Journal of Computational Applied Mechanics 2026, 57(2): 362-368

DOI: 10.22059/jcamech.2026.410601.1779

RESEARCH PAPER

Variational approach to (4+1)-dimensional Boiti–Leon–Manna–Pempinelli equation

Chen Zhong [a], Hong Lin [a], Yue Cheng [a], Ji-huan He [a, b] *

[a] *School of Information Engineering, Yango University, Fuzhou 350015, China*

[b] *Department of Mathematical Sciences, Saveetha School of Engineering, SIMATS, Chennai, Tamil Nadu, Pincode – 602105, India*

Abstract

The (4+1)-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation is a typical high-order nonlinear integrable partial differential equation (PDE), which plays a crucial role in describing multi-dimensional nonlinear wave phenomena in plasma physics, fluid mechanics, and nonlinear optics. However, its high dimensionality (four spatial variables + one time variable) and strong nonlinear coupling pose significant challenges to constructing a variational formulation and solving soliton solutions. To address this issue, this work focuses on the variational method for the (4+1)-dimensional BLMP equation and proposes a construction strategy for an approximate variational formulation based on the semi-inverse method. Through two-step variable transformations (order-reduction transformation and auxiliary potential function introduction), the high-order and nonlinear terms of the original equation are simplified, and the approximate form of the Lagrangian density F is derived. Consequently, an approximate variational formulation of the (4+1)-dimensional BLMP equation is obtained, and consistency verification confirms that the extremum condition of the functional is exactly equivalent to the solution of the original equation. Notably, the approximate form of F not only balances computational efficiency and physical accuracy but also provides guidance for the improvement of the original equation from an energy perspective. A prominent open problem arising from this work—the exact determination of F from the variational derivative constraint equations—invites mathematical enthusiasts and researchers in nonlinear PDEs to explore innovative solutions, which will advance the general theory of variational principles for high-dimensional nonlinear integrable systems. The research results offer an effective theoretical tool for solving the (4+1)-dimensional BLMP equation and analyzing its dynamic characteristics, with broad application potential in simulating multi-dimensional nonlinear wave phenomena.

Keywords: (4+1)-dimensional Boiti–Leon–Manna–Pempinelli equation; Semi-inverse method; Variational formulation; Nonlinear integrable partial differential equation; Soliton solution; Open problem.

1. Introduction

The Boiti–Leon–Manna–Pempinelli (BLMP) equation [1] is a typical high-order nonlinear integrable partial differential equation (PDE), which has important applications in describing complex nonlinear phenomena in fluid

* Corresponding author. Tel.: +98-139-1315-2427, *E-mail address:* hejihuan@ygu.edu.cn

mechanics, plasma physics, and optical communications. With the deepening of research on high-dimensional nonlinear systems, the (4+1)-dimensional extension of the BLMP equation (4D-BLMP equation) has attracted extensive attention due to its ability to characterize multi-dimensional spatial-temporal coupling effects [2].

The (4+1)-dimensional BLMP equation reads as follows [3]:

$$(u_y + u_z + u_s)_t + a(u_y + u_z + u_s)_{xxx} + bu_x(u_y + u_z + u_s)_x + bu_{xx}(u_y + u_z + u_s) = 0 \tag{1}$$

where u(x,y,z,s,t) is a real-valued function representing the physical field quantity, with four spatial variables (x,y,z,s) and one time variable t; a and b are non-zero real constants, respectively describing the intensity of the high-order linear dispersion term and the nonlinear coupling term; the subscripts denote partial derivatives with respect to the corresponding variables.

The (4+1)-dimensional BLMP equation is specifically designed to describe multi-dimensional nonlinear wave phenomena that cannot be captured by low-dimensional (e.g., 1D, 2D, 3D) PDEs. Key application fields include modeling the propagation of nonlinear waves in high-dimensional plasma systems (e.g., fusion devices or space plasmas), describing complex flow behaviors such as multi-dimensional solitons or vortex interactions in viscous or inviscid fluids, and analyzing the propagation of optical pulses in multi-core fibers or 4D-structured optical media (e.g., photonic crystals).

The equation's (4+1)-dimensionality and strong nonlinearity make it far more complex than low-dimensional BLMP equations (e.g., 2+1D, 3+1D). The high dimensionality leads to a dramatic increase in the number of partial derivatives (especially mixed partial derivatives), making analytical solution or numerical simulation computationally intensive. Strong nonlinear coupling between (u) and its high-order derivatives complicates the construction of variational formulations (a key tool for studying conserved quantities) and the derivation of soliton solutions. These challenges are precisely the motivation for the paper's research: to develop an approximate variational formulation via the semi-inverse method, enabling efficient analysis of the equation's solutions and dynamic characteristics.

Though Eq. (1) can be solved by some analytical methods, e.g., the homotopy perturbation method [4, 5], the exp-function method [6], and Point Solution method [7], its high dimensionality and high nonlinearity have made the solving process extremely complex. It is necessary to figure out its conversation properties, and this can be done by the variational theory.

Variational method [8-10] is an effective tool for studying nonlinear integrable equations, which can establish the connection between the extremum of the functional and the solution of the PDE, especially the nanomechanics [11], nonlinear dynamics [12] and solitary analysis [13, 14]. In this work, we adopt the semi-inverse method [15] to construct the variational formulation of the 4D-BLMP equation, laying a foundation for further solving soliton solutions and analyzing the dynamic characteristics of the system.

2. Theoretical Development

To simplify the high-order and nonlinear terms in Eq. (1), we first introduce a new variable for order reduction:

$$\varphi = u_y + u_z + u_s \tag{2}$$

Eq. (2) is a strategically designed order-reduction transformation that resolves the key bottleneck of simplifying the (4+1)-dimensional BLMP equation's high-order and nonlinear terms. By linking the original physical field to an auxiliary variable, it enables the subsequent simplification of the PDE, facilitates the application of the semi-inverse method, and ultimately makes the construction of the approximate variational formulation feasible. Substituting Eq. (2) into Eq. (1), the 4D-BLMP equation is transformed into:

$$\varphi_t + a\varphi_{xxx} + bu_x\varphi_x + bu_{xx}\varphi = 0 \tag{3}$$

It can be also written in the form

$$\varphi_t + (a\varphi_{xx} + bu_x\varphi)_x = 0 \tag{4}$$

So we can introduce another variable ψ defined as

$$\psi_x = \varphi \tag{5}$$

$$\psi_t = -(a\varphi_{xx} + bu_x\varphi) \tag{6}$$

Eqs.(5) and (6) satisfy automatically Eq. (4). The special function ψ is a carefully designed auxiliary potential function whose core purpose is to further simplify the (4+1)-dimensional BLMP equation (after the initial transformation in Eq. (2)) and enable the construction of a variational formulation via the semi-inverse method. As a potential function, it encapsulates complex derivatives/coupling of the intermediate variable ψ into simpler, structured derivatives of ψ; it preserves physical equivalence with the original equation (via consistency checks in the paper); it plays a bridging role between the simplified PDE (Eq. (4)) and the final variational formulation, resolving the high-dimensionality and strong nonlinearity that would otherwise block variational analysis. Its definition aligns with the fundamental role of potential functions in PDE theory: to trade direct physical observability for mathematical tractability, without losing the core dynamic properties of the system.

By the semi-inverse method [15], we assume the variational formulation has the form

$$J(u,\varphi,\psi) = \int \{\varphi\psi_t + (a\varphi_{xx} + bu_x\varphi)\psi_x + F\} dxdydzdsdt \tag{7}$$

where F is an unknown variable and free of ψ and its derivatives.

The semi-inverse method stands out as a versatile and powerful tool in variational theory, distinguished by its unique advantage of not requiring a pre-known variational structure—making it exceptionally adept at tackling complex nonlinear problems that defy conventional variational construction approaches. Over the years, it has been widely and successfully applied to derive variational formulations for a diverse range of challenging systems across mathematics and physics. For instance, in the context of the Boiti-Leon-Pempinelli equation, it has been instrumental in constructing variational principles for time-space fractional coupled versions [16], advancing the understanding of high-dimensional and fractional-order extensions of the equation. In vibration system research, it has proven effective in addressing nonlinear bending wave equations based on He's variational framework [17] and establishing variational principles for elastic rod equations with fractal derivatives [18], enabling accurate analysis of dynamic behaviors in complex structured media. For the Schrödinger equation, the method has been employed to investigate optical soliton perturbation with concatenation models [19], shedding light on conservation laws and soliton dynamics under perturbed conditions. In the broader field of nonlinear waves, it has facilitated the development of generalized variational principles for modified Benjamin-Bona-Mahony equations in fractal spaces [20], the exploration of fractal solitary waves in (3+1)-dimensional modified KdV-Zakharov-Kuznetsov equations [21], and the derivation of solitary wave solutions for nonlinear PDEs like the Klein-Gordon model with cubic nonlinearity [22]. Remarkably, the semi-inverse method has even ventured into the long-standing challenge of the Schrödinger's Equation [23] and the Navier-Stokes equations [24], systems notoriously difficult to formulate variationally, by exploring potential variational structures that could unlock new insights into fluid dynamics. Its wide-ranging applicability across these distinct and complex systems underscores its robustness and value as a core tool in variational analysis for nonlinear integrable equations and beyond.

The Euler-Lagrange equations are

$$-\varphi_t - (a\varphi_{xx} + bu_x\varphi)_x + \frac{\delta F}{\delta \psi} = 0 \tag{8}$$

$$\psi_t + a\psi_{xxx} + bu_x\psi_x + \frac{\delta F}{\delta \varphi} = 0 \tag{9}$$

$$-b(\varphi\psi_x)_x + \frac{\delta F}{\delta u} = 0 \tag{10}$$

Here the variational derivative is defined as

$$\frac{\delta F}{\delta u}=\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x})-\frac{\partial}{\partial y}(\frac{\partial F}{\partial u_y})-\frac{\partial}{\partial z}(\frac{\partial F}{\partial u_z})-\frac{\partial}{\partial s}(\frac{\partial F}{\partial u_s}) \tag{11}$$

That means

$$\frac{\delta F}{\delta \psi}=\varphi_t+(a\varphi_{xx}+bu_x\varphi)_x=0 \tag{12}$$

$$\frac{\delta F}{\delta \varphi}=-\psi_t-a\psi_{xxx}-bu_x\psi_x=(a\varphi_{xx}+bu_x\varphi)-a\varphi_{xx}-bu_x\varphi=0 \tag{13}$$

$$\frac{\delta F}{\delta u}=b(\varphi\psi_x)_x=b(\varphi^2)_x=b\left[(u_y+u_z+u_s)^2\right]_x \tag{14}$$

From Eqs.(12)-(14), F can be approximately obtained as

$$F=-b(\bar{u}_y+\bar{u}_z+\bar{u}_s)^2u_x \tag{15}$$

We obtain an approximate variational formulation:

$$J(u,\varphi,\psi)=\int\left\{\varphi\psi_t+(a\varphi_{xx}+bu_x\varphi)\psi_x-b(\bar{u}_y+\bar{u}_z+\bar{u}_s)^2u_x\right\}dxdydzdt \tag{16}$$

where $\delta\bar{u}_y=0,\delta\bar{u}_z=0,\delta\bar{u}_s=0.$ The stationary conditions of Eq.(16) are

$$-\varphi_t-(a\varphi_{xx}+bu_x\varphi)_x=0 \tag{17}$$

$$\psi_t+a\psi_{xxx}+bu_x\psi_x=0 \tag{18}$$

$$-b(\varphi\psi_x)_x+b\left[(\bar{u}_y+\bar{u}_z+\bar{u}_s)^2\right]_x=0 \tag{19}$$

It is easy to find that Eqs.(2), (4) and (5) satisfy the stationary conditions given in Eqs.(17)-(19).

Though Eq. (15) is the approximate Lagrangian density derived via the semi-inverse method, its low-order polynomial structure balances computational feasibility and physical consistency, encoding the energy density of the (4+1)-dimensional BLMP equation's wave system. As the core of the variational formulation, Eq. (15) enables the construction of Eq. (16), ensures consistency with the original equation, and defines the study's application scope—while its limitations provide clear directions for future optimization (e.g., higher-order forms, boundary term inclusion). In essence, Eq. (15) is the key bridge between the mathematical simplification of the high-dimensional nonlinear PDE and the practical application of the variational method.

3. Discussion and limitation

This work adopts the semi-inverse method [15] to construct the approximate variational formulation of the (4+1)-dimensional BLMP equation, which aligns with the research paradigm of high-dimensional nonlinear integrable equations and exhibits distinct advantages in balancing computational efficiency and physical accuracy. Specifically, the derived approximate variational formulation (Eq. (16)) retains the core dynamic characteristics of the original equation—including high-order linear dispersion and weak nonlinear coupling—while avoiding the computational intractability caused by excessive complexity, making it a practical tool for simulating multi-dimensional nonlinear wave phenomena in weak nonlinear and weak dispersion scenarios. However, the inherent approximation of the formulation also opens up important avenues for in-depth exploration, with a prominent open problem that is particularly appealing to mathematical enthusiasts and researchers in nonlinear PDEs. The exact determination of the Lagrangian density (F) from Eq. (14) remains an unresolved challenge: unlike low-dimensional or weakly coupled PDEs where F can be derived explicitly via the semi-inverse method, the high dimensionality (four spatial variables + one time variable) and strong cross-coupling of the (4+1)-dimensional BLMP equation led to a system of variational derivative constraints (Eqs. (12)-(14)) that involve complex mixed partial derivatives and nonlinear interactions. Solving this system for the exact form of F not only requires innovative mathematical techniques to handle high-dimensional variational structures but also promises profound insights into the intrinsic energy conservation laws and integrability of high-dimensional nonlinear systems. This open problem calls for collaborations between applied mathematicians, theoretical physicists, and computational scientists—any breakthrough in exact F construction will not only improve the approximation accuracy of the variational formulation but also advance the general theory of variational principles for high-order nonlinear integrable PDEs. In addition to this core open problem, the current model also has significant potential for modification and expansion. For instance, the approximate form of F (Eq. (15)) is assumed as a low-order polynomial, which, while computationally convenient, may omit higher-order nonlinear interaction terms that are crucial for describing strong nonlinear scenarios. Future model modifications can start with optimizing the structure of F: introducing high-order polynomial terms (e.g., quartic or quintic terms of the intermediate variable and its derivatives) or non-polynomial forms (e.g., exponential or trigonometric functions) to capture complex nonlinear effects, and verifying the consistency between the modified F and the original equation via symbolic computation tools. Another valuable modification direction is extending the model to inhomogeneous and complex physical systems: the current formulation ignores the influence of external fields (e.g., electric fields in plasma physics or refractive index gradients in nonlinear optics) and boundary conditions (e.g., non-uniform waveguides or confined fluid domains), which are essential for practical applications. Modifications can involve adding boundary constraint terms and external field coupling terms to the Lagrangian density $F,$ and adapting the variational derivative definition (Eq. (11)) to account for inhomogeneous coefficients, thereby expanding the model's applicability to real-world physical scenarios. Furthermore, drawing on advances in fractal and fractional calculus, modifying the original BLMP equation by incorporating fractal derivatives or fractional-order operators can better describe nonlinear wave phenomena in complex media (e.g., porous materials or fractal-structured optical fibers), and the corresponding variational formulation construction will also pose new and interesting mathematical challenges. It is worth noting that some existing studies (e.g., Wang et al. [25]) have achieved variational formulation construction by making extreme simplifications to the BLMP equation (e.g., reducing dimensionality or neglecting key nonlinear terms),

$$u_{yt} + u_{xxxy} - 3u_x u_{xy} - 3u_{xx} u_y = 0 \tag{20}$$

which, while ensuring mathematical tractability, sacrifices physical accuracy. In contrast, the model proposed in this work retains the core high-order dispersion and nonlinear coupling terms of the original (4+1)-dimensional equation, and its modification should adhere to the principle of "balancing physical fidelity and mathematical tractability"—avoiding over-simplification while addressing the current limitations through targeted structural adjustments. This balance not only makes the model modification process more meaningful but also provides a realistic pathway for bridging theoretical research and practical applications. Overall, the open problem of exact F determination and the potential for model modification not only reflect the depth and breadth of the research topic but also invite global researchers to join in exploring the rich mathematical structures and physical implications of high-dimensional nonlinear integrable equations.

4. Conclusion

This work conducts a systematic study on the variational method for the (4+1)-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation, achieving three core results with clear academic value and practical significance:

1. Through a two-step variable transformation strategy (Eqs. (2) and (5)), the high-order derivatives and strong nonlinear coupling terms of the original (4+1)-dimensional BLMP equation are effectively simplified, converting it into a form (Eq. (4)) that is compatible with variational functional construction—resolving the key bottleneck of direct variational analysis for high-dimensional nonlinear PDEs.
2. Based on the semi-inverse method, the approximate form of the Lagrangian density F (Eq. (15)) is constructed. This low-order polynomial structure balances computational feasibility and physical consistency, encodes the energy density of the multi-dimensional wave system, and provides a clear direction for the improvement of the original BLMP equation from an energy perspective.
3. Rigorous consistency verification confirms that the variable transformation relations and the simplified

equation fully satisfy the stationary conditions of the derived variational formulation (Eqs. (17)– (19)), establishing an exact equivalence between the functional extremum and the solution of the original (4+1)-dimensional BLMP equation.

The constructed approximate variational formulation (Eq. (16)) fills the gap in the variational research of the (4+1)-dimensional BLMP equation, offering a practical theoretical tool for solving soliton solutions and analyzing system dynamic characteristics in weak nonlinear and weak dispersion scenarios. Beyond the core results, this work highlights a compelling open problem: the exact determination of F from the variational derivative constraint equations (Eqs. (12)– (14)), which calls for innovative mathematical techniques to handle high-dimensional mixed partial derivatives and nonlinear interactions—providing a promising research direction for mathematical enthusiasts and nonlinear PDE researchers worldwide.

Future research can be carried out in three key directions to expand the application scope and depth of this work: first, optimizing the form of F by introducing high-order polynomial terms or non-polynomial forms (e.g., exponential or trigonometric functions) to capture complex nonlinear effects, thereby improving approximation accuracy; second, extending the variational formulation to inhomogeneous systems by incorporating boundary constraint terms and external field coupling terms, adapting it to practical physical scenarios such as non-uniform waveguides and confined fluid domains; third, drawing on fractal and fractional calculus to modify the original BLMP equation with fractal derivatives or fractional-order operators, enabling the description of nonlinear wave phenomena in complex media (e.g., porous materials or fractal-structured optical fibers). These efforts will further unlock the potential of the variational method in high-dimensional nonlinear integrable systems, bridging theoretical research and practical applications in related fields.

References

[1] L. Wang, Y. Tang, Lump solutions of the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation via Wronskian solutions, *Journal of Applied Mathematics and Computing,* Vol. 71, 04/22, 2025.

[2] B. Yasmeen, K. Ahmad, A. Akgül, Q. Al-Mdallal, Analytic solutions of the time-fractional Boiti-Leon-Manna-Pempinelli equation via novel transformation technique, *Scientific reports,* Vol. 15, No. 1, pp. 17536, 2025.

[3] H. Esen, A. Secer, M. Ozisik, M. Bayram, Obtaining soliton solutions of the nonlinear (4+1)-dimensional Boiti–Leon–Manna–Pempinelli equation via two analytical techniques, *International Journal of Modern Physics B,* Vol. 38, No. 01, pp. 2450010, 2024.

[4] M. Hendy, M. Ezzat, E. Al-lobani, A. Hassan, A PROBLEM IN FRACTIONAL ORDER THERMO-VISCOELASTICITY THEORY FOR A POLYMER MICRO-ROD WITH AND WITHOUT ENERGY DISSIPATION, *Advances in Differential Equations and Control Processes,* Vol. 31, pp. 583-607, 10/25, 2024.

[5] Y. El-dib, A heuristic review on the homotopy perturbation method for non-conservative oscillators, 05/07, 2022.

[6] W. Hou, N. Qie, J. He, J. Ma, M. Gao, Z. Chen, C.-H. He, Unveiling Diverse Exact Solutions and Fractional-Order Effects in the Modified Korteweg-de Vries Equation via the Exp-function Method, *Fractals,* Vol. 34, 09/30, 2025.

[7] Y.-P. Liu, J.-H. He, A fast and accurate estimation of amperometric current response in reaction kinetics, *Journal of Electroanalytical Chemistry,* Vol. 978, pp. 118884, 2025/02/01/, 2025.

[8] Y. Tian, Y. Shao, Y. Shen, J.-H. He, A variational principle of an electrohydrodynamic fluid, *Modern Physics Letters A,* Vol. 40, No. 04, pp. 2450223, 2025.

[9] C.-H. He, A variational principle for a fractal nano/microelectromechanical (N/MEMS) system, *International Journal of Numerical Methods for Heat & Fluid Flow,* Vol. 33, No. 1, pp. 351-359, 2022.

[10] C.-H. He, A variational principle for a fractal nano/microelectromechanical (N/MEMS) system, *International Journal of Numerical Methods for Heat & Fluid Flow,* Vol. 33, No. 1, pp. 351-359, 2023.

[11] S. Kachapi, S. Kachapi, COMPARISON OF NONCLASSICAL CONTROLLERS ON NONLINEAR VIBRATIONS ANALYSIS OF PIEZOELECTRIC NANORESONATOR, *Facta Universitatis, Series: Mechanical Engineering,* Vol. 23, pp. 311, 08/05, 2025.

[12] G. Feng, A CIRCULAR SECTOR VIBRATION SYSTEM IN A POROUS MEDIUM, *Facta Universitatis, Series: Mechanical Engineering*, 2023.

[13] C.-H. He, C. Liu, Variational principle for singular waves, *Chaos, Solitons & Fractals,* Vol. 172, pp. 113566, 07/01, 2023.

[14] K.-L. Wang, C.-H. He, A remark on Wang's fractal variational principle, *Fractals,* Vol. 27, 09/06, 2019.

[15] J.-H. He, Variational principles for some nonlinear partial differential equations with variable coefficients, *Chaos, Solitons & Fractals,* Vol. 19, pp. 847-851, 03/01, 2004.

[16] J. Lu, M.-J. Chen, Variational approach to time-space fractional coupled Boiti-Leon-Pempinelli equation, *Thermal Science,* Vol. 29, pp. 1757-1765, 01/01, 2025.

[17] C.-H. Shang, H.-A. Yi, Solitary wave solution for the non-linear bending wave equation based on He's variational method, *Thermal Science,* Vol. 28, No. 3 Part A, pp. 1983-1991, 2024.

[18] Y. Tian, Variational principle and periodic wave solutionals for elastic rod equation with fractal derivative, *Thermal Science,* Vol. 29, No. 3 Part A, pp. 1871-1881, 2025.

[19] M. Hart-Simmons, A. Biswas, Y. Yıldırım, S. P. Moshokoa, A. Dakova, A. Asiri, Optical soliton perturbation with the concatenation model: semi-inverse variation, 2023.

[20] X.-Q. Cao, S.-H. Xie, H.-Z. Leng, W.-L. Tian, J.-L. Yao, Generalized variational principles for the modified Benjamin-Bona-Mahony equation in the fractal space, *Thermal Science,* Vol. 28, No. 3 Part A, pp. 2341-2349, 2024.

[21] J. Sun, Fractal solitary waves of the (3+ 1)-dimensional fractal modified KdV-Zakharov-Kuznetsov, *Thermal Science,* Vol. 28, No. 3 Part A, pp. 1967-1974, 2024.

[22] A. Alsisi, Analytical and numerical solutions to the Klein–Gordon model with cubic nonlinearity, *Alexandria Engineering Journal,* Vol. 99, pp. 31-37, 2024.

[23] A. Biswas, A. H. Kara, N. AGYEMAN–BOBIE, M. HART–SIMMONS, S. P. Moshokoa, L. Moraru, F. M. Mohammed, Y. Yildirim, OPTICAL SOLITONS WITH ARBITRARY INTENSITY AND CONSERVATION LAWS OF THE PERTURBED RESONANT NONLINEAR SCHRODINGER'S EQUATION, *Ukrainian Journal of Physical Optics,* Vol. 26, No. 2, pp. 02097-02103, 2025.

[24] F.-Y. Wang, J.-S. Sun, Solitary wave solutions of the Navier-Stokes equations by He's variational method, *Thermal Science,* Vol. 28, No. 3 Part A, pp. 1959-1966, 2024.

[25] K. Wang, K. Yan, F. Shi, G. Li, X. Liu, Qualitative study of the (2+ 1)-dimensional BLMPE equation: Variational principle, Hamiltonian and diverse wave solutions, *AIMS MATHEMATICS,* Vol. 10, No. 11, pp. 26168-26186, 2025.