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# Power Series -Aftertreatment Technique for Nonlinear Cubic Duffing and Double-Well Duffing Oscillators

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**Abstract**

Modeling of large amplitude of structures such as slender, flexible cantilever beam and fluid-structure resting on nonlinear elastic foundations or subjected to stretching effects often lead to strongly nonlinear models of Duffing equations which are not amendable to exact analytical methods. In this work, explicit analytical solutions to the large amplitude nonlinear oscillation systems of cubic Duffing and double-well Duffing oscillators are provided using power series-aftertreatment technique. The developed analytical solutions are valid for both small and large amplitudes of oscillation. The accuracy and explicitness of the analytical solutions are carried out to establish the validity of the method. Good agreements are established between the solution of the new method and established exact analytical solution. The developed analytical solutions in this work can serve as a starting point for a better understanding of the relationship between the physical quantities of the problems as it provides continuous physical insights into the problems than pure numerical or computation methods.

Keyword: Nonlinear; Duffing Oscillators; Explicit analytical solutions; Power-series; Aftertreatment technique.

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## 1. Introduction

The dynamic analysis of large amplitude of structures has been an area of research interests over many decades. Modeling of structures such as slender, flexible cantilever beam carrying a lumped mass with rotary inertia at the intermediate point along its span effects often lead to strongly nonlinear models which are not amendable to exact analytical methods. This fact is also established in the development of governing equations of motion for fluid-structure interaction resting on nonlinear elastic foundations or subjected to stretching effects. Several attempts have been made to solve the nonlinear models or to develop exact analytical solution. In the cases where the exact analytical solutions are presented either in implicit or explicit form, it involves complex mathematical analysis with possession of high skills in mathematics. Also, such solutions do not provide general exact analytical solutions since they often come with conditional statements which make them limited in used. Application of analytical methods such as Exp-function method, He's Exp-function method, improved F-expansion method, Lindstedt-Poincare techniques, parameter-expansion method, quotient trigonometric function expansion method [1-5] to the nonlinear equation present analytical solutions either in implicit or explicit form which often involved complex mathematical analysis leading to analytic expression involving a large number terms. Furthermore, the methods are time-consuming task accompanied with possessing high skills in mathematics. Also, they do not provide general analytical solutions since the solutions often come with conditional statements (i.e. except in limited circumstances where exact analytical solutions are possible) which make them limited in used as many of the conditions with the exact solutions do not meet up with the practical applications since they give approximated solutions that hardly provide an all-encompassing understanding of the nature of systems in response to parameters affecting nonlinearity. Also in practice, analytical solutions with a large number of terms and conditional statements for the solutions are not convenient for use by designers and engineers [6]. Consequently, recourse has always been made to numerical methods such as Runge-Kutta method' finite difference method, finite element method etc. or approximate analytical methods such as Adomian decomposition method (ADM), differential transformation method (DTM), homotopy perturbation method (HPM), homotopy analysis method (HAM), variational iteration method (VIM) [7-43]. However, the determination of Adomian polynomials as carried out in ADM, the restrictions of HPM to weakly nonlinear problems, the lack of rigorous theories or proper guidance for choosing initial approximation, auxiliary linear operators, auxiliary functions, and auxiliary parameters in HAM, operational restrictions to small domains and the search for a particular value for the auxiliary parameter that will satisfy second the boundary condition which leads to additional computational cost in using DTM, the step-by-step integrations and large expressions of terms in VIM limit the applications of the approximation methods. Additionally, most of the approximate methods give accurate predictions only when the nonlinearities are weak, they fail to predict accurately for strong nonlinear models. Also, the methods often involved complex mathematical analysis leading to an analytic expression involving a large number terms and when such methods as HPM, HAM, ADM and VIM are routinely implemented, they can sometimes lead to erroneous results [44]. Therefore, the classical way for finding analytical solutions is obviously still very important since they provide good insights into the significance of various system parameters affecting the phenomena. It is more convenient to use analytical expressions in engineering calculations than experimental or numerical studies. Indisputably, for a better understanding of the relationship between physical quantities/properties, analytical solutions are the obvious starting points [45]. It is convenient for parametric studies, accounting for the physics of the problem and appears more appealing than the numerical solutions. It appears more appealing than the numerical solution as it helps to reduce the computation costs, simulations and task in the analysis of real-life problems. Therefore, an analytical solution is required for the problem. In this research, analytical solutions are provided to the nonlinear Duffing oscillators and double-well Duffing oscillators. Simplicity, flexibility in application, and avoidance of complicated numerical integration are some of the added advantages over the previous methods. It provides complementary advantages of higher accuracy, reduced computation cost, and task as compared to the other methods as found in literature. The developed analytical solutions are compared with the numerical results and the results of approximate analytical solutions and good agreements reached. The analytical solutions can serve as a starting point for a better understanding of the relationship between the physical quantities of the problems as it provides continuous physical insights into the problem than pure numerical or computation methods. Therefore, an exact analytical solution is required for the problem.

Power series method as an approximate analytical method has been used over many decades for solving both linear and nonlinear differential equations as it appeared in many engineering and scientific research analysis. Unfortunately, in the case of oscillatory systems, the truncated series obtained by the method is periodic only in a very small region. This drawback is not only peculiar to the power series solution method, other approximate analytical methods such as HPM, HAM, ADM and VIM also have the same short-coming for oscillatory systems.

To overcome this difficulty in the approximate analytical solutions, an after-treatment technique (AT) has been used to obtain approximate periodic solutions in a wide range of solution. In modifying the Adomian decomposition method to provide periodic solutions in a large region, Venkatarangan and Rajakshmi [46] and Jiao et al. [47] developed AT techniques which are based on using Pade approximation method, Laplace transform and its inverse. Although, the AT techniques were found to be effective in many cases, it has some disadvantages, not only it required a huge amount of computational work to provide accurate approximations for the periodic solutions but also there is a difficulty of obtaining the inverse Laplace transform which greatly restricts the application area of their technique [48]. Consequently, Elhalim and Emad [48] developed a comparatively simple aftertreatment technique after applying differential transformation method to non-linear oscillatory systems. Although, the DTM has proven to be more effective than the other approximate analytical solutions as it does not require many computations as carried out in ADM, HAM, HPM, and VIM. However, the transformation of the nonlinear equations and the development of equivalent recurrence equations for the nonlinear equations using DTM proved somehow difficult in some nonlinear system such as in rational Duffing oscillator, irrational nonlinear Duffing oscillator, finite extensibility nonlinear oscillator. Therefore the quest for comparatively simple, flexible, generic and high accurate analytical solutions continues. In this work, power series solution method is used to develop an accurate analytical solution for the nonlinear cubic Duffing equation. Because of the limitation of the truncated series obtained by the method as it is periodic only in a very small region, the aftertreatment technique as developed by Elhalim and Emad [48] is applied. The approach used in this present work has to advantages over previously developed methods for the nonlinear problems as shown in literature. The accuracy and explicitness of the analytical solutions were carried out to establish the validity of the method. In conclusion, good agreements are established. The analytical solutions can serve as a starting point for a better understanding of the relationship between the physical quantities of the problems as it provides continuous physical insights into the problem than pure numerical or computation methods.

## 2. Development of Analytical Solutions for the Cubic Duffing and double Cubic Duffing Equation

Considered an undamped, unforced nonlinear cubic Duffing oscillator

$$\ddot{u}(\tau)+\alpha u(\tau)+\beta u^3(\tau)=0 \tag{1}$$

While for the double-well Duffing equation, we have

$$\ddot{u}(\tau)-\alpha u(\tau)+\beta' u^3(\tau)=0 \tag{2}$$

where $\beta'$ is a positive constant which may not be a small value.

In both cases, the initial conditions are

$$u(0)=A \quad \dot{u}(0)=0 \tag{3}$$

Assume that the solution of Eq. (1) can be expressed by the following power series

$$u(t)=\sum_{k=0}^{\infty} a_k t^k \tag{4}$$

Substituting the power series of Eq. (4) into Eq. (1), the coefficient $a_k$ can be determined as

$$a_0=A \quad a_1=0 \quad a_2=\frac{-A}{2}\left(\alpha+\beta A^2\right) \quad a_3=0$$

$$a_4=\frac{A}{24}\left[\left(\alpha+\beta A^2\right)(\alpha+3\beta A^2)\right] \quad a_5=0 \tag{5}$$

$$a_6=\frac{-A\left(\alpha+\beta A^2\right)}{720}\left[\alpha+24\beta A^2+27\beta^2A^4\right]$$

Substituting the power series of Eq. (5) into Eq. (4), we have

$$\begin{aligned} u(t)=A-\frac{A}{2}\left(\alpha+\beta A^2\right)t^2+\frac{A}{24}\left[3\beta A^2\left(\alpha+\beta A^2\right)+\alpha\left(\alpha+\beta A^2\right)\right]t^4 \\ -\frac{A\left(\alpha+\beta A^2\right)}{720}\left[\alpha+24\beta A^2+27\beta^2A^4\right]t^6+\ldots \end{aligned} \tag{6}$$

While for the double-well Duffing Equation

$$\begin{aligned} u(t)=A-\frac{A}{2}\left(\beta A^2-\alpha\right)t^2+\frac{A}{24}\left[3\beta A^2\left(\beta A^2-\alpha\right)-\alpha\left(\beta A^2-\alpha\right)\right]t^4 \\ -\frac{A\left(\beta A^2-\alpha\right)}{720}\left[24\beta A^2+27\beta^2A^4-\alpha\right]t^6+\ldots \end{aligned} \tag{7}$$

The power series solution gives solution in the form of truncated series. This truncated series is periodic only in a very small region. In order to make the solution periodic over a large range, we applied Cosine-after treatment (CAT-technique). If the truncated series in Eq. (6) is expressed in even-power, only, of the independent variable *t*, i.e.

$$\Phi_N(t)=\sum_{k=0}^{N} a_{2k}t^{2k} \quad a_{2k+1}=0, \\ \forall k=0,1,\ldots,\frac{N}{2}-1,\ where\ N\ is\ even \tag{8}$$

The CAT- technique is based on the assumption that this truncated series can be expressed as another finite series in terms of the cosine-trigonometric functions with different amplitude and arguments

$$\Phi_N(t)=\sum_{j=1}^{N}\lambda_j cos\left(\Omega_j t\right), \quad where\ n\ is\ finite \tag{9}$$

On expanding both sides of Eq. (9) as power series of $t$ and equating the coefficient of like powers, we have

$$\begin{aligned} &t^0;\ \textstyle\sum_j^n \lambda_j = a_0,\\ &t^2;\ \textstyle\sum_j^n \lambda_j\Omega_j^2 = A\left(\alpha+\beta A^2\right)\\ &t^4;\ \textstyle\sum_j^n \lambda_j\Omega_j^4 = A\left[\left(\alpha+\beta A^2\right)(\alpha+3\beta A^2)\right]\\ &t^6;\ \textstyle\sum_j^n \lambda_j\Omega_j^6 = A\left(\alpha+\beta A^2\right)\begin{bmatrix}\alpha+24\beta A^2\\ +27\beta^2A^4\end{bmatrix}\\ &\ldots \end{aligned} \tag{10}$$

For practical application, it is sufficient to express the truncated series $\Phi_N(t)$ in terms of two cosines with different amplitudes and arguments as

$$\Phi_6(t)=\sum_{j=1}^{2}\lambda_j cos\left(\Omega_j t\right). \tag{11}$$

Therefore, we have

$$\lambda_1+\lambda_2=a_0 \tag{12a}$$

$$\lambda_1\Omega_1^2+\lambda_2\Omega_2^2=A\left(\alpha+\beta A^2\right) \tag{12b}$$

$$\lambda_1\Omega_1^4+\lambda_2\Omega_2^4=A\left[\left(\alpha+\beta A^2\right)(\alpha+3\beta A^2)\right] \tag{12c}$$

$$\lambda_1\Omega_1^6+\lambda_2\Omega_2^6= \\ A\left(\alpha+\beta A^2\right)\left[\alpha+24\beta A^2+27\beta^2A^4\right] \tag{12d}$$

On solving the above Eq. (12a-12d), we have

$$\lambda_1=A\left\{\frac{4\alpha+5\beta A^2+\sqrt{\left(2\alpha+3\beta A^2\right)\left(8\alpha+9\beta A^2\right)}}{2\sqrt{\left(2\alpha+3\beta A^2\right)\left(8\alpha+9\beta A^2\right)}}\right\} \tag{13a}$$

$$\lambda_2=-A\left\{\frac{4\alpha+5\beta A^2-\sqrt{\left(2\alpha+3\beta A^2\right)\left(8\alpha+9\beta A^2\right)}}{2\sqrt{\left(2\alpha+3\beta A^2\right)\left(8\alpha+9\beta A^2\right)}}\right\} \tag{13b}$$

$$\Omega_1=\pm\sqrt{5\alpha+6\beta A^2-\sqrt{\left(2\alpha+3\beta A^2\right)\left(8\alpha+9\beta A^2\right)}} \tag{13c}$$

$$\Omega_2=\pm\sqrt{5\alpha+6\beta A^2+\sqrt{\left(2\alpha+3\beta A^2\right)\left(8\alpha+9\beta A^2\right)}} \tag{13d}$$

Therefore, the approximated periodic solution for *u(t)* is given as

$$u(t)=A\left\{\begin{aligned}&\left\{\frac{4\alpha+5\beta A^2+\sqrt{\left(2\alpha+3\beta A^2\right)\left(8\alpha+9\beta A^2\right)}}{2\sqrt{\left(2\alpha-3\beta A^2\right)\left(8\alpha-9\beta A^2\right)}}\right\}cos\left\{\sqrt{5\alpha+6\beta A^2-\sqrt{\left(2\alpha+3\beta A^2\right)\left(8\alpha+9\beta A^2\right)}}\right\}t\\ &-\left\{\frac{4\alpha+5\beta A^2-\sqrt{\left(2\alpha-3\beta A^2\right)\left(8\alpha+9\beta A^2\right)}}{2\sqrt{\left(2\alpha-3\beta A^2\right)\left(8\alpha+9\beta A^2\right)}}\right\}cos\left\{\sqrt{5\alpha+6\beta A^2+\sqrt{\left(2\alpha+3\beta A^2\right)\left(8\alpha+9\beta A^2\right)}}\right\}t\end{aligned}\right\} \tag{14}$$

While for the double-well Duffing equation, we have the solution as

$$u(t)=A\left\{\begin{aligned}&\left\{\frac{5\beta' A^2-4\alpha+\sqrt{\left(3\beta' A^2-2\alpha\right)\left(9\beta' A^2-8\alpha\right)}}{2\sqrt{\left(2\alpha+3\beta' A^2\right)\left(8\alpha+9\beta' A^2\right)}}\right\}cos\left\{\sqrt{6\beta' A^2-5\alpha-\sqrt{\left(3\beta' A^2-2\alpha\right)\left(9\beta' A^2-8\alpha\right)}}\right\}t\\ &-\left\{\frac{5\beta' A^2-4\alpha-\sqrt{\left(3\beta' A^2-2\alpha\right)\left(9\beta' A^2-8\alpha\right)}}{2\sqrt{\left(2\alpha+3\beta' A^2\right)\left(8\alpha+9\beta' A^2\right)}}\right\}cos\left\{\sqrt{6\beta' A^2-5\alpha+\sqrt{\left(3\beta' A^2-2\alpha\right)\left(9\beta' A^2-8\alpha\right)}}\right\}t\end{aligned}\right\} \tag{15}$$

For the validation of the developed solution, the exact solution to the nonlinear problem was also developed using Jacobi elliptic functions. The exact solution is given as

$$u_{exact}(t) = \sqrt{\frac{A}{cn\left(\omega t, k^2\right)}} \qquad (16)$$

where $cn\left(\omega t, k^2\right)$ is the $cn$ Jacobian elliptic function that has a period in $\omega t$ equal to $4K\left(k^2\right)$, and $4K\left(k^2\right)$ is the complete elliptic integral of the first kind for the modulus $k$ and $\omega$. And

$$\omega = i\sqrt{\alpha + \beta A^2} \;\; k^2 = \sqrt{\frac{2\alpha + \beta A^2}{2(\alpha + \beta^2 A^2)}} \qquad (17)$$

Alternatively, we have

$$u_{exact}(t) = Acn\left(\omega t, k^2\right) \qquad (18)$$

where $\omega = \sqrt{\alpha + \beta A^2} \;\; k^2 = \sqrt{\frac{\beta A^2}{2(\alpha + \beta A^2)}}$

For the double-well Duffing equation, the exact analytical solutions are given as

$$u_{exact}(t) = \sqrt{\frac{A}{cn\left(\omega t, k^2\right)}} \qquad (19)$$

and

$$\omega = i\sqrt{\beta' A^2 - \alpha} \;\; k^2 = \sqrt{\frac{\beta' A^2 - 2\alpha}{2(\beta'^2 A^2 - \alpha)}} \qquad (20)$$

Alternatively, we have

$$u_{exact}(t) = Acn\left(\omega t, k^2\right) \qquad (21)$$

where $\omega = \sqrt{\beta' A^2 - \alpha} \;\; k^2 = \sqrt{\frac{\beta' A^2}{2(\beta'^2 A^2 - \alpha)}}$

### 3. Development of exact analytical solutions for the natural frequencies of the structures

The natural frequency analysis is the sine qua non for the analysis of stability, it must therefore be carried out in the dynamic response of the structures.

$$\ddot{u}(\tau) + \alpha u(\tau) - \beta u^3(\tau) = 0 \qquad (22)$$

Integrating Eq. (22) with respect to τ, we have

$$\frac{1}{2}\dot{u}^2(\tau) + \frac{\alpha}{2}u^2(\tau) - \frac{\beta}{4}u^4(\tau) = c \qquad (23)$$

where $c$ is a constant. On imposing the initial condition, we have

$$c = \frac{\alpha}{2}A^2 - \frac{\beta}{4}A^4 \qquad (24)$$

Substituting Eq. (24) into Eq. (23), we have

$$\frac{1}{2}\dot{u}^2(\tau) + \frac{\alpha}{2}u^2(\tau) - \frac{\beta}{4}u^4(\tau) = \frac{\alpha}{2}A^2 - \frac{\beta}{4}A^4 \qquad (25)$$

which gives

$$dt = \frac{du}{\sqrt{\alpha(A^2 - u^2) - \frac{\beta}{2}(A^4 - u^4)}} \qquad (26)$$

Integrating Eq. (26)

$$\int_0^{\frac{T_p}{4}} dt = \int_0^A \frac{du}{\sqrt{\frac{(K+C)}{M}(A^2 - u^2) - \frac{V}{2M}(A^4 - u^4)}} \qquad (27)$$

Then, we have

$$T_p(A) = 4\int_0^A \frac{du}{\sqrt{\alpha(A^2 - u^2) - \frac{\beta}{2}(A^4 - u^4)}} \qquad (28)$$

On substituting, $u = Asint$ ,we have

$$T_p = 4\int_0^{\frac{\pi}{2}} \frac{A\cos t dt}{\sqrt{\alpha A^2(1 - sin^2 t) - \frac{\beta}{2}A^4(1 - sin^4 t)}} \qquad (29)$$

which gives

$$T_p = 4\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{\left[\alpha - \frac{\beta A^2}{2}\right] - \frac{\beta A^2}{2} sin^2 t}} \qquad (30)$$

and

$$T_p = 4\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{\left[\alpha - \frac{\beta A^2}{2}\right]}\sqrt{1 - \left\{\frac{\frac{\beta A^2}{2}}{\left[\alpha - \frac{\beta A^2}{2}\right]}\right\} sin^2 t}} \qquad (31)$$

Which can be written as

$$T_p = \frac{4}{\xi_1}\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - \xi_2^2 sin^2 t}} \qquad (32)$$

where $\xi_1 = \sqrt{\left[\alpha - \frac{\beta A^2}{2}\right]} \;\; \xi_2 = \sqrt{\frac{\frac{\beta A^2}{2}}{\left[\alpha - \frac{\beta A^2}{2}\right]}}$

The above Eq. (32) is called the complete elliptic integral of first kind

$$T_p = \frac{4}{\xi_1}\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - \xi_2^2 sin^2 t}} \quad \xi_2^2 < 1 \qquad (33)$$

In order to evaluate the integral, we expand the integral in the form

$$\frac{1}{\sqrt{1-\xi_2^2 sin^2 t}}=1+\frac{\xi_2^2}{2}sin^2 t+\frac{3\xi_2^2}{8}sin^4 t+\frac{5\xi_2^6}{16}sin^6 t+\frac{35\xi_2^8}{128}sin^8 t+... \quad (34)$$

The above series is uniformly convergent for all $\xi_2$, and may, therefore, be integrated term by term. Then, we have

$$T_p=\frac{4}{\xi_1}\int_0^{\frac{\pi}{2}}\left(1+\frac{\xi_2^2}{2}sin^2 t+\frac{3\xi_2^4}{8}sin^4 t+\frac{5\xi_2^6}{16}sin^6 t+\frac{35\xi_2^8}{128}sin^8 t+...\right)dt \quad (35)$$

which can be expressed as

$$T_p=\frac{2\pi}{\xi_1}+\frac{2\pi}{\xi_1}\left(\left(\frac{1}{2}\right)^2\xi_2^2+\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\xi_2^4+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2\xi_2^6+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)^2\xi_2^8+...+\left(\prod_{n=1}^{N}\frac{2n-1}{2n}\right)^2\xi_2^{2N}\right) \quad (36)$$

But $T_p(A)=\frac{2\pi}{\omega}\to\omega=\frac{2\pi}{T_p(A)}$

$$\omega=\frac{\xi_1}{1+\left(\left(\frac{1}{2}\right)^2\xi_2^2+\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\xi_2^4+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2\xi_2^6+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)^2\xi_2^8+...+\left(\prod_{n=1}^{N}\frac{2n-1}{2n}\right)^2\xi_2^{2N}\right)} \quad (37)$$

It can easily be seen that as the nonlinear term tends to zero, the frequency ratio of the nonlinear frequency to the linear frequency, $\frac{\omega}{\omega_b}$ tends to 1.

$$\lim_{\xi_2\to 0}\frac{\omega}{\omega_b}=1 \quad (38)$$

Also, as the amplitude A tends to zero, the frequency ratio of the nonlinear frequency to the linear frequency, $\frac{\omega}{\omega_b}$ tends to 1.

$$\lim_{A\to 0}\frac{\omega}{\omega_b}=1 \quad (39)$$

Where $\omega_b=\sqrt{\alpha}$

For very large values of the amplitude A, we have

$$\lim_{A\to\infty}\frac{\omega}{\omega_b}=\infty \quad (40)$$

Alternatively, we can have

$$\omega_{1,exact}=\frac{\pi\xi_1}{2\left[\int_0^{\pi/2}\left(1-\xi_2^2 sin^2 t\right)^{-\frac{1}{2}}dt\right]} \quad (41)$$

Using term-by-term series integration method, we also developed an approximation analytical solution for the nonlinear natural frequency as

$$\omega_1=\frac{\xi_1}{\left(1+0.25000\xi_2+0.11680\xi_2^{\,2}+0.59601\xi_2^{\,3}-1.90478\xi_2^{\,4}+2.04574\xi_2^{\,5}\right)} \quad (42)$$

The ratio of the nonlinear frequency, $\omega_1$ to the linear frequency, $\omega_b$

$$\frac{\omega_1}{\omega_b}=\frac{1}{\left(1+0.25000\xi_2+0.11680\xi_2^{\,2}+0.59601\xi_2^{\,3}-1.90478\xi_2^{\,4}+2.04574\xi_2^{\,5}\right)} \quad (43)$$

Also, it can easily be seen that as the nonlinear term tends to zero, the frequency ratio of the nonlinear frequency to the linear frequency, $\frac{\omega}{\omega_b}$ tends to 1.

$$\lim_{\xi_2\to 0}\frac{\omega}{\omega_b}=1 \quad (44)$$

Also, as the amplitude A tends to zero, the frequency ratio of the nonlinear frequency to the linear frequency, $\frac{\omega}{\omega_b}$ tends to 1.

$$\lim_{A\to 0}\frac{\omega}{\omega_b}=1 \tag{45}$$

Also, it should be pointed out that when the nonlinear term, $\beta$ is set to zero, we recovered the linear natural frequency

$$\omega_b=\sqrt{\alpha} \tag{46}$$

## 4. Results and Discussion

Fig. 1 shows the comparison of the linear vibration with nonlinear oscillation. It could be seen in the figure that the discrepancy between the linear and nonlinear amplitudes increases with time.

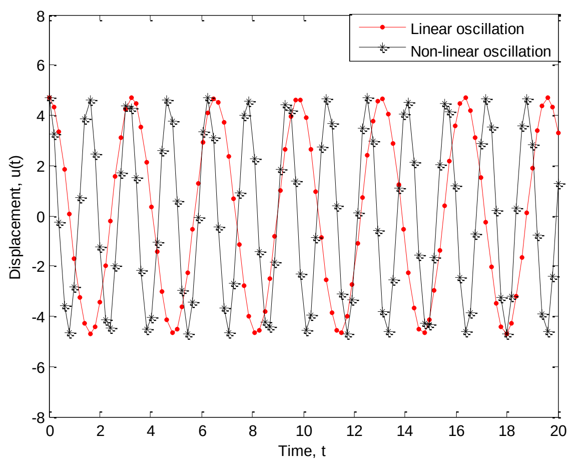


Fig. 1 Comparison of displacement time history for the linear and nonlinear oscillation

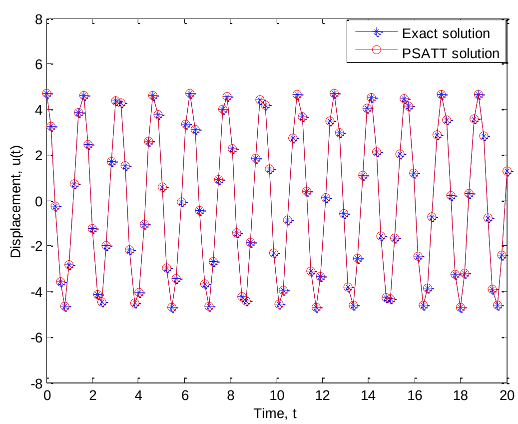


Fig. 2 Comparison between the obtained results and the exact solution for the linear vibration

Fig. 2 presents the comparison of exact solution and the Power series –Aftertreatment techniques (PSATT) solution results for the nonlinear models. The results show that good agreements are established reached and good agreements are established reached.

Fig. 3 shows a phase-space/plane curve of $u_t$ versus $u$ which shows the behaviour of the oscillator. The phase plots show the behaviour of the oscillator when the amplitude is varied. It is periodic with center (0, 0) where stability conditions can be observed. This situation is common in unforced, undamped cubic Duffing oscillators.

Fig. 4 shows the effect of amplitude on the frequency ratio. It can be seen from the figure, in contrast to linear systems, the nonlinear frequency is a function of amplitude so that the larger the amplitude, the more pronounced the discrepancy between the linear and the nonlinear frequencies becomes.

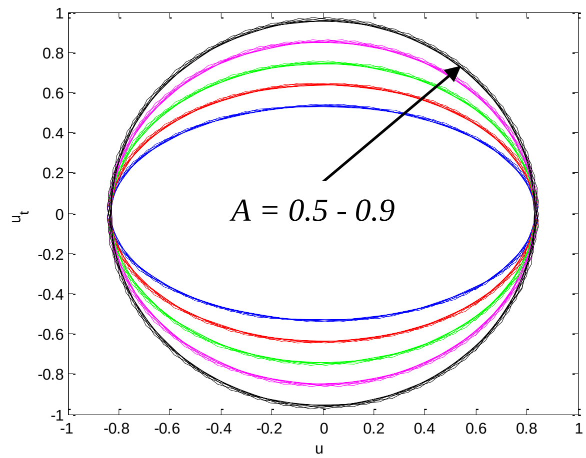


Fig. 3Phase-space curve of $u_t$ versus u

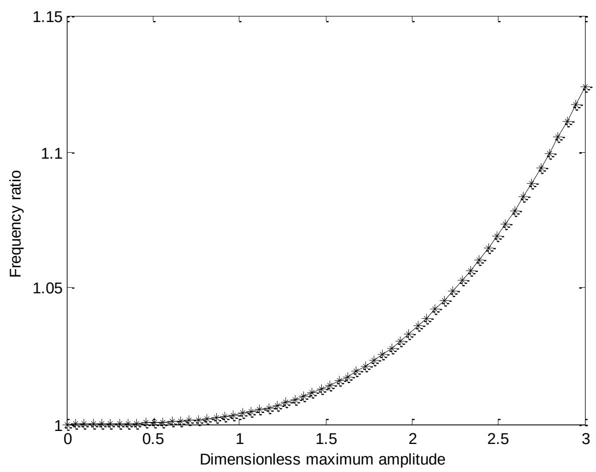


Fig. 4nonlinear amplitude-frequency response curve

## 5. Conclusion

In this work, analytical solutions to large amplitude nonlinear oscillation systems have been provided using power series-aftertreatment technique. The developed analytical solutions are shown to be valid for both small and large amplitudes of oscillation. The accuracy and explicitness of the analytical solutions were carried out to establish the validity of the method. Good agreements are established between the method and the exact solution. The analytical solutions can serve as a starting point for a better understanding of the relationship between the physical quantities of the problems as it provides continuous physical insights into the problems than pure numerical or computation methods.

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