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Nonlinear Vibration and Stability Analysis of Beam on the Variable Viscoelastic Foundation

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Abstract

The aim of this study is the investigation of the large amplitude deflection of an Euler-Bernoulli beam subjected to an axial load on a viscoelastic foundation with the strong damping. In order to achieve this purpose, the beam nonlinear frequency has been calculated by homotopy perturbation method (HPM) and Hamilton Approach (HA) and it was compared by the exact solutions for the different boundary conditions such as simple-simple, clamped-simple and clamped-clamped which showed a good accuracy in results. In addition, to find the deflection of the nonlinear Euler-Bernoulli beam, the problem has been solved based on homotopy perturbation method and modified differential transform method (MDTM) and finally, the results were compared by Rung-Kutta exact solutions. The derived deflection results by two mentioned methods had a good agreement with the exact RK4 solutions. By considering the paper results, buckling force is increased for each case permanently by increase in the boundary rigidity for a constant value of system amplitude (A). As a final comparison, in based on paper results, the buckling force is arisen by increasing the system amplitude for each case.

Keywords: Euler-Bernoulli beam, Homotopy perturbation method, Padé approximants, Modified differential transform method, Variable foundation, Vibration and buckling analysis.

1. Introduction

As Euler-Bernoulli beams are widely implemented in different engineering fields, their dynamical and vibrational investigations are the most important goal of any study project. Although many analytical, semi analytical and numerical researches on mechanical and structural elements have been performed yet, finding the new reliable solutions to solve some more complicated problems such as nonlinear beams in different load and boundary conditions are very useful. Nowadays, researchers focus on considerations of the strong nonlinearity induced by the large amplitude vibrations of the mechanical components. In case of

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different nonlinear beam solutions, Sedighi and Shirazi presented an analytical solution of cantilever beam vibration with nonlinear boundary condition by considering the dead zone effect concept [1]. Barari et al. [2, 3] studied on the beam deformation problem using homotopy perturbation method (HPM). Also, in some other researches perturbation method is applied to obtain nonlinear frequency and dynamic system response [4-6]. Durmaz and Kaya [7] and the other researchers [8-11] presented some investigations based on energy balance method (EBM) to investigate the nonlinear behavior of the oscillators. They derived first-, second- and third-order energy balance method for a cubic-quintic Duffing oscillator. Nonlinear and linear structures solving is investigated by several literatures [12-33] which their results have a good accuracy with the exact solutions. Rezazadeh et al. [34] presented this method to derive the micro-beam frequencies parametrically. Kacar et al. [35] studied on the beam analysis on the variable winkler elastic foundation by using the differential transform method (DTM). This method also is applied by some other researches [36-39]. The most controversial problem in DTM is the solutions stability for the large range of time which depends on the other system parameters. Karimian and Azimi [20] used differential transform method and Hamilton Approach to solve Euler-Bernoulli beams subjected to axial load. Abdelhafez [40] performed a solution for Duffing and Van der pol equations under damping effects by using modified differential transform method (MDTM) and based on Padé approximants concept. The similar researches have been performed recently [41-43]. In case of large amplitude vibration of beams, Mirzabeigy and Madoliat [44] studied on an Euler-Bernoulli beam resting on the variable elastic foundation. They derived the nonlinear frequencies by helping perturbation method. Senalp et al. [45] performed a comparison between linear and nonlinear response of Euler-Bernoulli beam on a viscoelastic foundation by using Galerkin method and finite element method (FEM). Based on this overview, as the large amplitude vibration of Euler-Bernoulli beam with the strong damping on the viscoelastic foundation has not been performed, this subject is investigated now.

In this paper, the governing motion equation of the nonlinear Euler-Bernoulli beam resting on the variable linear and nonlinear viscoelastic foundation has been derived by helping Hamilton principles and it has been solved by HPM and MDTM. Consequently, comparing the results by Rung-Kutta solution shows the good accuracy. Also, the nonlinear frequencies have been obtained by HPM and Hamilton Approach (HA) at the different simply support-simply support (S-S), simply support-clamped support (S-C) and clamped support-clamped support (C-C) boundary

conditions that they have complete agreement in comparison with the exact solutions in the other previous works.

2. Mathematical Modelling

Fig.1 shows a schematic of an Euler-Bernoulli beam on the nonlinear viscoelastic foundation with the variable coefficients. In this paper, the beam is subjected by the axial load. The spatial coordinate $\bar{x}$ along the beam length is applied to use in the mathematical formulations. The beam length L, mass per unit volume ρ, beam thickness h, modulus of elasticity E, moment of inertia I and the rectangular cross section with the area of A are the other specifications of the beam.
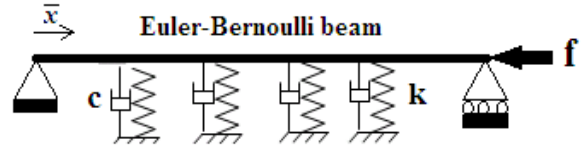

Figure 1: An Euler-Bernoulli beam subjected by axial load on the viscoelastic foundation

In order to derive the motion equation, one can be able to use Hamilton's principle as bellow

$$\delta \int_{t_1}^{t_2} (T + W - \Pi) d\bar{t} = 0 \qquad (1)$$

Where the kinetic energy T, the strain energy Π without axial displacement assumption and the external work W are given as follows

$$T = \frac{\rho A}{2} \int_0^L (\frac{\partial \bar{y}(\bar{x}, \bar{t})}{\partial \bar{t}})^2 d\bar{x} \qquad (2)$$

$$\Pi = \frac{EI}{2} \int_0^L \left(\frac{\partial^2 \bar{y}(\bar{x}, \bar{t})}{\partial \bar{x}^2} \right)^2 d\bar{x} +$$

$$\frac{EA}{2} \int_0^L \left(\frac{1}{2} (\frac{\partial \bar{y}(\bar{x}, \bar{t})}{\partial \bar{x}})^2 \right)^2 d\bar{x} + \qquad (3)$$

$$\frac{1}{2} \int_0^L k(\bar{x}) \bar{y}(\bar{x}, \bar{t})^2 d\bar{x}$$

$$W = \frac{f}{2} \int_0^L (\frac{\partial \bar{y}(\bar{x}, \bar{t})}{\partial \bar{x}})^2 d\bar{x} \qquad (4)$$

However, the motion equation of system is as follow

$$EI\frac{\partial^4\overline{y}(\overline{x},\overline{t})}{\partial\overline{x}^4}+\rho A\frac{\partial^2\overline{y}(\overline{x},\overline{t})}{\partial\overline{t}^2}+c(\overline{x})\frac{\partial\overline{y}(\overline{x},\overline{t})}{\partial\overline{t}}+$$

$$k(\overline{x})\overline{y}(\overline{x},\overline{t})+f\frac{\partial^2\overline{y}(\overline{x},\overline{t})}{\partial\overline{x}^2}- \qquad (5)$$

$$\frac{EA}{2L}\frac{\partial^2\overline{y}(\overline{x},\overline{t})}{\partial\overline{x}^2}\int_0^L\left(\frac{\partial\overline{y}(\overline{x},\overline{t})}{\partial\overline{x}}\right)^2 d\overline{x}=0$$

In the last equation, $k(\overline{x})$ and $c(\overline{x})$ can be considered as

$$k(\overline{x})=kg(\overline{x}),c(\overline{x})=cg(\overline{x}) \qquad (6)$$

Where k and c are the constant coefficients of stiffness and damping, also $g(\overline{x})$ is the spatial coordinate function along the beam length. In order to make the beam equation form easier, dimensionless parameters are considered as follow

$$x=\frac{\overline{x}}{L}, \qquad y=\frac{\overline{y}}{R}, \qquad t=\overline{t}\sqrt{\frac{EI}{\rho AL^4}},$$

$$F=\frac{fL^2}{EI}, \qquad C=\frac{fL^2}{EI}, \qquad K=\frac{kL^4}{EI} \qquad (7)$$

where $R=\sqrt{\dfrac{I}{A}}$ is the gyration radius of the cross section.

The form of solution based on variables separation method is assumed as follow

$$y(x,t)=\psi(x)\cdot U(t) \qquad (8)$$

Where $\psi(x)$ will be determined based on the different boundary conditions. Implementing the weighted residual Bubnov-Galerkin method leads to

$$\int_0^1(\frac{\partial^4 y(x,t)}{\partial x^4}+\frac{\partial^2 y(x,t)}{\partial t^2}+C\frac{\partial y(x,t)}{\partial t}+Kg(x)y(x,t)+$$

$$F\frac{\partial^2 y(x,t)}{\partial x^2}-\frac{1}{2}\frac{\partial^2 y(x,t)}{\partial x^2}\int_0^1\left(\frac{\partial y(x,t)}{\partial x}\right)^2 dx)\psi(x)dx=0 \qquad (9)$$

Consequently, the time-dependent nonlinear equation will be derived as follow

$$\ddot{U}(t)+\gamma\dot{U}(t)+\alpha U(t)+\beta U(t)^3=0 \qquad (10)$$

Where

$$\gamma=C\frac{\int_0^1 g(x)\psi(x)^2 dx}{\int_0^1\psi(x)^2 dx},$$

$$\alpha=\alpha_1+\alpha_2+\alpha_3, \qquad \alpha_1=\frac{\int_0^1(\frac{d^4\psi(x)}{dx^4})\psi(x)dx}{\int_0^1\psi(x)^2 dx},$$

$$\alpha_2=F\frac{\int_0^1(\frac{d^2\psi(x)}{dx^2})\psi(x)dx}{\int_0^1\psi(x)^2 dx}, \qquad \alpha_3=K\frac{\int_0^1 g(x)\psi(x)^2 dx}{\int_0^1\psi(x)^2 dx} \qquad (11)$$

$$\beta=-\frac{1}{2}\frac{\int_0^1(\frac{d^2\psi(x)}{dx^2})\left(\int_0^1(\frac{d\psi(x)}{dx})^2 dx\right)\psi(x)dx}{\int_0^1\psi(x)^2 dx}$$

In this study, the main purpose is to investigate the dynamic response of Eq.10 by an analytical solution and then comparing it with the numerical solution. Perturbation method will be implemented to find an analytical solution for the large amplitude oscillation (Eq.10) under the initial conditions as bellow

$$U(0)=A_0 \qquad , \qquad \dot{U}(0)=0 \qquad (12)$$

In the above initial conditions, A_0 is the dimensionless initial displacement of the midpoint of Euler-Bernoulli beam.

In this paper, three stiffness functions are assumed to define winkler foundation. The three mentioned cases are as follows:

Case 1: $K(x)=k(1-0.2x)$

Case 2: $K(x)=k(1-0.2x^2)$ (12-a)

Case 3: $K(x)=k\pi\sin(\pi x)$

3. Solution Methods

As can be observed, Eq.10 has a nonlinear form that it makes some difficulties to solve it by the ordinary analytical methods. Therefore, in order to derive the solution of the governing motion Eq.10 and the nonlinear frequencies in the various mentioned boundary conditions in Table 1, HPM, HA and MDTM are applied in this paper. The nonlinear frequencies are derived by HPM, HA and the response of the system is obtained by HPM, MDTM and RK4.

Table 1: The first eigenmode of beam with the different BCs

BC	$\psi(x)$		
SS-SS	$\sin(\lambda x)$	,	$\lambda = 3.14$
SS-CS	$[\cos(\lambda x) - \cosh(\lambda x)] - \dfrac{\cos\lambda - \cosh\lambda}{\sin\lambda - \sinh\lambda}[\sin(\lambda x) - \sinh(\lambda x)]$	,	$\lambda = 3.93$
CS-CS	$[\cosh(\lambda x) - \cos(\lambda x)] - \dfrac{\cosh\lambda - \cos\lambda}{\sinh\lambda - \sin\lambda}[\sinh(\lambda x) - \sin(\lambda x)]$	,	$\lambda = 4.73$

3.1. *Homotopy Perturbation Method*

In order to solve the problem by implementing HPM, Eq.10 is divided to the linear and nonlinear parts. Remembering this equation and its corresponding initial conditions as follow

$$\ddot{U}(t) + \gamma\dot{U}(t) + \alpha U(t) + \beta U(t)^3 = 0,$$

$$U(0) = A_0, \qquad \dot{U}(0) = 0$$

So as to solve the last equation by perturbation method, the parameter U is defined as

$$U = \varepsilon T \tag{13}$$

Substituting Eq.13 into Eq.10 leads to

$$\ddot{T}(t) + 2\xi\omega\dot{T}(t) + \alpha T(t) + \varepsilon^2\beta T(t)^3 = 0$$

$$T(0) = \frac{A_0}{\varepsilon} \qquad \dot{T}(0) = 0 \tag{14}$$

Where $\xi = \dfrac{\gamma}{2\omega}$ is the damping ratio of the above system.

Considering the form of system solution T, and the constant coefficient α as follow

$$T = T_0 + \varepsilon^2 T_1 + \varepsilon^4 T_2 + \cdots \tag{15}$$

$$\alpha = \omega^2 + \varepsilon^2\omega_1 + \varepsilon^4\omega_2 + \cdots \tag{16}$$

Substituting Eqs.15 and 16 into Eq.14 and collecting the same power of ε leads to below equations

$$\ddot{T}_0 + 2\xi\omega\dot{T}_0 + \omega^2 T_0 = 0$$

$$T_0(0) = A \qquad \dot{T}_0(0) = 0 \tag{17}$$

And

$$\ddot{T}_1 + 2\xi\omega\dot{T}_1 + \omega^2 T_1 + \omega_1 T_0 + \beta T_0^3 = 0$$

$$T_1(0) = 0 \qquad \dot{T}_1(0) = 0 \tag{18}$$

While $\xi < \omega$, the solution of Eq.17 will be as follow

$$T_0 = \frac{A}{\varepsilon}e^{-\xi\omega t}\cos(\omega_d t + \varphi) = \frac{A}{\varepsilon}e^{-\xi\omega t}\cos\psi \tag{19}$$

Where ω_d and φ are the damped frequency and phase of system. Substituting Eq.19 into Eq.18 gives

$$\ddot{T}_1 + 2\xi\omega\dot{T}_1 + \omega^2 T_1 + \omega_1(\frac{A}{\varepsilon}e^{-\xi\omega t}\cos(\omega_d t + \varphi)) +$$
$$\beta(\frac{A}{\varepsilon}e^{-\xi\omega t}\cos(\omega_d t + \varphi))^3 = 0 \tag{20}$$

The expanded form of $\cos^3(\omega_d t + \varphi)$ and the Taylor series of $e^{-2\xi\omega t}$ are as follows

$$\cos^3(\omega_d t + \varphi) = \frac{3}{4}\cos(\omega_d t + \varphi) + \frac{1}{4}\cos 3(\omega_d t + \varphi) \tag{21}$$

$$e^{-2\xi\omega t} = 1 - 2\xi\omega t + \frac{(2\xi\omega t)^2}{2!} - \cdots \tag{22}$$

Substituting Eqs.21, 22 into Eq.20 and after some essential mathematical calculation, Eq.20 will be shown as

$$\ddot{T}_1 + 2\xi\omega\dot{T}_1 + \omega^2 T_1 + (\omega_1 + \frac{3\beta A^2}{4\varepsilon^2})\frac{A}{\varepsilon}e^{-\xi\omega t}\cos(\omega_d t + \varphi) +$$
$$\frac{\beta A^3}{4\varepsilon^3}e^{-\xi\omega t}\cos 3(\omega_d t + \varphi) = 0 \tag{23}$$

In order to eliminate the secular terms should be

$$\omega_1 = -\frac{3\beta A^2}{4\varepsilon^2} \tag{24}$$

Nonlinear frequency by substituting Eq.24 into Eq.16 for the first approximation will be derived as

$$\omega_{PM} = \sqrt{(\alpha + \frac{3}{4}\beta A^2)} \tag{25}$$

The last equation is the system nonlinear frequency. Furthermore, the system response based on HPM can be expressed as follow

$$U(t,\varepsilon) = e^{-\xi t}(c_{10}\cos\omega_d t + c_{20}\sin\omega_d t) + \varepsilon u_1(c_1, c_2, t) + \varepsilon^2 u_2 + \cdots \tag{26}$$

If perturbed Duffing equation with the damper is presented as

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = -\varepsilon f(u,\dot{u}) \tag{27}$$

Differentiating Eq.26 twice with respect to t and substituting it in Eq.27 for the coefficients of ε gives

$$\frac{\partial^2 u_1}{\partial t^2} + 2\xi\frac{\partial u_1}{\partial t} + \omega^2 u_1 = -\beta e^{-3\xi t}(c_{10}\cos\omega_d t + c_{20}\sin\omega_d t)^3 \tag{28}$$

Where c_{10} and c_{20} can be obtained from the initial conditions. By solving Eq.28 u_1 can be found as

$$u_1 = -\frac{e^{-\xi\omega t}}{16\omega^2}(\frac{3c_1 c_2^2(\xi\omega\cos\omega_d t - \omega_d\sin\omega_d t)}{\xi\omega} + \frac{3c_2^3(\xi\omega\sin\omega_d t + \omega_d\cos\omega_d t)}{\xi\omega} + \frac{(c_1^3 - 3c_1 c_2^2)[((\xi\omega)^2 - 2\omega_d^2)\cos 3\omega_d t - 3\xi\omega\omega_d\sin 3\omega_d t]}{(\xi\omega)^2 + 4\omega_d^2} + \frac{(3c_1^2 c_2 - c_2^3)[((\xi\omega)^2 - 2\omega_d^2)\sin 3\omega_d t + 3\xi\omega\omega_d\cos 3\omega_d t]}{(\xi\omega)^2 + 4\omega_d^2}) \tag{29}$$

Where c_1, c_2 are as follows

$$c_1 = c_{10} + \frac{3\beta c_{10}^2(\xi\omega c_{10} + \omega_d c_{20})(1 - e^{-2\xi\omega t})}{16\xi\omega^3} \tag{30}$$

$$c_2 = c_{20} + \frac{3\beta c_{10}^2(\xi\omega c_{20} - \omega_d c_{10})(1 - e^{-2\xi\omega t})}{16\xi\omega^3} \tag{31}$$

It is obvious, when $\beta = 0$, c_1 and c_2 are equal to c_{10} and c_{20}. Finally, the first approximate solution will be found as bellow

$$U(t,\varepsilon) = e^{-\xi\omega t}(c_{10}\cos\omega_d t + c_{20}\sin\omega_d t) + \varepsilon u_1 \tag{32}$$

3.2 Hamilton Approach

The nonlinear frequency of Euler-Bernoulli beam can be easily derived from the Hamilton Principles. Hamiltonian form of the Eq.10 is as follow

$$H = \frac{1}{2}u'^2 + \frac{1}{2}\alpha u^2 + \frac{1}{4}\beta u^4 \tag{33}$$

Substituting initial condition below

$$u(0) = A_0 \qquad u'(0) = 0 \tag{34}$$

Then

$$H = \frac{1}{2}u'^2 + \frac{1}{2}\alpha u^2 + \frac{1}{4}\beta u^4 = \frac{1}{2}\alpha A_0^2 + \frac{1}{4}\beta A_0^4 \tag{35}$$

Where, the first term of the left-hand side of the above equation shows kinetic energy and the next two terms illustrate the total system potential energy.

If the form of trial function is assumed as

$$U(t) = A_0\cos(\omega t) \tag{36}$$

Substituting Eq.36 into Eq.35 can be showed residual equation as follow

$$R(t) = \frac{1}{2}A_0^2(\omega^2\sin^2\omega t + \alpha\cos^2\omega t + \frac{1}{2}\beta A_0^2\cos^4\omega t - \alpha - \frac{1}{2}\beta A_0^2) \tag{37}$$

The first-order approximation is as follow

$$\int_0^{\frac{\pi}{2\omega}} R(t)\cos(\omega t)dt = 0 \tag{38}$$

Solving Eq.38 leads to the nonlinear frequency as follow

$$\omega_H = \sqrt{\alpha + \frac{49}{70}\beta A_0^2} \tag{39}$$

3.3 DTM and MDTM

Differential Transform Method (DTM) presents a good approximate about a specific point by helping the power series concept. It is one of the most powerful methods to achieve the accurate solutions of the system in the limited time domain. So as to apply this

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method to unlimited time domain, DTM has some difficulties and because of this reason MDTM can be used to obtain more accurate solution than DTM. However, the first step is to use DTM to derive the differential transform of the system response $x(t)$. In order to aim this goal, $X(k)$ is defined as follow

$$X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0} \quad (40)$$

Also, the invers transform of $X(k)$ is

$$x(t) = \sum_{k=0}^{\infty} X(k)t^k \quad (41)$$

Using Eqs.40 and 41

$$x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0} \quad (42)$$

Some primary relations of DTM are presented in Table 2.

Table 2: Some primary relations of DTM

Original Function	Transformed Function
$x(t) = \alpha g(t) \pm \beta h(t)$	$X(k) = \alpha G(k) \pm \beta H(k)$
$x(t) = g(t)h(t)$	$X(k) = \sum_{l=0}^{k} G(l)H(k-l)$
$x(t) = g(t)h(t)f(t)$	$X(k) = \sum_{l=0}^{k}\sum_{m=0}^{k-l} G(l)H(m)F(k-l-m)$
$x(t) = \dfrac{d^m g(t)}{dt^m}$	$X(k) = \dfrac{(k+m)!}{k!}G(k+m)$
$x(t) = e^t$	$X(k) = \dfrac{1}{k!}$

.

In order to obtain more precise solution results, MDTM is implemented in this study based on Padé approximation concept. In order to apply this method, Laplace transform of Eq.41 should be taken as follow

$$L[x(t)] = L[\sum_{k=0}^{\infty} X(k)t^k] \quad (43)$$

$f(t)$ can be obtained by substituting $s = \frac{1}{t}$ in the derived Laplace transform equation. In based on Padé approximation concept, $L(n,m)$ is equal to $f(t)$ and illustrated by the quotient of two polynomials $P_n(t)$ and $Q_m(t)$ of degrees n and m as follow

$$L(n,m) = f(t) = \frac{P_n(t)}{Q_m(t)} \quad (44)$$

Where

$$P_n(t) = p_0 t^0 + p_1 t^1 + p_2 t^2 + \cdots + p_n t^n \quad (45)$$

$$Q_n(t) = 1 + q_1 t^1 + q_2 t^2 + \cdots + q_m t^m \quad (46)$$

Solving Eq.44 leads to obtain the coefficients $[p_0, p_1,..., p_n]$ and $[q_1, q_2,..., q_m]$. Consequently, after

specify all coefficients, $L_s(n,m)$ can be derived by substituting $t = \frac{1}{s}$ in Eq.44. Finally, the modified solution of $x(t)$ can be obtained by applying invers Laplace transform of $L_s(n,m)$.

$$x(t) = L^{-1}[L_s(n,m)] \quad (47)$$

4. Results and Discussions

In order to have a good comparison between the different methods, the numerical analysis has been performed helping the appropriate data that satisfy the most important goals of this study such as the nonlinear variable viscoelastic foundation model and the strong damping conditions.

Table 3 shows the nonlinear natural frequencies for S-S, S-C and C-C boundary conditions by applying the linear and nonlinear elastic foundation model. The results that summarized in Table 3 show the comparison between the linear frequencies and system nonlinear frequencies for the different boundary conditions and two cases foundation functions and also stiffness amounts based on HPM and HA (first order energy balance method). The obtained ratio of the nonlinear to linear frequencies in this paper is

compared by another literature [44] in Table 4. It should be mentioned that the results which presented in reference [44] are for an Euler-Bernoulli beam without axial load and elastic foundation. There is no comparison data for an Euler-Bernoulli beam resting on viscoelastic foundation.

Table 3: Nonlinear natural frequencies for different boundary conditions by applying the linear and nonlinear elastic foundation model.

Boundary Condition	Foundation function	$g(x)=1-0.2x$		$g(x)=1-0.2x^2$	
	Stiffness amount	K=10	K=100	K=10	K=100
S-S	Linear frequency	10.315	13.690	10.336	13.848
	Nonlinear(HPM) frequency	11.166	14.341	11.185	14.492
	Nonlinear (HA) frequency	11.111	14.299	11.130	14.450
S-C	Linear frequency	15.698	18.061	15.712	18.182
	Nonlinear(HPM) frequency	17.207	19.387	17.220	19.499
	Nonlinear (HA) frequency	17.110	19.301	17.123	19.414
C-C	Linear frequency	22.573	24.301	22.583	24.395
	Nonlinear(HPM) frequency	23.797	25.442	23.807	25.532
	Nonlinear (HA) frequency	23.717	25.368	23.727	25.457

Table 4**:** Ratio of the nonlinear to linear frequencies in this paper and another literature [44].

Boundary Condition	Present	Ref. [44]	Error
S-S	1.0825	1.0892	0.6%
S-C	1.0961	-	-
C-C	1.0542	1.0550	0.07%

Table 3 indicates the greater amount of the nonlinear frequency with respect to the linear frequency in each boundary condition. Additionally, the rate of frequency is increased by raising the rigidity of the boundary conditions. The above data is derived by assuming the amplitude amount A=1 and the axial load F=0.

Table 5 shows the rate of buckling force (F) in each boundary condition with respect to variation of system amplitude (A). As can be seen in Table 5, the buckling force is increased by arising in rigidity of the boundary condition for constant system amplitude in each case. Moreover, by increasing the system amplitude the buckling force is increased for each case.

Table 5: Rate of buckling force (F) in each boundary condition with respect to the variation of system amplitude (A).

	Buckling Force (F) in S-S			Buckling Force (F) in S-C			Buckling Force (F) in C-C		
	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
A=0.1	19.007	19.447	36.908	28.391	28.772	44.083	48.049	48.419	63.635
A=1	20.838	21.279	38.740	32.662	33.043	48.354	52.616	52.987	68.202
A=5	63.238	65.679	83.140	136.197	136.578	151.890	163.338	163.709	178.925

Table 6 shows the nonlinear damped frequency in the various boundary conditions and variable viscoelastic foundation functions that derived for the different amount of damping coefficients. The results below indicate the decrease in the nonlinear damped frequency by increasing the amount of damping coefficient

Table 6: Nonlinear damped frequency in the various boundary conditions and variable viscoelastic foundation functions that derived for the different amount of damping coefficients.

Boundary Condition	Foundation function	$g(x) = 1 - 0.2x$		$g(x) = 1 - 0.2x^2$	
	Amount of Damping Coefficient (C)	K=10	K=100	K=10	K=100
S-S	1	10.305	13.683	10.325	13.840
	5	10.067	13.504	10.063	13.646
	10	9.281	12.929	9.197	13.020
S-C	1	15.692	18.055	15.705	18.176
	5	15.541	17.924	15.539	18.033
	10	15.059	17.509	15.008	17.577
C-C	1	22.568	24.297	22.578	24.390
	5	22.460	24.197	22.459	24.280
	10	22.120	23.881	22.082	23.932

Fig. 2 shows the variations of the nonlinear damped frequency (ω_d) with respect to the dimensionless stiffness for the different boundary conditions. As shown in Fig. 2, by increasing the amount of the dimensionless stiffness, the nonlinear damped frequency is increased and for the constant stiffness, the amount of the nonlinear damped frequency has the greater rate for the fully clamped boundary conditions.

Fig. 3 indicates the variations of the nonlinear damped frequency (ω_d) with respect to the dimensionless system amplitude for the different boundary conditions. As shown in Fig. 3, by increasing the amount of dimensionless amplitude, the nonlinear damped frequency is increased and for the constant amplitude, the amount of nonlinear damped frequency has the greater rate for the fully clamped boundary conditions.

Fig. 4 shows the variations of the nonlinear damped frequency (ω_d) with respect to the axial load for the different boundary conditions S-S, S-C and C-C. It is obvious that, where the buckling load has the maximum amount, the nonlinear frequency is equal to zero. Fig. 4 indicates that the linear and parabolic elastic foundation have approximately the same frequency results with respect to the axial load variations. In contrast, the harmonic elastic foundation leads to the greater noticeable frequency amount with respect to the other mentioned foundation functions.
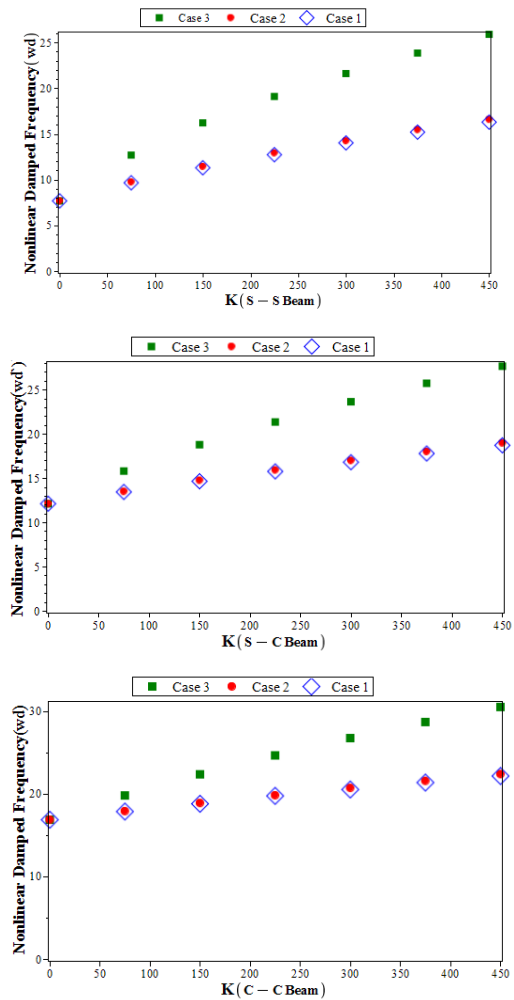

Figure 2: variations of the nonlinear damped frequency (ω_d) with respect to the dimensionless stiffness for the different boundary conditions
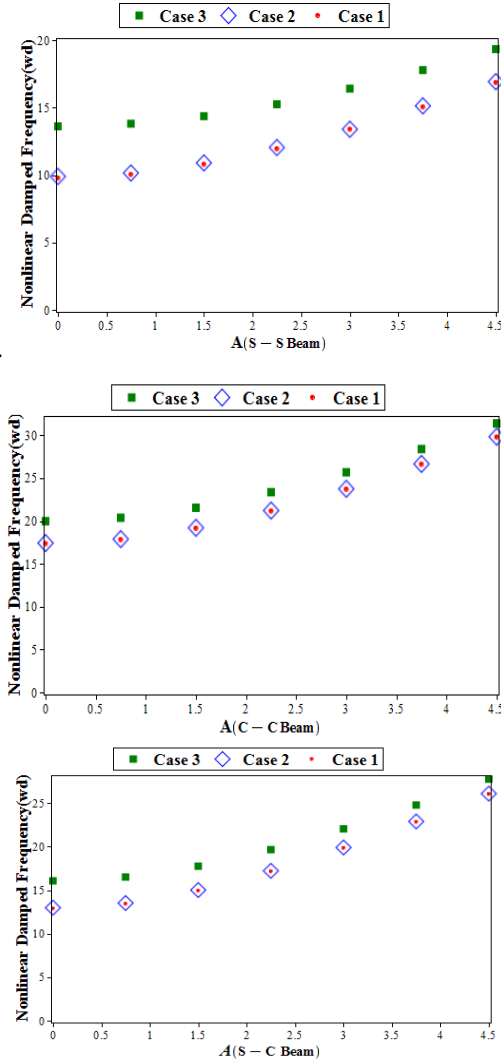
Figure 3: variations of the nonlinear damped frequency (ω_d) with respect to the dimensionless system amplitude for the different boundary conditions

Figure 4: variations of the nonlinear damped frequency (ω_d) with respect to the axial load for the different boundary conditions

Fig. 5 presents the system response by implementing DTM, HPM and RK4. Although there is a good agreement between HPM and RK4 method, DTM cannot follow them for a large range of time. This problem is improved by using MDTM approach. As can be observed in Fig. 6, applying MDTM makes the response very close to the exact solution which derived by Rung-kutta method and also the analytical homotopy perturbation solution. In fact, as DTM response results are valid just in a small range of time, MDTM is implemented in this study to obtain precise solution in a wide time-domain. Fig. 6 shows the good agreement between HPM, MDTM and the solution obtained by RK4. In order to derive the above data (Fig. 6) in based on Eq. 10 some numerical inputs are applied as $\alpha = 315$, $\beta = 1$, $\gamma = 21$, $A_0 = 2$ •
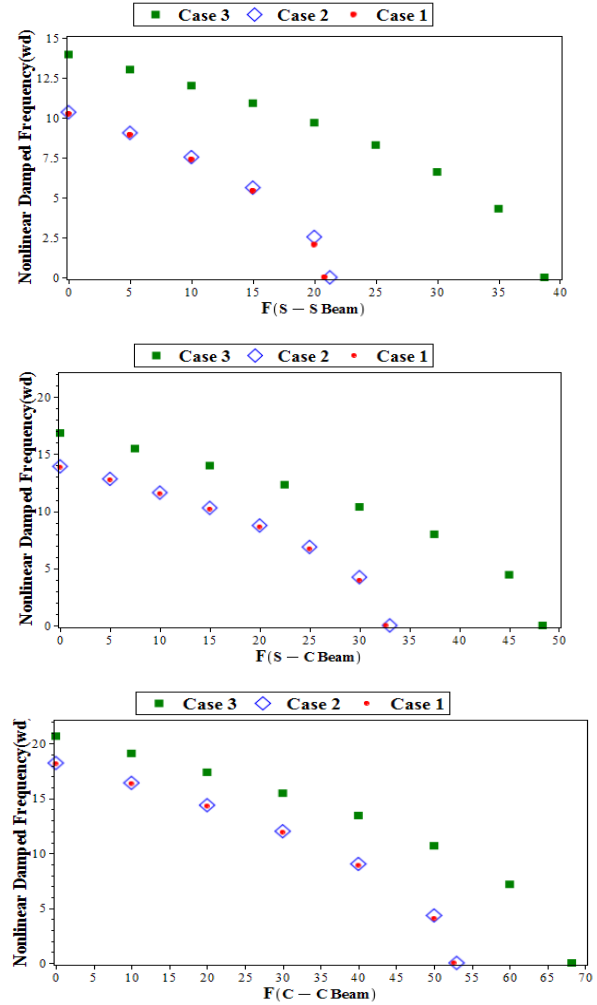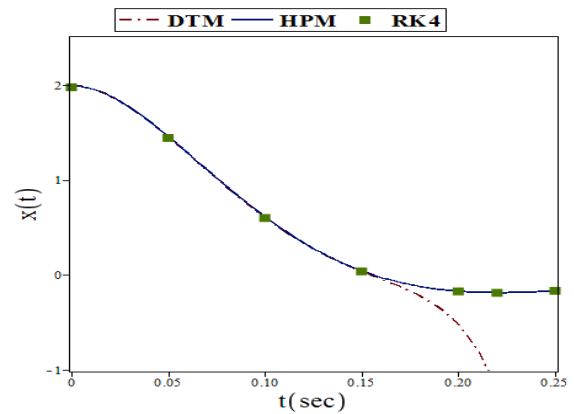

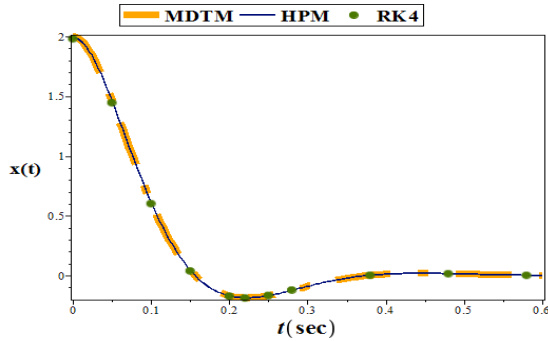Figure 5: system response by implementing DTM, HPM and RK4

Figure 6: improved system response by implementing MDTM and comparing it with HPM and RK4

5. Conclusions

In this paper, free nonlinear vibration of an Euler-Bernoulli beam resting on the viscoelastic foundation induced by axially load with the immovable different S-S, S-C and C-C boundary conditions is studied. The variable viscoelastic foundation is implemented in the linear, parabolic and harmonic functions. The nonlinear damped frequencies are obtained by HPM and HA and finally, they are compared by the other previous works that they have a good agreement with the applied methods and the exact solutions. The nonlinear system response is derived by HPM, DTM and MDTM which they are compared by the solutions obtained by RK4 method. The first order homotopy perturbation method gives the precise results with respect to RK4 solution. Although DTM solutions for the small range of time have a good accuracy with respect to the other solutions, but by increasing the time, the convergence of the results is missed. In order to improve the results, MDTM is applied and by using this method the poor accuracy in a large time domain is compensated. All the results for the nonlinear damped frequency show the noticeable ascent in the frequency by any increase in the system stiffness. Result illustrates by increasing the amount of damping coefficient, the nonlinear damped frequency is decreased gradually and also by stimulating system near to main eigenmode (harmonic mode) the nonlinear damped frequency is increased dramatically. Considering the paper results, buckling force is arisen for each case permanently by increase in the boundary rigidity for a constant value of system amplitude (A). Moreover, in based on paper results, the buckling force is arisen by increasing the system amplitude for each case.

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