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A new approach for nonlinear vibration analysis of thin and moderately thick rectangular plates under inplane compressive load

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Abstract

In this study, a hybrid method is proposed to investigate the nonlinear vibrations of pre- and post-buckled rectangular plates for the first time. This is an answer to an existing need to develop a fast and precise numerical model which can handle the nonlinear vibrations of buckled plates under different boundary conditions and plate shapes. The method uses the differential quadrature element, arc-length, harmonic balance and direct iterative methods. The governing differential equations of plate vibration have been extracted considering shear deformations and the initial geometric imperfection. The solution is assumed to be the sum of the static and dynamic parts which upon inserting them into the governing equations, convert them into two sets of nonlinear differential equations for static and dynamic behaviors of the plate. First, the static solution is calculated using a combination of the differential quadrature element method and an arc-length strategy. Then, putting the first step solutions into the dynamic nonlinear differential equations, the nonlinear frequencies and modal shapes of the plate are extracted using the harmonic balance and direct iterative methods. Comparing the obtained solutions with those published in the literature confirms the accuracy and the precision of the proposed method. The results show that an increase in the nonlinear vibration amplitude increases the nonlinear frequencies.

Keywords:

Buckled plate; Differential quadrature element method; Direct iterative method; Harmonic balance method; Nonlinear vibration.

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1. Introduction

Basically, many natural phenomena behave in a nonlinear manner. Nonlinear vibration of different structures has been an interesting research field among the researchers. Also, plates are one of the most frequently used structures in industrial applications. Therefore, investigating the nonlinear vibration of plate structures under the application of different loads has been the focus of many studies. Wah[1] proposed an approximated formula for large amplitude vibrations of rectangular plates using the Berger equations and showed that these equations can be applied to dynamic problems. Mei[2] investigated the large amplitude vibration of beams and plates using the finite element method. He extracted the governing nonlinear differential equations of plates using Berger equations and studied different combinations of simply-supported and clamped boundary conditions. Yamaki and Chiba [3] used Marguerre equations to study the nonlinear vibration of a fully clamped isotropic plate under a distributed periodic load considering initial geometric imperfection and initial edge displacements. Mei and Decha-Umphai [4] studied nonlinear forced vibration of rectangular plates under harmonic forces using the finite element method. Kapania and Yang [5] presented a finite element model to study the buckling, post-buckling and nonlinear vibration of initially imperfect plates. They showed that considering the inplane displacements and inertia into governing equations decreases the nonlinear frequencies. Hui-shen [6] used Von-Karman theory of large deformations to study the post-buckling of rectangular plates under uniaxial compression combined with lateral pressure. He used Galerkin method to model the lateral pressure as an initial geometric imperfection and solved the obtained equations using perturbation method. Woo and Nair [7] investigated large amplitude vibrations of thin plates taking into account the nonlinear terms of Von-Karman equations. They used the Galerkin method to convert the governing equations into a duffing equation and solved it using the harmonic balance method. Esmaeilzadeh and Jalali [8] studied nonlinear vibrations of a simply-supported viscoelastic rectangular plate using Voight-Kelvin model**.** Chen et al.[9] investigated the nonlinear vibration of plates using finite element and harmonic balance method. Azrar et al.[10] studied free and forced nonlinear vibrations of a rectangular plate under a harmonic force using an asymptotic numerical method. Ribeiro [11] studied geometrically nonlinear vibration of thick plates using finite element. Bikri et al. [12] used Simpson's rule to investigate large amplitude vibration of an isotropic rectangular plate. Amabili [13-15] studied large amplitude oscillations of a rectangular plate under an external force with a near to resonance frequency. Zarubinskaya and Van Horssen [16] investigated nonlinear vibration of simply supported square plates on nonlinear elastic foundation. Girish and Ramchandra [17] investigated the vibration of thermally postbuckled rectangular composite plates considering shear deformation and initial geometric imperfection by Galerkin method. Fung and Chen [18] used average stress method to extract the nonlinear differential equations of rectangular FGM plates and solved them by Galerkin and Runge-Kuta methods. They studied the effects of initial stress and initial geometric imperfection on nonlinear behavior of plates and found that tensile stress could increase the frequencies and compressive stress decrease them. Shooshtari and Khadem [19] used the multiple scale method to solve the nonlinear vibration of a rectangular plate. They used the Galerkin method to convert the governing differential equations into a third-order duffing equation and solved it with the multiple scale method. Bakhtiari-Nejad and Nazari [20] studied the nonlinear vibration of isotropic plates with viscoelastic laminate using the methods of multiple scale and finite difference. Houmat [21] studied the nonlinear vibration of composite annular elliptical plates considering the shear deformations, rotary inertia and geometrical nonlinearity using harmonic balance method. Singha and Daripa [22] used finite element method to study the nonlinear vibration and dynamic stability of isotropic and composite plates under inplane periodic loads. Rashidi et al. [23] considered shear deformation and rotary inertia in governing equations of nonlinear vibration of an isotropic rectangular plate. They used Galerkin and homotopy perturbation methods to solve obtained equations. Hashemi and Jaberzadeh [24] proposed a method for investigating the nonlinear vibration of plates using the finite strip method. They used a combination of trigonometric functions and polynomials to extract nonlinear frequencies and modal shapes of the plate. Malekzadeh [25, 26] used the differential quadrature method and first-order shear deformation theory to study nonlinear vibrations of thin to moderately thick composite plates. Ma et al. [27] used multiple scale method to study the nonlinear dynamic of stiffened plate. They considered the stiffeners as Euler beams and used Lagrange equation and modal superposition method to derive the dynamic equilibrium equations of the stiffened plate according to energy of the system. Yazdi [28-32] used homotopy perturbation method to investigate nonlinear vibrations of FGM plates. He confirmed that homotopy perturbation method with two terms has good precision. Detroux et al [30-32] used harmonic balance method to investigate nonlinear behavior of some mechanical system. Daneshmehr et al. [33] applied Eringen's nonlocal theory to study the small scale effects on natural frequencies of nanoplates made of functionally graded materials. They used higher order shear deformation plate theory and generalized differential quadrature method to obtain accurate results. Liu et al. [34] used differential quadrature method to investigate the buckling and postbuckling of piezoelectric nanoplates based on the nonlocal Mindlin plate model. They observed that an increase in nonlocal parameters leads to smaller buckling and postbuckling loads and the positive/negative electric voltage decrease/increase buckling and postbuckling loads. KrishnaBhaskar and MeeraSaheb [35] proposed coupled displacement field (CDF) method to investigate nonlinear vibration of Mindlin plates. Their method simplified the vibration problem due to the reduction in number of undetermined coefficients compared with conventional Rayleigh-Ritz method..

To the best of authors knowledge, there is a need to propose a fast and accurate numerical method to investigate nonlinear vibrations of buckled plates with none of the restrictions of the other references. This work focuses on the application of a novel method for investigating the nonlinear vibrations behavior of rectangular buckled plates under uniaxial compressive load. After extracting the governing differential equations, the solution is considered as the sum of the static (time independent) and dynamic (time dependent) responses. Using this assumption, the governing equations are converted into two nonlinear differential equation sets; a time independent and a time dependent equation set. The former set is discretized using the differential quadrature element method (DQEM) to obtain a nonlinear system of algebraic equations which will be solved by an arc-length strategy to find the post-buckling state of the plate. Then, the nonlinear vibration equations about the buckled plate are discretized using DQEM and the harmonic balance method which will result in a nonlinear eigenvalue problem. This nonlinear eigenvalue problem will be solved by the direct iterative method to find the nonlinear natural frequencies of the buckled plate. To validate the results of the proposed method they are compared with those obtained from the published literature through a series of case studies. Also, the effects of several parameters on the nonlinear frequencies of buckled plate are investigated.

2. Governing Equations

Fig. 1 shows an initially imperfect plate having dimensions *a* and *b*, and thickness *h*. the plate is under the compressive inplane load, *P*, acting across the width.

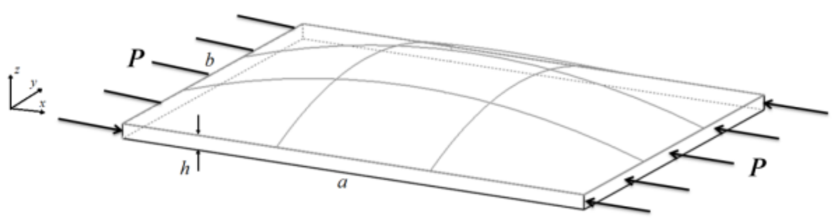

Figure 1. A rectangular plate with initial imperfection under uniaxial compressive load.

Considering the Von-Karman strain-displacement relations for large deformations and the first-order shear deformation theory, the relation between stresses and displacements can be obtained as equation 1.

$$\sigma_{xx}=\frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x}+z\frac{\partial \alpha}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial w}{\partial x}\frac{\partial w_0}{\partial x}+\nu(\frac{\partial v}{\partial y}+z\frac{\partial \beta}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{\partial w}{\partial y}\frac{\partial w_0}{\partial y})\right)$$

$$\sigma_{yy}=\frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial y}+z\frac{\partial \beta}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{\partial w}{\partial y}\frac{\partial w_0}{\partial y}+\nu(\frac{\partial u}{\partial x}+z\frac{\partial \alpha}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial w}{\partial x}\frac{\partial w_0}{\partial x})\right)$$

$$\sigma_{zz}=\frac{E}{1-\nu^2}\left(\nu(\frac{\partial u}{\partial x}+z\frac{\partial \alpha}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial w}{\partial x}\frac{\partial w_0}{\partial x})+\nu(\frac{\partial v}{\partial y}+z\frac{\partial \beta}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{\partial w}{\partial y}\frac{\partial w_0}{\partial y})\right) \tag{1}$$

$$\tau_{xy}=\frac{E}{2\ \ 1+\nu}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+z\frac{\partial \alpha}{\partial y}+z\frac{\partial \beta}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w_0}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial w_0}{\partial x}\right)$$

$$\tau_{xz}=\frac{E}{2\ \ 1+\nu}\left(\frac{\partial w}{\partial x}+\alpha\right)$$

$$\tau_{yz} = \frac{E}{2\;1+\nu}\left(\frac{\partial w}{\partial y} + \beta\right)$$

where u, v and w are displacements in x, y and z directions, and α and β are the rotations about the x and y axes, respectively. w_0 is the initial geometric imperfection, and E and ν represent the Young's modulus and the Poison's ratio. The inplane forces and moments of the plate can be determined by integrating these stresses.

$$\begin{aligned}
N_x &= \frac{Eh}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial w_0}{\partial x} + \nu\frac{\partial w}{\partial y}\frac{\partial w_0}{\partial y}\right] \\
N_y &= \frac{Eh}{1-\nu^2}\left[\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial y}\frac{\partial w_0}{\partial y} + \nu\frac{\partial w}{\partial x}\frac{\partial w_0}{\partial x}\right] \\
N_{xy} &= \frac{Eh}{2\;1+\nu}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w_0}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial w_0}{\partial x}\right] \\
M_x &= \frac{Eh^3}{12\;1-\nu^2}\left[\frac{\partial \alpha}{\partial x} + \nu\frac{\partial \beta}{\partial y}\right] \\
M_y &= \frac{Eh^3}{12\;1-\nu^2}\left[\frac{\partial \beta}{\partial y} + \nu\frac{\partial \alpha}{\partial x}\right] \\
M_{xy} &= \frac{Eh^3}{24\;1+\nu}\left[\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}\right] \\
Q_x &= K_s Gh\left[\alpha + \frac{\partial w}{\partial x}\right] \\
Q_y &= K_s Gh\left[\beta + \frac{\partial w}{\partial y}\right]
\end{aligned} \tag{2}$$

here, K_s represents the shear correction factor and G is the shear modulus. Using these forces and moments, the governing differential equations of motion turn out to be:

$$\begin{aligned}
&\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \mu\frac{\partial^2 u}{\partial t^2} \\
&\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = \mu\frac{\partial^2 v}{\partial t^2} \\
&\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = \mu\frac{\partial^2 w}{\partial t^2} - \left[N_x\frac{\partial^2\;w+w_0}{\partial x^2} + 2N_{xy}\frac{\partial^2\;w+w_0}{\partial x \partial y} + N_y\frac{\partial^2\;w+w_0}{\partial y^2}\right] \\
&\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_x\frac{\partial^2 \alpha}{\partial t^2}
\end{aligned} \tag{3}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{y}}{\partial y}-Q_{y}=I_{y}\frac{\partial^2\beta}{\partial t^2}$$

where $I_x = I_y = \rho h^3 / 12$, $\mu = \rho h$ and ρ is the density of the plate. Inserting equation (2) into equation (3), the nonlinear differential equations of the plate are obtained based on the displacement field. In order to solve this system of nonlinear differential equations, the solution is assumed to be composed of:

$$\begin{aligned} w &= w_s + w_d \\ u &= u_s + u_d \\ v &= v_s + v_d \\ \alpha &= \alpha_s + \alpha_d \\ \beta &= \beta_s + \beta_d \end{aligned} \tag{4}$$

where subscripts *s* and *d* stand for the static solution (time independent) and dynamic solution (time dependent), respectively. Inserting equation (4) into the governing equations and eliminating the time dependent terms, the nonlinear differential equations of equilibrium are obtained as equations (A1)-(A5) in appendix A.

Deformation of the plate under different compressive load ratios can be obtained solving equations (A1) to (A5). To solve them, they are discretized using differential quadrature element method (Bellman et al. [36], Quan and Chang [37], Wang and Wang [38], Wang [39]). The resulting nonlinear algebraic equations will be solved using an arc-length strategy (Wempner [40], Riks [41], Forde and Stiemer [42]) which has the ability to pass the bifurcation points if the load passes the buckling load. Upon solution of the equilibrium problem, inserting equation (4) one more time into the governing equations and eliminating terms which only depend on the static solutions, and considering large amplitude vibrations about the buckled shape of the plate, the nonlinear differential equations of post-buckled plate vibrations can be expressed as equations (B1)-(B5) in appendix B. These equations can be presented in matrix form as:

$$\left[m\right]\ddot{\phi} + \left[KL\right]+\left[KNL\right] \phi = 0 \tag{5}$$

where [*KL*] and [*KNL*] are linear and nonlinear stiffness matrices and {*ø*} is the displacement vector in the form of $\{u,v,w,\alpha,\beta\}^T$. In order to solve these equations and extract the nonlinear frequencies and modal shapes of the buckled plate, the harmonic balance method (Krylov and Bogoliubov [43], Detroux et al.[30-32] and the differential quadrature element method are used to obtain a nonlinear eigenvalue problem in the form of:

$$\left(-\left[m\right]\omega^2+\left(\left[KL\right]+\frac{3}{4}\left[KNL\right]\right)\right) \phi = 0 \tag{6}$$

Coefficient ¾ in Eq. (6) has been induced by using the harmonic balance method and elimination of the higher-order terms. As mentioned earlier, the direct iterative method (Ribeiro and Petyt [44]) has been used to solve this nonlinear eigenvalue problem. First, the nonlinear terms are eliminated and the resulted linear eigenvalue problem is solved. Then, nonlinear terms are calculated using linear modal shapes. Next, frequencies and modal shapes of the new eigenvalue problem are extracted and the computed nonlinear modal shapes are used to calculate the nonlinear terms. This process continues until the convergence criteria (Equation (7)) is reached.

$$\left|\frac{\omega_{i+1}^2-\omega_i^2}{\omega_{i+1}^2}\right| \le Err_{all} \tag{7}$$

where *i* and *i+1* subscripts refer to i^{th} and $(i+1)^{th}$ iterations. Err_{all} is the relative error and its value may be selected between 10^{-6} and 10^{-3}.

The solution of this nonlinear eigenvalue problem provides the nonlinear frequencies and corresponding modal shapes of the plate under the compressive inplane load .

3. Results

In order to verify the precision and the consistency of the proposed approach, the formulation developed in the preceding section is applied on a square simply-supported plate and the results are compared with those published in the literature. Table 1 shows the result for a plate having Poison's ratio equal to 0.3 under no-load condition. The ratio of the first nonlinear frequency to linear frequency for different vibration amplitudes of the plate's center are compared with the results obtained by Wah [1], Mei[2] and KrishnaBhaskar and MeeraSaheb[35]. As mentioned earlier, Mei[2] used finite element method to solve this problem while Wah[1] used Galerkin method and KrishnaBhaskar and MeeraSaheb[35] used CDF method to find the nonlinear frequencies of the unloaded plate.

Table 1. The first nonlinear frequency against the maximum amplitude at the center for a square plate.

	$\frac{W_a}{h}$	Ref [35]	Ref [2]	Ref [1]	Present
$\frac{\omega_{NL}}{\omega_L}$	0	1	1	1	1
	0.2	1.0203	1.0182	1.0222	1.0194
	0.4	1.0789	1.0709	1.0858	1.0750
	0.6	1.1699	1.1530	1.1833	1.1599
	0.8	1.2865	1.2590	1.3067	1.2668
	1	1.4227	1.3825	1.4491	1.3892

where W_a/h is the ratio of nonlinear vibration amplitude to the thickness and ω_{NL}/ω_L is the ratio of nonlinear frequency to linear frequency. Table 1 shows that the proposed algorithm is in very good agreement with those obtained by other references ([1],[2],[35]). However, as the relative amplitude of vibration increases, the results obtained by current study diverge from those of Wah[1] but still are in very good agreement with Mei[2]. Differences between the presented results and Wah[1] are because of the in-plane inertia that has been considered in this study. Mei[2], too had taken into account the effects of inplane displacements and inertia, and observed that the consideration of these effects could reduce the nonlinear frequencies.

Figure 2 shows the variation of vibration amplitude ratio against the frequency ratio for different aspect ratio of the square plate. The plate is simply-supported with length to thickness ratio equal to 240 (i.e. a/h=240). Comparison of the results of the current study and those obtained by the Mei[2] shows the accuracy of the proposed method. It can be seen that increasing the length to width ratio results in an increase in the nonlinear frequency. The figure also shows that an increase in the aspect ratio induces a shift of the backbone curve to the right.

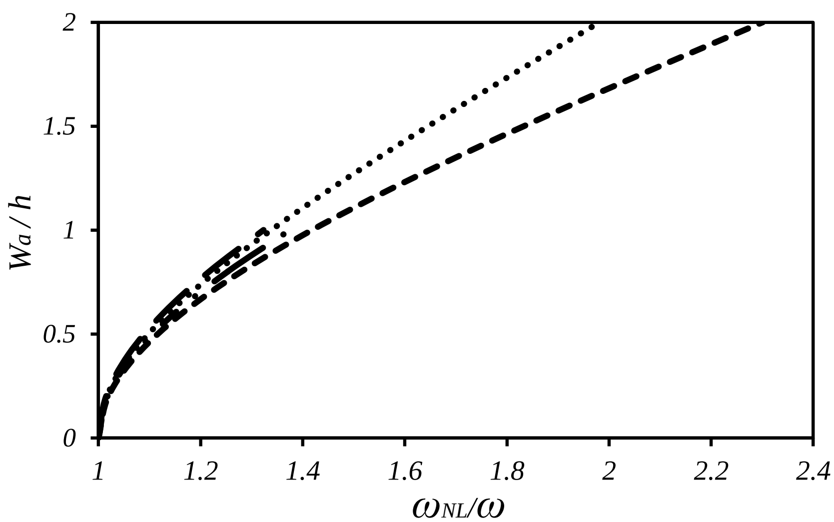

Figure 2: Variation of maximum amplitude versus the frequency ratio for different aspect ratio.
(a/b=1 (…), *a/b=2* (---), *a/b=1 Mei [2]* (-.), *a/b=2 Mei[2]* (-..))

Figure 3 shows the variations of the first four nonlinear natural frequencies of a plate in terms of the compressive load for different nonlinear vibration amplitudes. The square plate is simply supported at its boundaries. The figure shows that an increase in compressive load in pre-buckling state decrease the frequencies as a result of the bending stiffness reduction, and increase them after the buckling load because of the domination of the stretching stiffness over bending stiffness. The same behavior can be observed for linear vibrations. The figure also shows that increasing the nonlinear vibration amplitude increases the corresponding frequencies due to the increase in the bending stiffness.

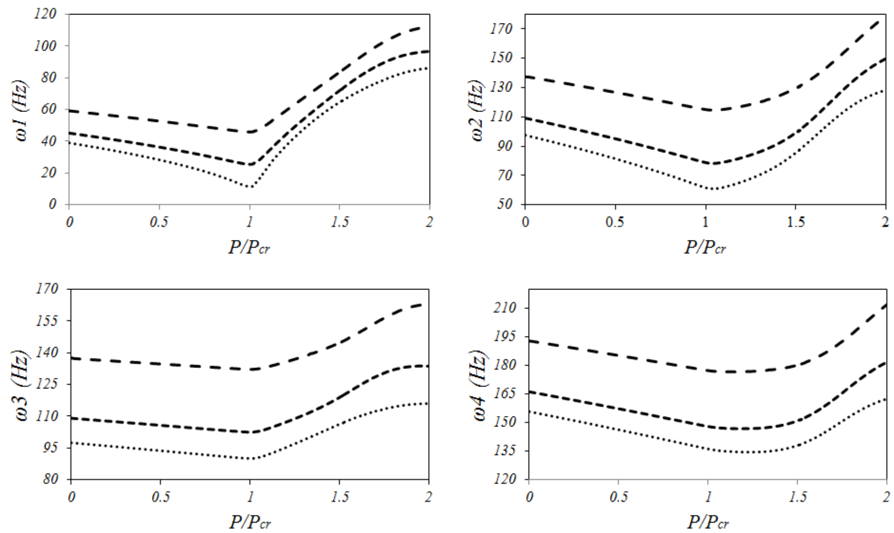

Figure 3: variation of first four natural frequencies of a plate against compressive load for different vibration amplitude.

(linear (…), $W_a/h=0.5$ (---), $W_a/h=1$ (_))

The first four nonlinear modal shapes of the plate under different buckling states are presented in figure 4. Figure 4(a) represents the modal shapes at the pre-buckling state with $P/P_{cr}=0$, where Figure 4(b) shows the same modes at the post-buckled state of $P/P_{cr}=2$. The vibration amplitude for both cases is fixed at W_a/h=0.5.

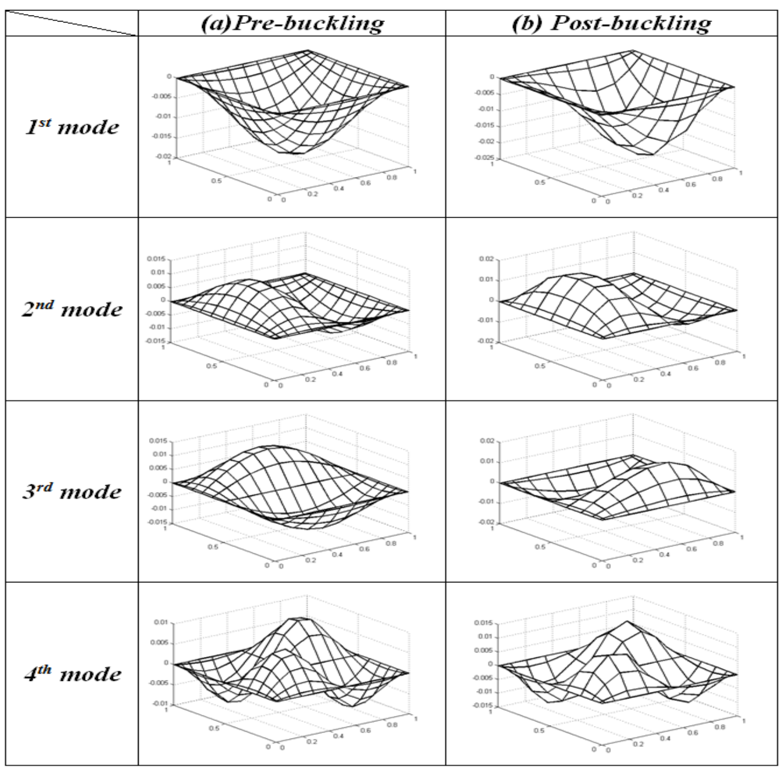

Figure 4: Nonlinear modal shapes of the simply supported plate: (a) Pre-buckling (b) Post-buckling.

Figure 5 shows the effects of initial imperfection amplitude on the first four nonlinear natural frequencies of a simply supported square plate for different inplane compressive loads. As it can be seen from the figure, the variation of frequencies is insignificant for small imperfection amplitudes, and it becomes negligible at large compressive loads. This is due to the fact that after buckling, the plate curvature increases and the small initial imperfection becomes negligible as compared with it. However, for large imperfection amplitudes, the variation in frequencies become more significant and are comparable as the load increases over the buckling state. Figure 6 shows the effect of plate thickness on the relative change in the first nonlinear frequency compare with the linear one for different compressive applied load. The variation in nonlinear frequencies for the thicker plates is more than the thin plates. It could be seen that the máximum change of first nonlinear frequency of plate for a moderately thick plate (h/a=0.1) is about 19.43% but for a thin plate (h/a=0.001) in the same load ratio, it is about 3.22%.

Figure 7 shows the variation of the first four nonlinear frequencies of a fully clamped square plate under the applied inplane compressive load for different nonlinear vibration amplitudes. It can be seen from the figure that as the compressive inplane load increases from zero to the buckling load, the reduction in plate stiffness causes all natural frequencies to be smoothly decreased. After the buckling, the curves for the odd frequencies increase rapidly, while the even modes experience some more decrement (unlike simply supported plate). The change in frequencies depends on the variation of bending and stretching stiffnesses. The figure shows that for the second and forth modes, the reduction of frequencies after the buckling load continue because they are bending modes. However, for the first and third modes, as they are bending-stretching modes, the frequencies increase after the buckling load due to the fact that the dynamic stretching-induced stiffness dominates the elastic bending stiffness.

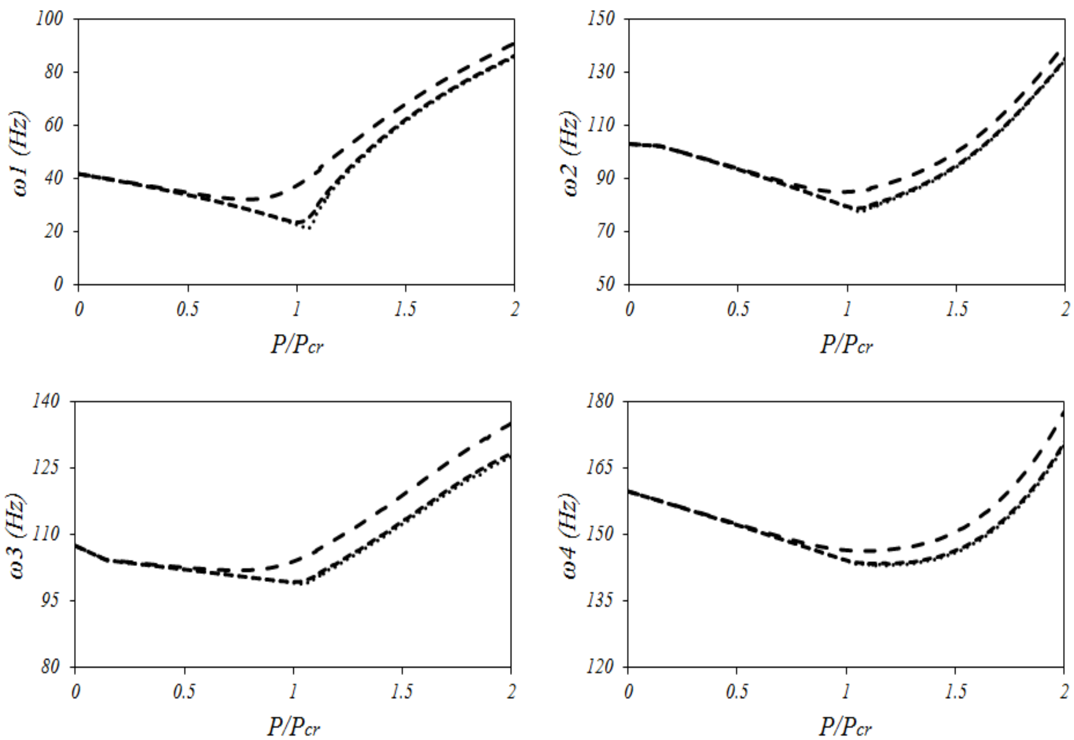

Figure 5: variation of the first four nonlinear frequencies of a simply supported square plate with applied load for different initial imperfection. (*$W_0/h=0.001$* (…), *$W_0/h=0.01$* (---), *$W_0/h=0.1$*(_))

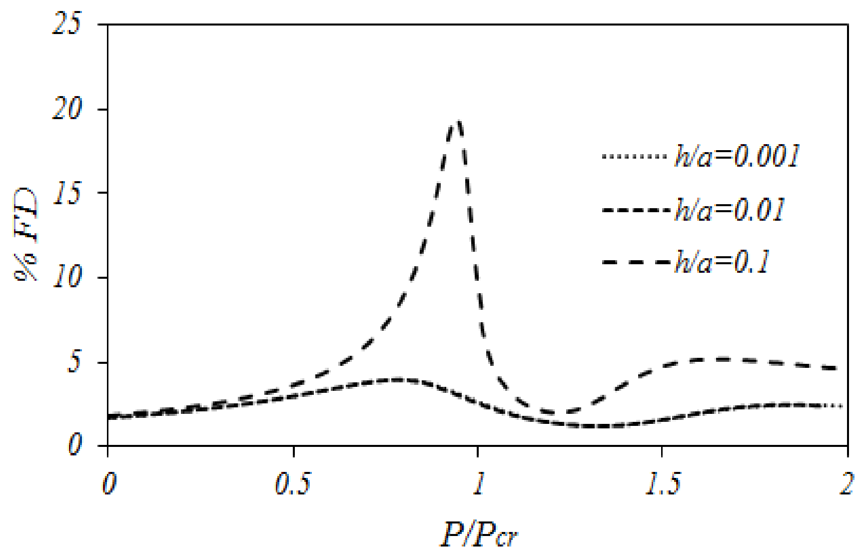

Figure 6: Effect of plate thickness on first nonlinear frequency. (h/a=0.001 (…), h/a=0.01 (---), h/a=0. 1 (_))

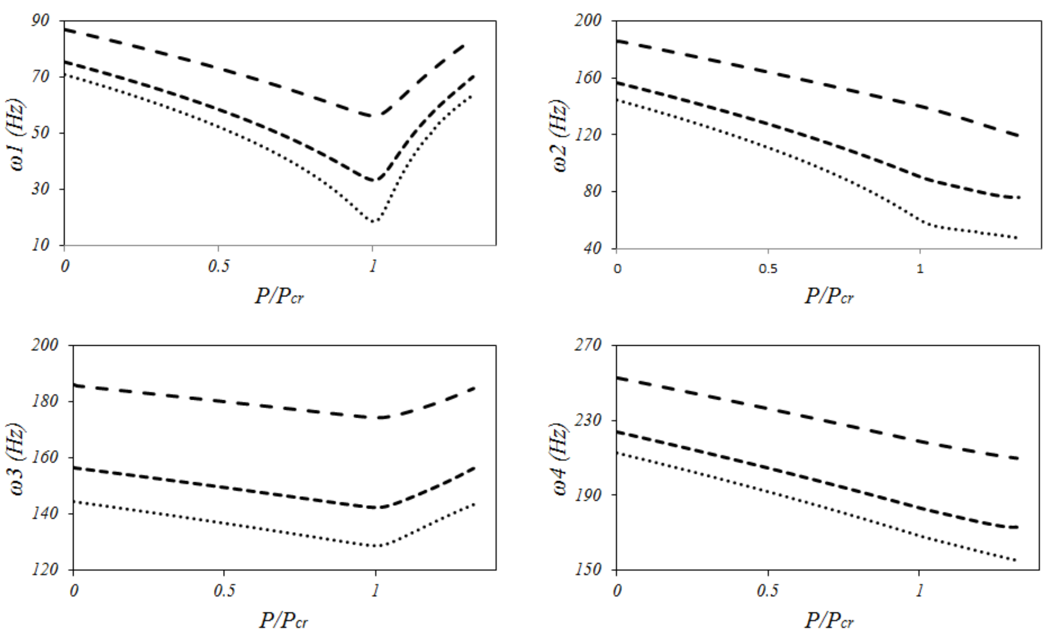

Figure 7: variation of the first four nonlinear frequencies of a fully clamped square plate with the compressive load for different amplitude. (linear (…), $W_a/h=0.5$ (---), $W_a/h=1$ (_))

Figure 8 shows the first four modal shapes for a fully clamped square plate in Pre-buckling ($P/P_{cr}=0$) and Post-buckling ($P/P_{cr}=1.2$) states, respectively.

4. Conclusions

In this study, nonlinear vibrations of plates under the action of inplane compressive load have been investigated by a hybrid method which uses the differential quadrature element, the arc-length, the harmonic balance and the direct iterative methods. The governing differential equations were divided into two sets of static (post-buckling) and dynamic (vibration) equations. The static equations were discretized using the differential quadrature element method, and solved by an arc-length strategy to find the state of equilibrium under the applied compressive inplane load. Next, inserting the static solution into the dynamic equations and using the harmonic balance and the differential quadrature element method, the governing dynamic equations were transformed into a nonlinear eigenvalue problem and solved by the direct iterative method.

Several case studies were performed to show the integrity of the proposed method. Also, the effect of some parameters influencing the nonlinear natural frequencies for simply supported and clamped boundary conditions were examined. It was realized that the nonlinear frequencies are functions of the compressive load and vibration amplitude. Moreover, the investigation shows that the frequencies are sensitive to the initial imperfection amplitudes (especially after the buckling).

It can be concluded that for simply supported plate, increasing the compressive load decreases the nonlinear frequencies in the pre-buckling domain and increases them in the post-buckling domain as well. Also, the plate thickness influences the frequencies; the thicker plate, the larger change in nonlinear frequencies. However, the results are different for the clamped plates. The presence of stretching stiffness causes the increase in frequencies after the buckling for some modes, while the influence of bending stiffness continues to decrease the frequencies for other modes.

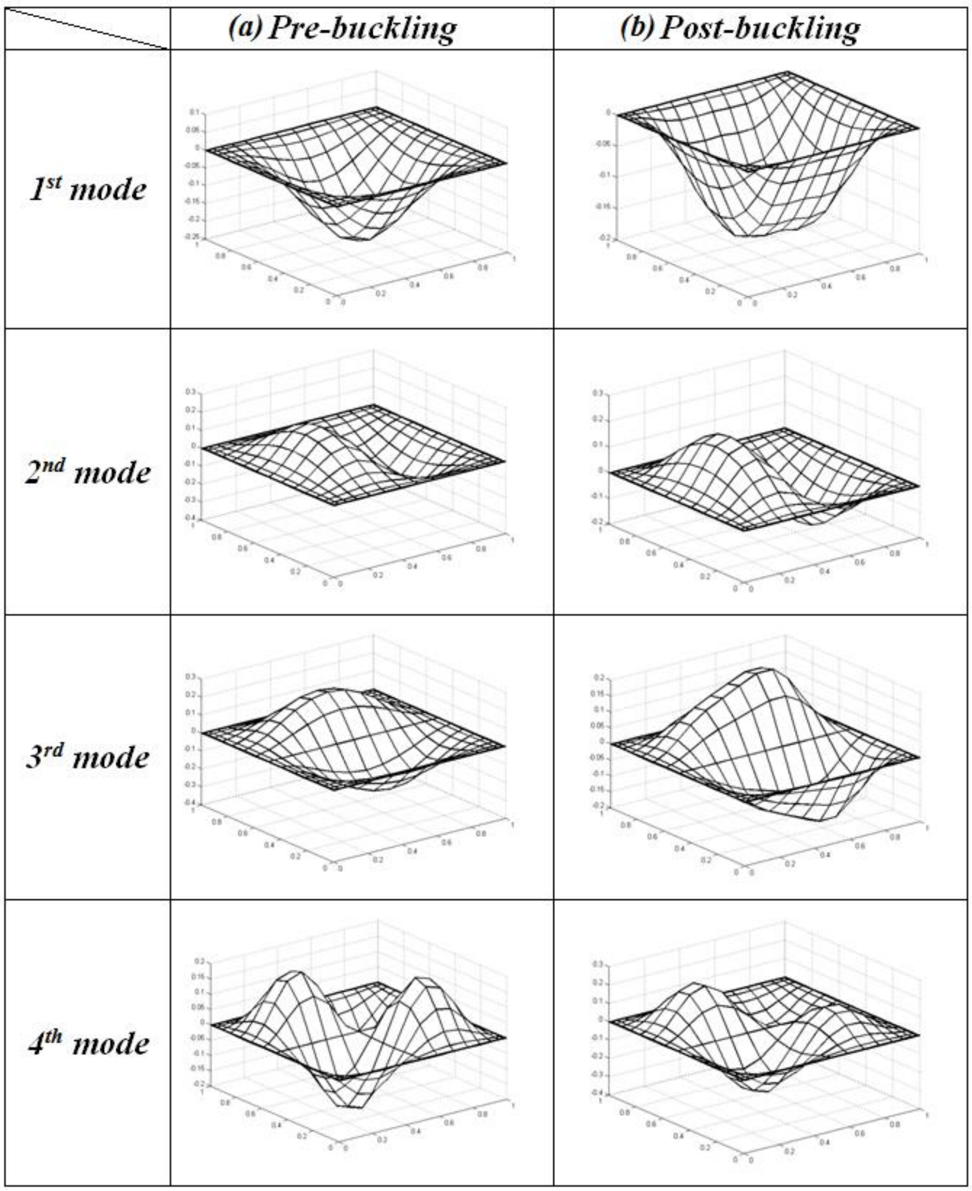

Figure 8: nonlinear modal shapes of the fully clamped plate: (a) Pre-buckling (b) Post-buckling.

5. Appendix A: Nonlinear Differential Equations of Equilibrium

$$
\begin{aligned}
&\frac{Eh}{1-\nu^2}\Big[u_{s,xx}+w_{s,xx}w_{s,x}+w_{s,xx}w_{0,x}+w_{s,x}w_{0,xx}++\nu\,(v_{s,xy}+w_{s,xy}w_{s,y}+w_{s,xy}w_{0,y}+w_{s,y}w_{0,xy})\Big]+\\
&\frac{Eh}{2(1+\nu)}\Big[u_{s,yy}+v_{s,xy}+w_{s,xy}w_{s,y}+w_{s,x}w_{s,yy}+w_{0,xy}w_{s,y}+w_{0,x}w_{s,yy}+w_{s,xy}w_{0,y}+w_{s,x}w_{0,yy}\Big]=0
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
&\frac{Eh}{1-\nu^2}\Big[v_{s,yy}+w_{s,yy}w_{s,y}+w_{s,yy}w_{0,y}+w_{s,y}w_{0,yy}+\nu\,(u_{s,xy}+w_{s,xy}w_{s,x}+w_{s,xy}w_{0,x}+w_{s,x}w_{0,xy})\Big]+\\
&\frac{Eh}{2(1+\nu)}\Big[u_{s,xy}+v_{s,xx}+w_{s,xy}w_{s,x}+w_{s,x}w_{s,xy}+w_{0,xx}w_{s,y}+w_{0,x}w_{s,xy}+w_{s,xx}w_{0,y}+w_{s,x}w_{0,xy}\Big]=0
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
&K_sGh\,(\alpha_{s,x}+w_{s,xx})+K_sGh\,(\beta_{s,y}+w_{s,yy})+\frac{Eh}{1-\nu^2}(u_{s,xx}w_{s,x}+w_{s,xx}w_{s,x}^2+w_{s,xx}w_{0,x}w_{s,x}\\
&+w_{s,x}^2w_{0,xx}+\nu(v_{s,xy}w_{s,x}+w_{s,xy}w_{s,x}w_{s,y}+w_{s,xy}w_{s,x}w_{0,y}+w_{0,xy}w_{s,x}w_{s,y}))+\frac{Eh}{1-\nu^2}(u_{s,xx}w_{0,x}\\
&+w_{s,xx}w_{0,x}w_{s,x}+w_{s,xx}w_{0,x}^2+w_{0,xx}w_{0,x}w_{s,x}+\nu(v_{s,xy}w_{0,x}+w_{s,xy}w_{0,x}w_{s,y}+w_{s,xy}w_{0,x}w_{0,y}+\\
&w_{0,xy}w_{0,x}w_{s,y}))+\frac{Eh}{1-\nu^2}(u_{s,x}w_{s,xx}+\frac{1}{2}w_{s,xx}w_{s,x}^2+w_{s,xx}w_{0,x}w_{s,x}+\nu(v_{s,y}w_{s,xx}+\frac{1}{2}w_{s,xx}w_{s,y}^2+\\
&w_{s,y}w_{s,xx}w_{0,y}))+\frac{Eh}{1-\nu^2}(u_{s,x}w_{0,xx}+\frac{1}{2}w_{0,xx}w_{s,x}^2+w_{0,xx}w_{0,x}w_{s,x}+\nu(v_{s,y}w_{0,xx}+\frac{1}{2}w_{0,xx}w_{s,y}^2+\\
&w_{s,y}w_{0,xx}w_{0,y}))+\frac{Eh}{1-\nu^2}(v_{s,yy}w_{s,y}+w_{s,yy}w_{s,y}^2+w_{s,yy}w_{0,y}w_{s,y}+w_{s,y}^2w_{0,yy}+\nu(u_{s,xy}w_{s,y}+\\
&w_{s,xy}w_{s,x}w_{s,y}+w_{s,xy}w_{s,y}w_{0,x}+w_{0,xy}w_{s,x}w_{s,y}))+\frac{Eh}{1-\nu^2}(v_{s,yy}w_{0,y}+w_{s,yy}w_{0,y}w_{s,y}+w_{s,yy}w_{0,y}^2\\
&+w_{0,yy}w_{0,y}w_{s,y}+\nu(u_{s,xy}w_{0,y}+w_{s,xy}w_{0,y}w_{s,x}+w_{s,xy}w_{0,x}w_{0,y}+w_{0,xy}w_{0,y}w_{s,x}))+\frac{Eh}{1-\nu^2}(v_{s,y}w_{s,yy}+\\
&\frac{1}{2}w_{s,yy}w_{s,y}^2+w_{0,yy}w_{s,y}w_{s,y}+\nu(u_{s,x}w_{s,yy}+\frac{1}{2}w_{s,yy}w_{s,x}^2+w_{s,x}w_{s,yy}w_{0,x}))+\frac{Eh}{1-\nu^2}(v_{s,y}w_{0,yy}+\frac{1}{2}w_{0,yy}w_{s,y}^2+\\
&w_{0,yy}w_{0,y}w_{s,y}+\nu(u_{s,x}w_{0,yy}+\frac{1}{2}w_{0,yy}w_{s,x}^2+w_{s,x}w_{0,yy}w_{0,x})+\frac{Eh}{2(1+\nu)}(u_{s,xy}w_{s,y}+v_{s,xx}w_{s,y}+w_{s,xx}w_{s,y}^2+w_{s,xy}w_{s,x}w_{s,y}+\\
&w_{0,xx}w_{s,y}^2+w_{s,xy}w_{0,x}w_{s,y}+w_{s,xx}w_{0,y}w_{s,y}+w_{s,xy}^2w_{s,x}w_{s,y}))+\frac{Eh}{2(1+\nu)}(u_{s,xy}w_{0,y}+v_{s,xx}w_{0,y}+w_{s,xx}w_{s,y}w_{0,y}+\\
&w_{s,xy}w_{s,x}w_{0,y}+w_{s,xy}w_{s,y}w_{0,y}+w_{s,xy}w_{0,x}w_{0,y}+w_{s,xx}w_{0,y}^2+w_{s,xy}^2w_{0,y}w_{s,x}))+\frac{Eh}{2(1+\nu)}(u_{s,yy}w_{s,x}+\\
&v_{s,xy}w_{s,x}+w_{s,xy}w_{s,x}w_{s,y}+w_{s,yy}w_{s,x}^2+w_{s,xy}w_{s,x}w_{0,y}+w_{s,yy}w_{s,x}^2))++\frac{Eh}{2(1+\nu)}(u_{s,yy}w_{0,x}+v_{s,xy}w_{0,x}+\\
&w_{s,xy}w_{0,x}w_{s,y}+w_{s,yy}w_{0,x}w_{s,x}+w_{s,yy}w_{0,x}^2+w_{s,xy}w_{0,x}w_{0,y}+w_{s,yy}w_{s,x}w_{0,x}))+\frac{Eh}{(1+\nu)}(u_{s,y}w_{s,xy}+\\
&v_{s,x}w_{s,xy}+w_{s,xy}w_{s,x}w_{s,y}+w_{s,xy}w_{0,x}w_{s,y}+w_{s,xy}w_{s,x}w_{0,y}))+\frac{Eh}{(1+\nu)}(u_{s,y}w_{0,xy}+v_{s,x}w_{0,xy}+\\
&w_{0,xy}w_{s,x}w_{s,y}+w_{0,xy}w_{0,x}w_{s,y}+w_{0,xy}w_{s,x}w_{0,y}))=0
\end{aligned}
\tag{A3}
$$

$$
\frac{Eh^3}{12(1-\nu^2)}(\alpha_{s,xx}+\nu\beta_{s,xy})+\frac{Eh^3}{24(1+\nu)}(\alpha_{s,yy}+\beta_{s,xy})-K_sGh(\alpha_s+w_{s,x})=0
\tag{A4}
$$

$$
\frac{Eh^3}{12(1-\nu^2)}(\beta_{s,xx}+\nu\alpha_{s,xy})+\frac{Eh^3}{24(1+\nu)}(\alpha_{s,xy}+\beta_{s,xx})-K_sGh(\beta_s+w_{s,y})=0
\tag{A5}
$$

6. Appendix B: Nonlinear Differential Equations of Buckled Plate Vibration

$$
\begin{aligned}
&\frac{Eh}{1-\nu^2}\big[u_{d,xx} + w_{s,xx}w_{d,x} + w_{d,xx}w_{s,x} + w_{d,xx}w_{0,x} + w_{d,x}w_{0,xx} + w_{d,xx}w_{d,x} + \nu\;\; v_{d,xy} + w_{d,xy}w_{s,y} + w_{s,xy}w_{d,y} + w_{d,xy}w_{d,y} + w_{d,xy}w_{0,y} + w_{d,y}w_{0,xy} \;\; + \\
&\frac{Eh}{2\;\;1+\nu}\big[u_{d,yy} + v_{d,xy} + w_{d,xy}w_{s,y} + w_{d,xy}w_{d,y} + w_{s,xy}w_{d,y} + w_{s,x}w_{d,yy} + w_{d,x}w_{s,yy} + w_{0,xy}w_{d,y} + w_{0,x}w_{d,yy} + w_{d,x}w_{d,yy} + w_{d,xy}w_{0,y} + w_{d,x}w_{0,yy}\big] = \mu\ddot{u}_d
\end{aligned}
\tag{B1}
$$

$$
\begin{aligned}
&\frac{Eh}{1-\nu^2}\big[v_{d,yy} + w_{d,yy}w_{s,y} + w_{s,yy}w_{d,y} + w_{d,yy}w_{d,y} + w_{d,yy}w_{0,y} + + w_{d,y}w_{0,yy} + \nu\;\; u_{d,xy} + w_{d,xy}w_{s,x} + w_{d,xy}w_{d,x} + w_{s,xy}w_{d,x} + w_{d,xy}w_{0,x} + w_{d,x}w_{0,xy} \;\; + \\
&\frac{Eh}{2\;\;1+\nu}\big[u_{d,xy} + v_{d,xx} + w_{d,xy}w_{s,x} + w_{s,xy}w_{d,x} + w_{s,x}w_{d,xy} + w_{d,x}w_{s,xy} + w_{0,xx}w_{d,y} + w_{0,x}w_{d,xy} + w_{d,xx}w_{0,y} + w_{d,xx}w_{d,y} + w_{d,x}w_{0,xy} + w_{d,x}w_{d,xy}\big] = \mu\ddot{v}_d
\end{aligned}
\tag{B2}
$$

$$
\begin{aligned}
&K_sGh\;\; \alpha_{d,x} + w_{d,xx} \;\; + K_sGh\;\; \beta_{d,y} + w_{d,yy} \;\; + \frac{Eh}{1-\nu^2}(u_{d,xx}w_{s,x} + u_{s,xx}w_{d,x} + u_{d,xx}w_{d,x} + w_{d,xx}w_{s,x}^2 + w_{s,xx}w_{d,x}^2 + w_{d,xx}w_{d,x}^2 + \\
&2w_{s,xx}w_{s,x}w_{d,x} + 2w_{d,xx}w_{s,x}w_{d,x} + w_{d,xx}w_{0,x}w_{s,x} + w_{s,xx}w_{0,x}w_{d,x} + w_{d,xx}w_{0,x}w_{d,x} + w_{0,xx}w_{d,x}^2 + 2w_{s,x}w_{d,x}w_{0,xx} + \nu(v_{d,xy}w_{s,x} + \\
&v_{s,xy}w_{d,x} + v_{d,xy}w_{d,x} + w_{d,xy}w_{s,x}w_{0,y} + w_{d,xy}w_{d,x}w_{d,y} + w_{d,xy}w_{d,x}w_{s,y} + w_{sxy}w_{d,x}w_{d,y} + w_{s,xy}w_{d,x}w_{0,y} + w_{0,xy}w_{d,x}w_{s,y} + \\
&w_{d,xy}w_{d,y}w_{s,x} + w_{0,xy}w_{s,x}w_{d,y} + w_{d,xy}w_{d,x}w_{0,y} + w_{0,xy}w_{d,x}w_{d,y}) + \frac{Eh}{1-\nu^2}(u_{d,xx}w_{0,x} + w_{d,xx}w_{0,x}w_{s,x} + w_{d,xx}w_{0,x}w_{d,x} + w_{s,xx}w_{0,x}w_{d,x} + \\
&w_{d,xx}w_{0,x}^2 + w_{0,xx}w_{0,x}w_{d,x} + \nu(v_{d,xy}w_{0,x} + w_{d,xy}w_{0,x}w_{s,y} + w_{d,xy}w_{0,x}w_{d,y} + w_{s,xy}w_{0,x}w_{d,y} + w_{d,xy}w_{0,x}w_{0,y} + w_{0,xy}w_{0,x}w_{d,y})) + \\
&\frac{Eh}{1-\nu^2}(u_{d,x}w_{s,xx} + u_{s,x}w_{d,xx}u_{d,x}w_{d,xx} + \frac{1}{2}w_{s,x}^2w_{d,xx} + \frac{1}{2}w_{d,x}^2w_{d,xx} + \frac{1}{2}w_{d,x}^2w_{s,xx} + w_{s,xx}w_{s,x}w_{d,x} + w_{d,xx}w_{s,x}w_{d,x} + w_{s,xx}w_{0,x}w_{d,x} + \\
&w_{d,xx}w_{0,x}w_{s,x} + w_{d,xx}w_{0,x}w_{d,x} + \nu(v_{d,y}w_{s,xx} + v_{s,y}w_{d,xx} + v_{d,y}w_{d,xx} + \frac{1}{2}w_{s,y}^2w_{d,xx} + \frac{1}{2}w_{d,y}^2w_{d,xx} + \frac{1}{2}w_{d,y}^2w_{s,xx} + w_{d,y}w_{s,xx}w_{0,y} + \\
&w_{s,y}w_{d,xx}w_{0,y} + w_{d,y}w_{d,xx}w_{sy} + w_{s,y}w_{s,xx}w_{d,y} + w_{0,y}w_{d,xx}w_{d,y}) + \frac{Eh}{1-\nu^2}(u_{d,x}w_{0,xx} + \frac{1}{2}w_{d,x}^2w_{0,xx} + w_{0,xx}w_{0,x}w_{d,x} + \nu(v_{d,y}w_{0,xx} + \\
&\frac{1}{2}w_{d,y}^2w_{0,xx} + w_{d,y}w_{0,xx}w_{0,y})) + \frac{Eh}{1-\nu^2}(v_{d,yy}w_{s,y} + v_{s,yy}w_{d,y} + v_{d,yy}w_{d,y} + w_{d,yy}w_{s,y}^2 + w_{d,yy}w_{d,y}^2 + 2w_{s,yy}w_{0,y}w_{d,y} + 2w_{d,yy}w_{s,y}w_{d,y} + \\
&w_{d,y}^2w_{s,yy} + w_{d,yy}w_{0,y}w_{s,y} + w_{d,yy}w_{0,y}w_{d,y} + w_{s,yy}w_{0,y}w_{d,y} + w_{d,y}^2w_{0,yy} + 2w_{s,y}w_{d,y}w_{0,yy} + \nu(u_{d,xy}w_{s,y} + u_{s,xy}w_{d,y} + u_{d,xy}w_{d,y} + \\
&w_{s,xy}w_{o,x}w_{d,y} + w_{s,xy}w_{d,y}w_{0,x} + w_{d,xy}w_{d,y}w_{d,x} + w_{d,xy}w_{s,y}w_{0,x} + w_{d,xy}w_{s,y}w_{dx} + w_{0,xy}w_{s,x}w_{d,y} + w_{s,xy}w_{d,y}w_{s,x} + w_{s,xy}w_{d,y}w_{d,x} + \\
&w_{d,xy}w_{d,y}w_{s,x} + w_{d,xy}w_{d,y}w_{0,x} + w_{0,xy}w_{d,y}w_{d,x})) + \frac{Eh}{1-\nu^2}(v_{d,yy}w_{0,y} + w_{d,yy}w_{0,y}w_{d,y} + w_{d,yy}w_{0,y}w_{s,y} + w_{s,yy}w_{0,y}w_{d,y} + w_{d,yy}w_{0,y}^2 + \\
&w_{0,yy}w_{0,y}w_{d,y} + \nu(u_{d,xy}w_{0,y} + w_{d,xy}w_{0,y}w_{s,x} + w_{d,xy}w_{0,y}w_{d,x} + w_{s,xy}w_{0,y}w_{d,x} + w_{d,xy}w_{0,x}w_{0,y} + w_{0,xy}w_{0,y}w_{d,x})) + \frac{Eh}{1-\nu^2}(v_{d,y}w_{d,yy} + \\
&v_{d,y}w_{s,yy} + v_{s,y}w_{d,yy} + \frac{1}{2}w_{d,yy}w_{d,y}^2 + \frac{1}{2}w_{d,yy}w_{s,y}^2 + w_{d,yy}w_{d,y}w_{s,y} + \frac{1}{2}w_{s,yy}w_{d,y}^2 + w_{s,yy}w_{d,y}w_{s,y} + w_{d,yy}w_{s,y}w_{0,y} + w_{d,yy}w_{d,y}w_{0,y} + \\
&w_{d,y}w_{s,yy}w_{s,y} + \nu(u_{d,x}w_{d,yy} + u_{s,x}w_{d,yy} + u_{d,x}w_{s,yy} + \frac{1}{2}w_{d,yy}w_{d,x}^2 + \frac{1}{2}w_{s,yy}w_{d,x}^2 + \frac{1}{2}w_{d,yy}w_{s,x}^2 + w_{d,x}w_{d,yy}w_{s,x} + w_{d,x}w_{s,yy}w_{0,x} + \\
&w_{d,x}w_{d,yy}w_{0,x} + w_{s,x}w_{d,yy}w_{0,x})) + \frac{Eh}{1-\nu^2}(v_{d,y}w_{0,yy} + \frac{1}{2}w_{0,yy}w_{d,y}^2 + w_{0,yy}w_{s,y}w_{d,y} + w_{0,yy}w_{0,y}w_{d,y} + \nu(u_{d,x}w_{0,yy} + w_{d,x}w_{0,yy}w_{s,x} + \\
&\frac{1}{2}w_{0,yy}w_{d,x}^2 + w_{d,x}w_{0,yy}w_{0,x}) + \frac{Eh}{2(1+\nu)}(u_{d,xy}w_{d,y} + u_{s,xy}w_{d,y} + u_{d,xy}w_{s,y} + v_{s,xx}w_{d,y} + v_{d,xx}w_{s,y} + v_{d,xx}w_{d,y} + w_{d,xx}w_{s,y}^2 + \\
&w_{d,xx}w_{d,y}^2 + 2w_{s,xx}w_{s,y}w_{d,y} + 2w_{d,xx}w_{s,y}w_{d,y} + w_{s,xx}w_{d,y}^2 + 2w_{0,xx}w_{s,y}w_{d,y} + w_{s,xy}w_{d,x}w_{s,y} + w_{d,xy}w_{d,x}w_{d,y} + w_{d,xy}w_{d,x}w_{s,y} + \\
&w_{s,xy}w_{d,x}w_{s,y} + w_{s,xy}w_{d,x}w_{d,y} + w_{d,xy}w_{s,x}w_{s,y} + w_{s,xy}w_{s,x}w_{d,y} + w_{d,xy}w_{0,x}w_{s,y} + w_{d,xy}w_{s,x}w_{d,y} + w_{0,x}w_{s,xy}w_{d,y} + w_{s,xx}w_{0,y}w_{d,y} + \\
&w_{d,xx}w_{0,y}w_{s,y} + w_{0,xy}w_{s,y}w_{d,x} + w_{0,xy}w_{s,x}w_{d,y} + w_{0,xx}w_{d,y}^2 + w_{d,xy}w_{0,x}w_{d,y} + w_{d,xx}w_{0,y}w_{d,y} + w_{0,xy}w_{d,x}w_{d,y})) + \frac{Eh}{2(1+\nu)}(u_{d,xy}w_{0,y} + \\
&v_{d,xx}w_{0,y} + w_{d,xx}w_{d,y}w_{0,y} + w_{s,xx}w_{d,y}w_{0,y} + w_{d,xx}w_{s,y}w_{0,y} + w_{s,xy}w_{d,x}w_{0,y} + w_{d,xy}w_{s,x}w_{0,y} + w_{0,xy}w_{d,y}w_{0,y} + w_{0,xy}w_{d,y}w_{d,y} + w_{d,xy}w_{s,y}w_{0,y} + \\
&w_{d,xy}w_{0,x}w_{0,y} + w_{d,xx}w_{0,y}^2 + w_{0,xy}^2w_{0,y}w_{d,x})) + \frac{Eh}{2(1+\nu)}(u_{d,yy}w_{d,x} + u_{d,yy}w_{s,x} + u_{s,yy}w_{d,x} + v_{d,xy}w_{d,x} + v_{s,xy}w_{d,x} + v_{d,xy}w_{s,x} + \\
&w_{d,xy}w_{d,x}w_{d,y} + w_{d,xy}w_{d,x}w_{sy} + w_{s,xy}w_{s,x}w_{d,y} + w_{s,xy}w_{d,x}w_{d,y} + w_{s,xy}w_{d,x}w_{s,y} + w_{d,xy}w_{s,x}w_{s,y} + w_{d,xy}w_{s,x}w_{d,y} + w_{d,yy}w_{d,x}^2 + w_{d,yy}w_{s,x}^2 + \\
&2w_{s,yy}w_{d,x}w_{s,x} + 2w_{d,yy}w_{d,x}w_{s,x} + w_{s,yy}w_{d,x}^2 + w_{d,xy}w_{s,x}w_{0,y} + w_{s,xy}w_{d,x}w_{0,y} + w_{d,yy}w_{s,x}^2 + w_{0,xy}w_{s,x}w_{d,y} + w_{0,xy}w_{d,x}w_{s,y} + w_{d,yy}w_{0,x}w_{s,x} + \\
&w_{0,xy}w_{d,x}w_{d,y} + w_{d,yy}w_{0,x}w_{d,x} + w_{d,xy}w_{d,x}w_{0,y} + w_{0,yy}w_{d,x}^2 + w_{d,x}w_{0,x}w_{s,yy} + 2w_{s,yy}w_{d,x}w_{s,y}) + \frac{Eh}{2(1+\nu)}(u_{d,yy}w_{0,x} + v_{d,xy}w_{0,x} + \\
&w_{d,xy}w_{0,x}w_{d,y} + w_{d,xy}w_{0,x}w_{s,y} + w_{s,xy}w_{0,x}w_{d,x} + w_{s,yy}w_{0,x}w_{d,x} + w_{d,yy}w_{0,x}w_{d,x} + w_{d,yy}w_{0,x}w_{s,x} + w_{d,yy}w_{0,x}^2 + w_{d,xy}w_{0,x}w_{0,y} + w_{0,yy}w_{d,x}w_{0,x} + \\
&w_{d,y}w_{0,xy}w_{0,x})) + \frac{Eh}{(1+\nu)}(u_{d,y}w_{d,xy} + u_{s,y}w_{d,xy} + u_{d,y}w_{s,xy} + v_{d,x}w_{d,xy} + v_{s,x}w_{d,xy} + v_{d,x}w_{s,xy} + w_{d,xy}w_{d,x}w_{d,y} + w_{d,xy}w_{d,x}w_{s,y} + \\
&w_{s,xy}w_{d,x}w_{d,y} + w_{d,xy}w_{s,x}w_{d,y} + w_{d,xy}w_{s,x}w_{s,y} + w_{s,xy}w_{d,x}w_{s,y} + w_{s,xy}w_{s,x}w_{d,y} + w_{d,xy}w_{0,x}w_{dy} + w_{d,xy}w_{0,x}w_{s,y} + w_{s,xy}w_{0,x}w_{d,y} + \\
&w_{d,xy}w_{d,x}w_{0,y} + w_{s,xy}w_{d,x}w_{0,y} + w_{d,xy}w_{s,x}w_{0,y})) + \frac{Eh}{(1+\nu)}(u_{d,y}w_{0,xy} + v_{d,x}w_{0,xy} + w_{0,xy}w_{s,x}w_{d,y} + w_{0,xy}w_{d,x}w_{s,y} + w_{0,xy}w_{0,x}w_{d,y} + \\
&w_{0,xy}w_{d,x}w_{0,y} + w_{0,xy}w_{d,x}w_{d,y})) = \mu\ddot{w}_d
\end{aligned}
\tag{B3}
$$

$$\frac{Eh^3}{12\ \ 1-\nu^2}(\alpha_{d,xx}+\nu\beta_{d,xy})+\frac{Eh^3}{24\ \ 1+\nu}\ \ \alpha_{d,yy}+\beta_{d,xy}\ \ -K_sGh(\alpha_d+w_{d,x})=I_x\ddot{\alpha}_d \quad \text{(B4)}$$

$$\frac{Eh^3}{12\ \ 1-\nu^2}(\beta_{d,xx}+\nu\alpha_{d,xy})+\frac{Eh^3}{24\ \ 1+\nu}\ \ \alpha_{d,xy}+\beta_{d,xx}\ \ -K_sGh(\beta_d+w_{d,y})=I_y\ddot{\beta}_d \quad \text{(B5)}$$

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