;$$

$$\omega = 2\pi \; rad/sec. \tag{23}$$

The exact solution of the above equation for given initial conditions is $U = \cos(\omega t)$. **Error! Reference source not found.** shows the responses obtained by different choices of constant parameters using $\Delta t = 0.2$ *sec*.

As illustrated in Fig. 5, the results of four different choices of constant parameters in the proposed method with four different numerical damping are demonstrated in which all produced less period elongation error than Bathe method. Another interesting fact about Fig. 5 is that Bathe method has only one constant level of numerical damping which in this example resulted in digression from the reference solution; this is while the proposed method with ($\beta = 1/4$; $\gamma = 1/2$) and ($\mu = 1/3$; $\upsilon = 1/6$) has desirable responses.

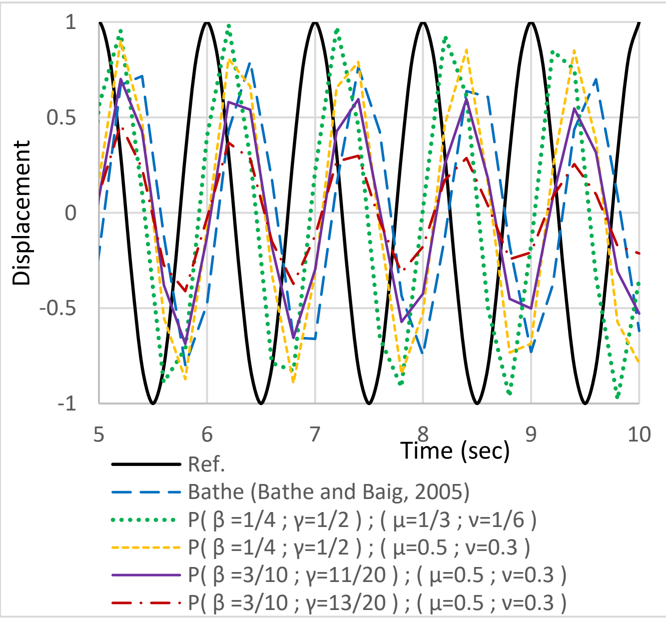

Fig. 5. A simple problem $\Delta t = 0.2$ *sec.*

It is also notable that, although not illustrated in Fig. 5, through various examples, it is found that with the choice of ($\beta = 0.19$; $\gamma = 0.55$) and ($\mu = 0.68$; $\upsilon = 0.43$), the proposed method resulted in responses very similar to (or even sometimes better than) Bathe method with less period elongation and equal numerical damping, which can be increased by increasing the value of μ until $\mu = 0.78$. This issue has been investigated practically in examples' section.

4. Flowchart of the proposed method

Computational steps of the proposed algorithm for the linear structural dynamic problems are presented in Fig. 6 in the form of flowchart.

Normally in nonlinear cases, in order to minimize the residual force vector, numerous iterations are applied on the incremental equation; depending on the equilibrium path tracing algorithm, the stiffness matrix can be updated in any of the presented stages. Please note that if any change occurs in the stiffness matrix, the calculated matrix $\hat{\mathbf{K}}^{-1}$ and vector $\hat{R}$ for each sub-step must be updated as well. It is also worthy of paying attention that in nonlinear cases the internal force vector is found using (in each sub-step):

$$F_{s_{t+\Delta t}} = F_{s_t} + \mathbf{K}\Delta U_{t+\Delta t} \tag{24}$$

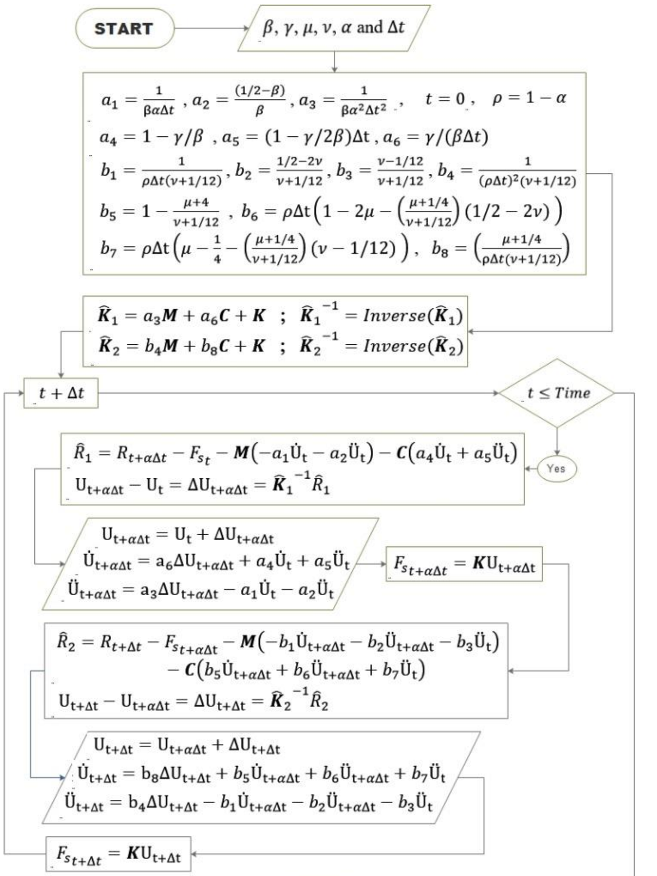

Fig. 6. Flowchart of the proposed method

5. Numerical Properties

A few examples are presented below in order to have a practical assessment of the method. Note that in all figures of this section, Newmark's average acceleration method (AA) is referred to as Newmark with ($\beta = 1/4$; $\gamma = 1/2$), Newmark's linear acceleration method is referred to as Newmark with ($\beta = 1/6$; $\gamma = 1/2$),the method proposed by Bathe and Baig [10] is referred to as Bathe method, and the method proposed by Shojaee et al [21] is referred to as MQB-Spline.

5.1. Example 1

This example demonstrates the robustness of the multi-time stepping methods and is chosen from [11]. As illustrated in Fig. 7 a simple three degree-of-freedom spring system is considered. Eq. (25) shows the governing equations of this problem:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

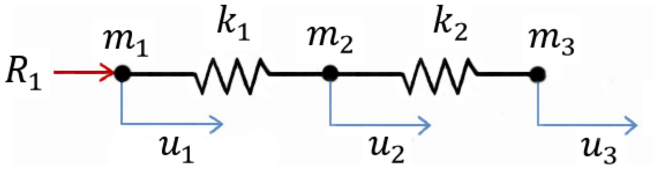

Fig. 7. A three degree-of-freedom spring system [11]

in which with $k_1 = 10^7$, $k_2 = 1$, $m_1 = 0$, $m_2 = 1$, $m_3 = 1$ and the prescribed displacement at node 1 is $u_1 = \sin(1.2t)$. Use of dynamic condensation in this problem leads to the following equations:

$$\begin{bmatrix} m_2 & 0 \\ 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} k_1 u_1 \\ 0 \end{bmatrix} \tag{26}$$

After solution of Eq. (26) is performed, the reaction force can be obtained using the following equation:

$$R_1 = m_1\ddot{u}_1 + k_1 u_1 - k_1 u_2 \tag{27}$$

In this example the mode superposition solution is referred to as reference solution. In order to comply with the reference paper [11], a time step size of 0.2618 sec. has been adopted. Fig. 8 shows the displacement responses of node 3 obtained by various methods over 10 sec.

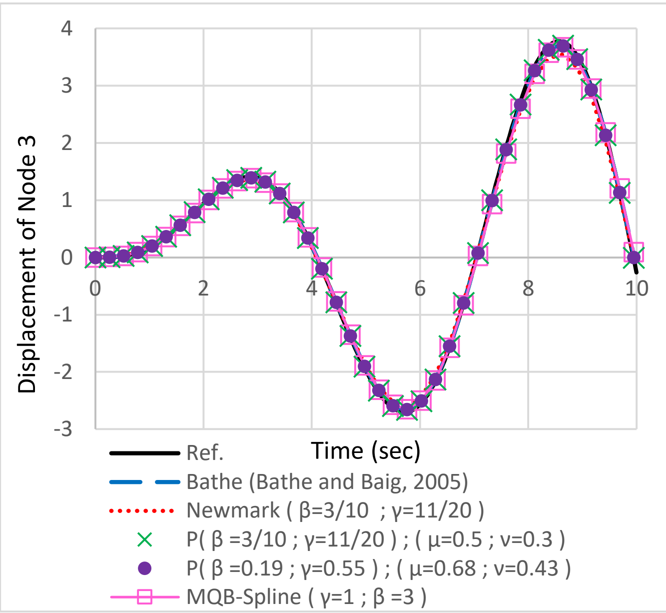

Fig. 8. Displacement of node 3 for various methods

It is perceptible from Fig. 8 that the responses are very close to each other. It is also notable that in this example, for the sake of having reasonable responses, it is required to have numerical damping; so the choice of ($\beta = 3/10$; $\gamma = 11/20$) and ($\mu = 0.5$; $\upsilon = 0.3$) is adopted for the proposed method. Additionally, as mentioned, the choice of ($\beta = 0.19$; $\gamma = 0.55$) and ($\mu = 0.68$; $\upsilon = 0.43$) yields similar (sometimes better) responses to the Bathe method; which is more clear in the following figures.

Figs. 9 and 10 illustrate the velocity responses of nodes 2 and 3 respectively. It can be seen from the Fig. 9 that the proposed method with ($\beta = 3/10$; $\gamma = 11/20$) and ($\mu = 0.5$; $\upsilon = 0.3$) takes about two steps to find the reference solution but for the Bathe method and the proposed method with ($\beta = 0.19$; $\gamma = 0.55$) and ($\mu = 0.68$; $\upsilon = 0.43$) it takes only one step to find the reference solution. Additionally, the large amount of error in MQB-Spline method's response cannot be disregarded. This is while the velocity responses of node 3, illustrated in Fig. 10, show no tangible differences.

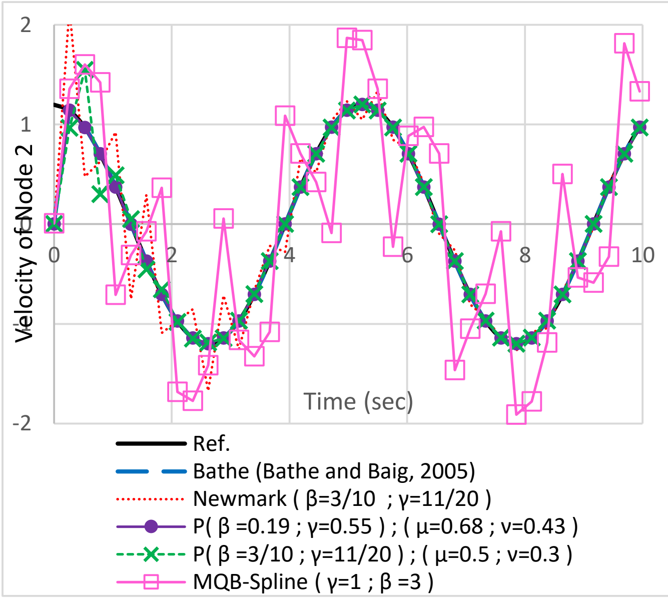

Fig. 9. The Velocity response of Node 2

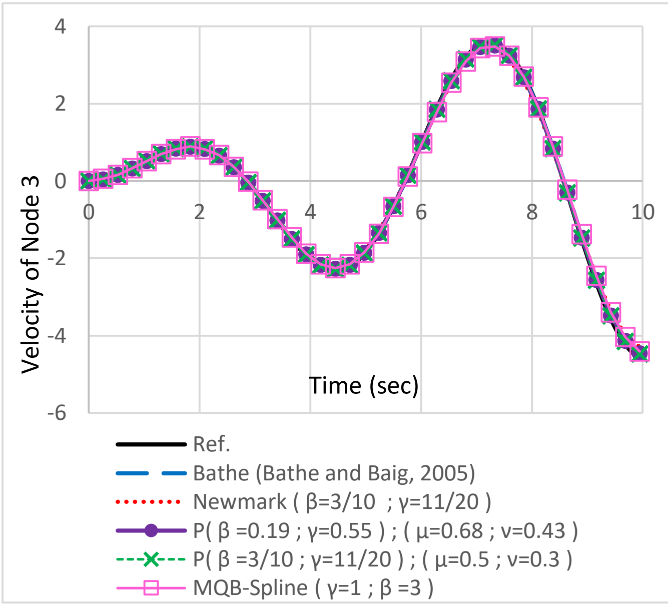

Fig. 10. The Velocity response of Node 3

Figs. 11 and 12 present the acceleration and reaction force response at nodes 2 and 1 respectively. According to these figures, although for the proposed method with ($\beta = 3/10$; $\gamma = 11/20$)

and ($\mu = 0.5 ; \upsilon = 0.3$) it takes a couple of steps to reach an agreement with reference solution, it produces less error than Bathe and MQB-Spline methods on the first step. Interestingly, this error is almost zero for the proposed method with ($\beta = 0.19$; $\gamma = 0.55$) and ($\mu = 0.68 ; \upsilon = 0.43$) which is in complete agreement with the reference solution from the first step. It is also worthy of paying attention that the amount of error in acceleration response of node 2 obtained by MQB-Spline method is considerably large.

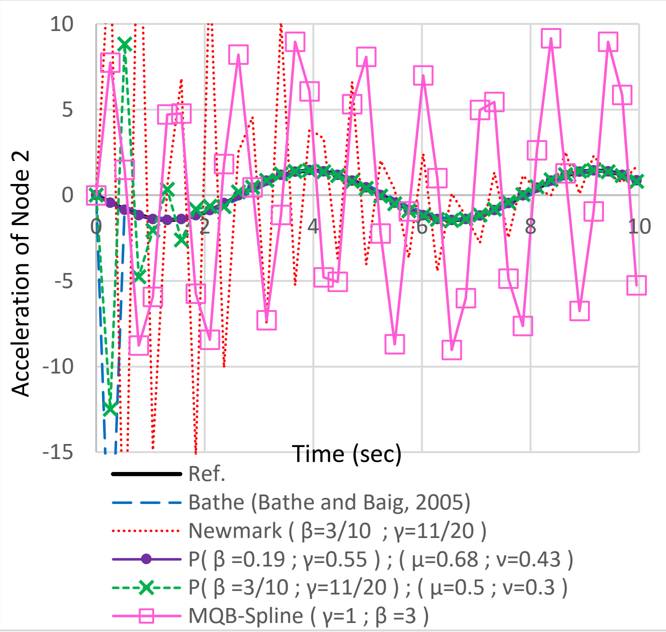

Fig. 11. The acceleration response of node 2

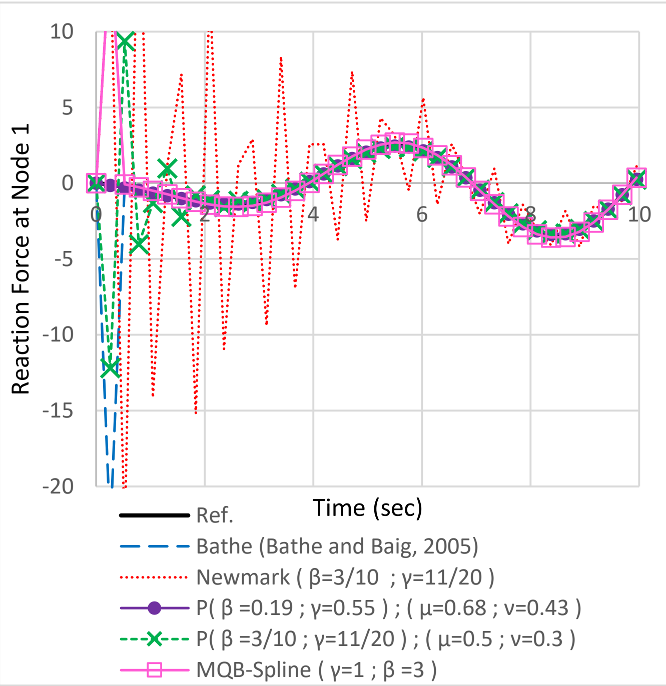

Fig. 12. The reaction force response at node 1

In order to have a quantitative assessment of the obtained results, Tables 1 and 2 present the results in last 3 seconds of analysis for velocity and acceleration of node 2, respectively.

It is notable that proposed (1) and (2) in Tables 1 and 2 represent the proposed method with ($\beta = 0.19 ; \gamma = 0.55$); ($\mu = 0.68 ; \upsilon = 0.43$) and ($\beta = 3/10 ; \gamma = 11/20$); ($\mu = 0.5 ; \upsilon = 0.3$), respectively. According to these tables, MQB-Spline and Newmark method produce very high amount of errors; and it is notable that these errors are even greater in initial time steps while the produced errors by proposed method and Bathe method are in a reasonable range.

5.2. Example 2

This example assesses the proposed method in a nonlinear SDOF problem where damping is present. The SDOF system is shown in Fig. 13 and the force-displacement behavior of this system is shown in Fig. 14 in which it is presented that the problem has elastic-perfect plastic behavior [22] (units are consistent).

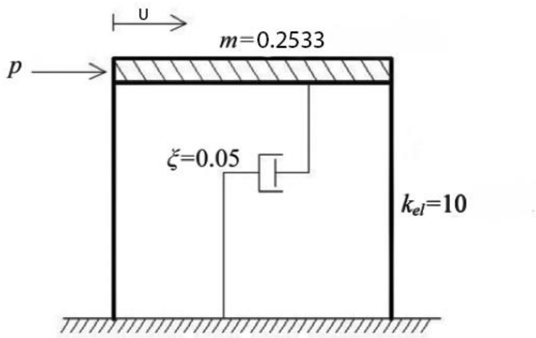

Fig. 13. A SDOF system with nonlinear behavior

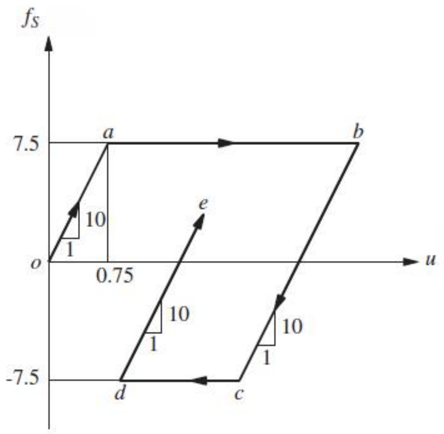

Fig. 14. The elastic-perfect plastic behavior [22]

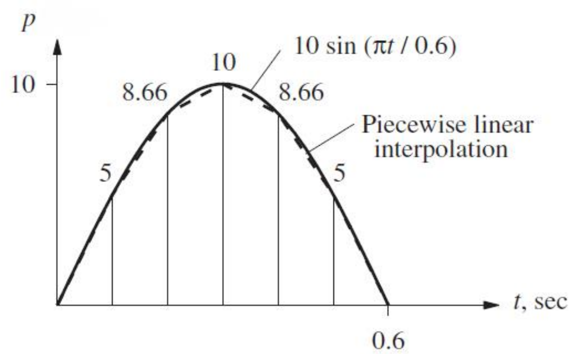

Fig. 15. The harmonic load acting on the system [22]

Table 1. Quantitative results of Velocity in Node 2

t_i	Ref.	Newmark		Bathe		MQB-Spline		Proposed (1)		Proposed (2)	
		Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value	Error %
7.0686	-0.7067	-0.804	13.7	-0.71	0.45	-1.048	48.34	-0.705	0.166	-0.705	0.28
7.3304	-0.9705	-0.939	3.18	-0.98	0.78	-0.699	27.95	-0.971	0.097	-0.970	0.02
7.5922	-1.1403	-1.140	0.01	-1.15	0.87	-0.073	93.58	-1.141	0.123	-1.140	0.03
7.854	-1.1999	-1.268	5.67	-1.209	0.82	-1.910	59.22	-1.200	0.039	-1.199	0.06
8.1158	-1.1438	-1.082	5.37	-1.150	0.62	-1.780	55.66	-1.141	0.206	-1.140	0.28
8.3776	-0.9772	-1.042	6.67	-0.979	0.23	-1.186	21.47	-0.970	0.639	-0.970	0.71
8.6394	-0.7046	-0.680	3.37	-0.712	1.05	0.504	171.5	-0.705	0.081	-0.705	0.03
8.9012	-0.3724	-0.372	0.04	-0.375	0.67	-0.536	44.07	-0.370	0.490	-0.371	0.51
9.163	-0.0043	-0.030	617.	-0.001	73.7	-0.585	1.E+4	0.001	110.5	0.000	1.E+2
9.4248	0.36426	0.418	14.9	0.372	2.34	-0.325	189.2	0.371	1.924	0.3706	1.76
9.6866	0.70931	0.672	5.16	0.710	0.12	1.813	155.6	0.705	0.476	0.7050	0.61
9.9484	0.97218	1.005	3.39	0.978	0.61	1.333	37.14	0.971	0.088	0.9703	0.19
Average Error		-	56.5	-	6.85	-	1204.	-	9.57	-	8.85

Table 2. Quantitative results of Acceleration in Node 2

t_i	Ref.	Newmark		Bathe		MQB-Spline		Proposed (1)		Proposed (2)	
		Value	Error %	Value	Error %	Value	Error %	Value	Error %	Value	Error %
7.0686	-1.1417	-2.776	143.	-1.238	8.46	4.981	536.3	-1.179	3.262	-1.150	0.77
7.3304	-0.8167	1.332	263.	-0.980	20.0	5.453	767.6	-0.862	5.510	-0.817	0.04
7.5922	-0.4130	-2.482	501.	-0.626	51.6	-4.859	1076.	-0.454	9.889	-0.403	2.25
7.854	0.03063	1.142	3627	-0.211	788.	-7.622	24980	-0.008	124.7	0.0492	60.6
8.1158	0.47132	0.355	24.6	0.225	52.3	2.622	456.4	0.445	5.545	0.4973	5.51
8.3776	0.86650	-0.013	101.	0.639	26.3	9.160	957.1	0.849	2.029	0.8967	3.49
8.6394	1.18788	2.522	112.	0.990	16.6	1.274	7.292	1.175	1.107	1.2084	1.73
8.9012	1.38081	0.079	94.2	1.245	9.86	-6.746	588.5	1.381	0.012	1.4017	1.52
9.163	1.44042	2.306	60.2	1.377	4.38	-0.949	1.6E2	1.456	1.085	1.4579	1.21
9.4248	1.36094	1.234	9.31	1.375	1.04	8.958	558.2	1.384	1.733	1.3713	0.76
9.6866	1.13955	0.754	33.9	1.238	8.66	5.857	414.0	1.182	3.695	1.1505	0.96
9.9484	0.81391	1.692	107.	0.980	20.4	-5.263	746.6	0.859	5.558	0.8171	0.39
Average Error		-	423.	-	84.1	-	2604.	-	6.611	-	13.69

The dynamic load acting on the structure is a half-harmonic force; as presented in Fig. 15. The problem is solved using Trapezoidal rule with $\Delta t = 0.1$ *sec.* and the proposed method with $\Delta t = 0.2$ *sec.* (the values of $0.5\Delta t$ are recorded as well) Please be noticed that in order to have a reference solution for the methods to be compared with, Trapezoidal rule with $\Delta t = 0.01$ *sec.*, which is a considerably short time step, is referred to as reference solution. Fig. 16 presents the displacement responses obtained by the mentioned methods.

It is perceptible from Fig. 16 that proposed method with the same solution effort with Trapezoidal rule, that is the adoption of 2 times longer time step, has yielded more accurate responses and is in closest agreement with the reference solution.

Fig. 17 presents the velocity responses of the SDOF system in time interval between 0.2 to 0.6 *sec.*. Again as this figure presents, the proposed method, though with a marginal difference, has yielded responses with higher accuracy.

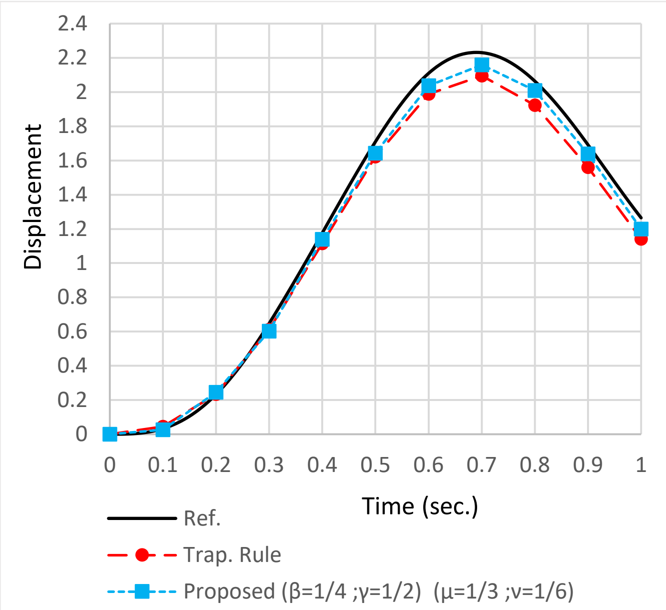

Fig. 16. Displacement responses of SDOF system under

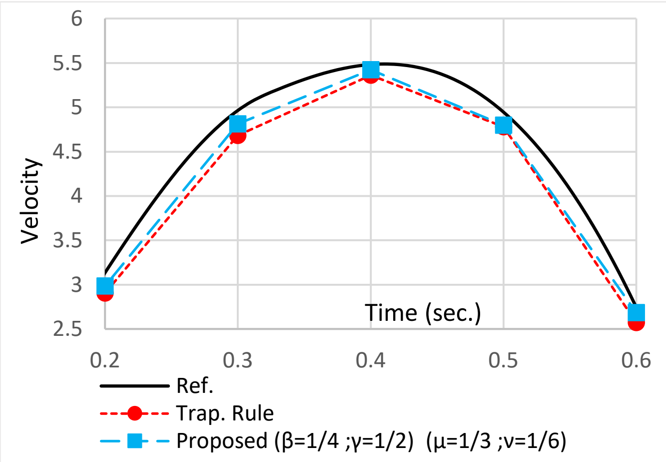

Fig. 17. Velocity responses of SDOF system under dynamic

5.3. Example 3

In this example, the behavior of the proposed method is investigated by an n-degree-of-freedom Mass-Spring system, which is shown in Fig. 18. This example challenges the methods with their amount of numerical damping; where even an unconditionally stable method like Trapezoidal rule grows without bound. This system can also be representative of shear systems in modelling buildings.

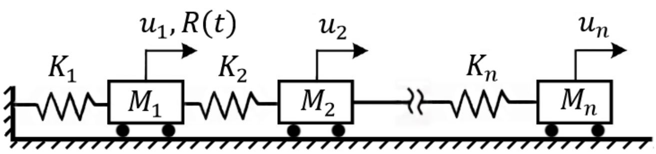

Fig. 18. n degree-of-freedom system

The system is considered to have nonlinear behavior with instantaneous softening stiffness. The following equation presents the nonlinear behavior of the system in mathematical terms:

$$K_i = K_i^0\left(1+\varphi(\Delta U_i)^2\right) \tag{28}$$

in which ΔU_i denotes drift of each d.o.f., K_i^0 is the initial stiffness, and φ is the coefficient which determines the intensity of softening or hardening of stiffness. It is obvious that the system becomes linear if $\varphi = 0$ is adopted. Please be noticed that the friction between system and ground is assumed to be zero. The properties of the problem are shown in Table 3 and Fig. 19 shows the displacement response for 20th d.o.f..

Table 3. Properties of the problem studied in Example 4 (units are

Property	n	$\mathbf{K_i^0}$ $\mathbf{k = 10^6}$	$\mathbf{M_i}$ $\mathbf{m = 10^3}$	$\mathbf{R(t)}$	φ
Value	20	$\frac{k}{i}$: i = 1 to 20	5m : i = 1 to 10 3m : i = 11 to 15 2m : i = 16 to 20	$-50\times10^3\times\sin(t)$	-170

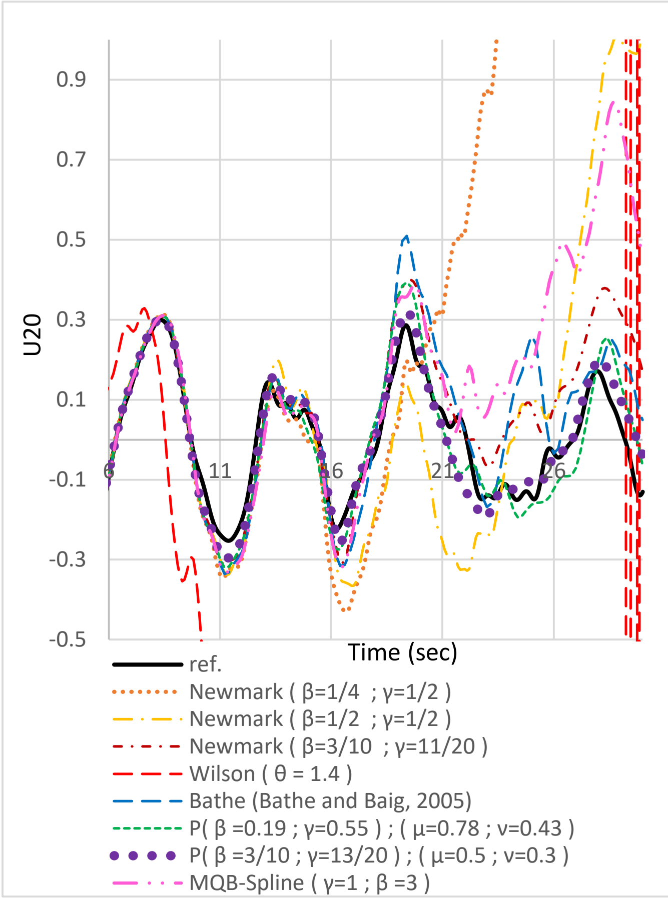

Fig. 19. Displacement response of 20th d.o.f in Example 3

Here, Newmark's average acceleration method with $\Delta t = 0.01$ *sec.* is regarded as reference solution. For other non-composite methods, including Newmark's-AA, the $\Delta t = 0.1$ sec is adopted; and for the composite methods $\Delta t = 0.2$ *sec.* has been utilized to solve the problem. As mentioned, this problem demands great numerical damping and that is why the Newmark method with ($\beta = 3/10$; $\gamma = 11/20$), has yielded better responses than other Newmark family members (as shown in Fig. 19). Regarding the proposed method, as described in precision section, the choice of ($\beta = 3/10$; $\gamma = 13/20$) and ($\mu = 0.5$; $\upsilon = 0.3$) provides the method with enough numerical damping to yield highly accurate responses. Finally, the choice of ($\beta = 0.19$; $\gamma = 0.55$) and ($\mu = 0.78$; $\upsilon = 0.43$) for the proposed method, as mentioned, is expected to yield similar responses to the Bathe method, has shown superiority in this example as well; the reason is that $\mu \succ 0.68$ has induced a numerical damping greater than Bathe method's. In addition, the responses obtained by MQB-Spline method show remarkably large digression from the reference solution.

6. Conclusion

A novel multi-time stepping method is proposed. The method utilizes the Newmark method in the first sub-step and applies a quadratic acceleration method on the second sub-step. The stability analysis of the proposed method proved the unconditional stability region, which is the same region determined for the constant parameters in quadratic acceleration method, provided that the Trapezoidal rule is adopted for the first sub-step. The precision analysis proved that less period elongation is produced with the proposed method in comparison with other methods.

As a conclusion to the proposed method, one definite advantage of this method is its control of numerical dissipation which is rare among multi-time stepping methods. The proposed method also showed minor period elongation than existing ones. Presence of four constant parameters in the proposed method can be seen as disadvantage at the first sight; but in order for the operator not to be confused with so much options of constant parameters, three distinct options with three levels of numerical damping, being zero, medium, and high, was suggested. The responses of proposed method obtained from solving high flexible and nonlinear problems, both being benchmark examples, proven the superiority of the proposed method in practice. Finally, through a nonlinear problem, where high amount of numerical damping was demanded, the numerical dissipation control in the proposed method tested and verified.

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