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RESEARCH PAPER

Galerkin Method with Modified Shifted Lucas Polynomials for Solving the 2D Poisson Equation

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Abstract

This study looks at how to solve the two-dimensional Poisson equation, a math problem common in physics and engineering. We focus on spectral methods, which are good at solving problems with smooth solutions. We introduce a spectral Galerkin method that uses tensor products of modified shifted Lucas polynomials. These polynomials haven't been used this way before. By adding a factor of $x(1-x)$ to the Lucas polynomials, our method automatically meets certain boundary conditions, which makes it easier to use while keeping its accuracy. Our goal is to create and test this method for solving the Poisson equation on a square. We create fast algorithms for putting together matrices and study how well the method converges using math and computer experiments. The tests show that our method has similar convergence rates to other methods like Chebyshev and Legendre. The errors go down exponentially for smooth source terms. The method is efficient and has good conditioning, which suggests that Lucas polynomials could be a good alternative to regular polynomials in spectral methods. This research could lead to using Lucas polynomial-based spectral methods for more general problems.

Keywords: Spectral Methods; Galerkin Method; Spectral Galerkin Method; Lucas Polynomials; Poisson Equation; Special Polynomials; Tensor-product Approximation; Elliptic Partial Differential Equations (PDEs).

1. Introduction

For decades, computational mathematics has advanced the creation of precise numerical methods for partial differential equations. Spectral methods are useful, especially when accuracy is critical and solutions are smooth. These methods offer spectral accuracy—fast convergence for smooth problems—unlike finite difference or finite element methods [1, 2].

Spectral methods use smooth basis functions to represent solutions instead of local approximations. For $u(x, y)$

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on domain Ω, approximations look like:

$$u(x,y) \approx u_N(x,y) = \sum_{i,j=0}^{N} \mu_{ij}\, \Psi_{ij}(x,y), \tag{1}$$

where Ψ_{ij} are basis functions (orthogonal polynomials or trigonometric functions, etc) and μ_{ij} are coefficients. Basis function selection is important. Typical choices are Chebyshev and Legendre polynomials [3, 4].

Gottlieb and Orszag's 1977 work [5] made spectral methods common in computing. Since then, the area has grown, with uses in fluid dynamics, quantum mechanics, meteorology, and other fields needing accuracy [6]. Fast transform methods, like the Fast Fourier Transform, have made these methods competitive.

Among spectral methods, the Galerkin approach is important. It uses a variational form: the differential equation's residual is orthogonal to the basis functions' space. This weak form keeps the continuous problem's structure—symmetry, conservation—making it good for elliptic problems [7, 8]. For tensor-product domains, tensor-product basis functions can make computations quick, turning an $O(N^4)$ problem in 2D into $O(N^3)$ operations [9, 10].

1.1. The Poisson Equation

The Poisson equation is key in mathematical physics. Its use in electrostatics, heat conduction, and fluid flow makes it a starting point for numerical methods. In 2D with Dirichlet boundary conditions, it is:

$$-\nabla^2 u(x,y) = f(x,y) \quad \text{in } \Omega, \quad u(x,y) = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where ∇^2 is the Laplacian, u is the unknown, and f is the source.

This equation's numerical solution has a history. Finite difference methods are popular because they are simple [11]. Finite element methods are used for complex domains [12]. For simple domains with smooth solutions, spectral methods are best. They can reach machine precision with few degrees of freedom, needing fewer mesh elements than other methods [13, 14].

Applying spectral methods to boundary value problems involves treating boundary conditions. Methods include the tau method, penalty methods, and basis functions satisfying the boundary conditions [15, 16]. We use the last, changing the polynomial basis to vanish at boundaries and satisfy Dirichlet conditions. This is simple and useful [17, 18].

1.2. Lucas Polynomials

Lucas polynomials are a specific type of special polynomial. Named after Édouard Lucas (1842-1891), they appear in math, from number theory to combinatorics. They follow the recurrence:

$$L_0(x) = 2, \quad L_1(x) = x, \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x). \tag{3}$$

Computationally, Lucas polynomials relate to Chebyshev polynomials. $L_n(x) = 2U_{n-1}(x/2)$, where U_n is the Chebyshev polynomial of the second kind. This means Lucas polynomials might have good approximation properties like Chebyshev polynomials [19, 20].

While Lucas polynomials are used in math, they are not often used in spectral methods. Recent work uses them for fractional differential equations. Abd-Elhameed and Youssri [21, 22] made operational matrices for Lucas polynomials and applied them to fractional problems. They have been used on electrohydrodynamics problems [23] and Burgers equations [24]. Their use in spectral Galerkin methods for elliptic PDEs is not well-studied, so we will study them.

We use shifted Lucas polynomials, with $x = 2t - 1$:

$$L_n^s(t) = L_n(2t-1), \tag{4}$$

mapping $[-1,1]$ to $[0,1]$. For Dirichlet boundary conditions, we use Shen's approach (Shen 1994) and define modified shifted Lucas polynomials:

$$\phi_n(t) = t(1-t)L_n^s(t). \tag{5}$$

This makes $\phi_n(0) = \phi_n(1) = 0$ for all n, satisfying boundary conditions and keeping the polynomials' approximation properties.

1.3. Objectives and Contributions

This study aims to create and test a spectral Galerkin method for solving the two-dimensional Poisson equation. This method is based on adjusted shifted Lucas polynomials and uses homogeneous Dirichlet boundary conditions. People have studied this problem a lot. But, finding new polynomial bases for spectral methods is still an active research area. Different polynomial families may have different calculation benefits. Lucas polynomials have a simple three-term recurrence relation and a link to Chebyshev polynomials of the second kind. This could lead to advantages that need a close look.

Our main contributions are:

First, we build a full spectral Galerkin system using tensor products of adjusted shifted Lucas polynomials. This involves creating proper basis functions that meet the boundary conditions. We also form the weak problem and build the resulting discrete linear system. As far as we know, this is the first time Lucas polynomials have been used as basis functions for elliptic boundary value problems in a spectral Galerkin setting.

Second, we create quick algorithms to calculate the stiffness and mass matrices in our system. By using the tensor-product structure of the two-dimensional basis and the three-term recurrence relation of Lucas polynomials, the calculation complexity is close to common spectral methods using standard orthogonal polynomials.

Third, we give a careful error analysis that shows our method converges at the best speed. The analysis gives clear limits on the expansion coefficients, truncation errors, and total residual error. It shows that the method reaches spectral accuracy for smooth solutions. The theory follows standard methods, but we pay attention to the specific features of the adjusted Lucas polynomial basis.

Finally, we check our theoretical results with thorough numerical tests. These tests use standard problems with known solutions. This lets us confirm the predicted convergence rates and see how well the method performs in practice, looking at accuracy and calculation speed.

The rest of this paper is set up as follows. Section 2 gives important background, like definitions from functional analysis and key things about standard and shifted Lucas polynomials. Section 3 creates the math system, building our spectral basis and forming the weak Galerkin system. Section 4 details the numerical solution scheme, including how to put together the matrix and carry out the steps. Section 5 gives the theoretical error analysis, setting limits on coefficients, truncation errors, and residuals. Section 6 gives examples that show how well the method works. Section 7 gives final thoughts and possible future research directions.

We hope this study helps spectral methods grow by showing that Lucas polynomials can be good basis functions for elliptic PDEs. We focus on the model Poisson problem in this paper. But, the methods we develop should work for a wider range of problems. We plan to study this in the future.

2. Essential Preliminaries

2.1. Preliminary Definitions and Analysis Foundations

Definition 1 ($L^2(\Omega)$ Space). Space of square-integrable functions:

$$L^2(\Omega) = \{v: \Omega \to \mathbb{R} \mid \| v \|_{L^2(\Omega)} < \infty\}, \quad \| v \|_{L^2(\Omega)} = \left(\iint_\Omega |v|^2 d\Omega\right)^{1/2}. \tag{6}$$

Equipped with inner product $\langle u, v\rangle_{L^2} = \iint_\Omega u\, v d\Omega$.

Definition 2 ($H^1(\Omega)$ Space). Sobolev space of functions with square-integrable first derivatives:

$$H^1(\Omega) = \{v \in L^2(\Omega) \mid \partial_x v, \partial_y v \in L^2(\Omega)\}, \tag{7}$$

where derivatives are weak derivatives.

Definition 3 ($H_0^1(\Omega)$ Space). Subspace of $H^1(\Omega)$ with vanishing trace on $\partial\Omega$:

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = 0\}. \tag{8}$$

2.2. Some Properties of the Standard and Shifted Lucas polynomials

2.2.1. Standard Lucas Polynomials

Definition 4. The Lucas polynomials $\{L_n(x)\}_{n=0}^{\infty}$ are defined in (Bergum and Jr. 1974) by the recurrence:

$$\begin{cases} L_0(x) = 2, \\ L_1(x) = x, \\ L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad n \geq 2. \end{cases} \tag{9}$$

Binet's formula for the Lucas polynomials $L_n(x)$ is given by [22]:

$$L_n(x) = \alpha(x)^n + \beta(x)^n \tag{10}$$

Where:

$$\alpha(x) = \frac{x + \sqrt{x^2 + 4}}{2} \quad \text{and} \quad \beta(x) = \frac{x - \sqrt{x^2 + 4}}{2} \tag{11}$$

and also have the following explicit power form representation [21]:

$$L_n(x) = n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(n-k-1)!}{k!\,(n-2k)!} x^{n-2k}. \tag{12}$$

2.2.2. Shifted Lucas Polynomials to $[0,1]$

Definition 5. The shifted Lucas polynomials $\{L_n^s(x)\}_{n=0}^{\infty}$ are defined by the affine transformation:

$$L_n^s(x) = L_n(2x - 1), \quad x \in [0,1]. \tag{13}$$

With Recurrence Relation:

$$\begin{cases} L_0^s(x) = 2, \\ L_1^s(x) = 2x - 1, \\ L_n^s(x) = (2x - 1)L_{n-1}^s(x) + L_{n-2}^s(x), \quad n \geq 2. \end{cases} \tag{14}$$

3. Mathematical Formulation

We consider the two-dimensional Poisson equation with homogeneous Dirichlet boundary conditions on the unit square $\Omega = (0,1) \times (0,1)$. This classical elliptic boundary value problem seeks a function $u: \Omega \to \mathbb{R}$ satisfying

$$\begin{gathered} -\nabla^2 u(x, y) = f(x, y) \quad \text{in} \quad \Omega, \\ u(x, y) = 0 \quad \text{on} \quad \partial\Omega, \end{gathered} \tag{15}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ denotes the Laplace operator and $f: \Omega \to \mathbb{R}$ is a prescribed source term. The choice of the unit square as our computational domain, while seemingly restrictive, allows us to fully exploit the tensor-product structure of our proposed spectral basis while maintaining sufficient generality for theoretical analysis and numerical validation.

To ensure well-posedness of the problem and optimal convergence of our spectral approximation, we assume that the source term satisfies $f \in L^2(\Omega)$, implying $\| f \|_{L^2(\Omega)} < \infty$. While classical solutions require $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, our variational formulation naturally seeks solutions in the Sobolev space $H_0^1(\Omega)$, which consists of functions in $H^1(\Omega)$ that vanish on the boundary in the trace sense. This weak formulation not only relaxes the regularity requirements but also provides the natural setting for our Galerkin spectral method. In what follows, we develop a spectral approximation scheme based on modified shifted Lucas polynomials that automatically incorporates the boundary conditions.

3.1. Spectral Basis Construction

The cornerstone of our numerical scheme is the construction of appropriate basis functions that combine the favorable properties of Lucas polynomials with automatic satisfaction of the boundary conditions. We begin with the shifted Lucas polynomials $L_k^s(x)$ on $[0,1]$, obtained from the standard Lucas polynomials through the transformation $x \mapsto 2x-1$. Since these polynomials do not inherently vanish at the boundaries, we introduce the modified shifted Lucas polynomials

$$\phi_k(x) = x(1-x)L_k^s(x), \quad k = 0,1,2,\dots \tag{16}$$

The multiplication by the factor $x(1-x)$ ensures that $\phi_k(0) = \phi_k(1) = 0$ for all k, thereby guaranteeing that any linear combination of these functions satisfies the homogeneous Dirichlet boundary conditions. These modified basis functions inherit several desirable properties: they belong to $C^\infty[0,1] \cap H_0^1(0,1)$, form a complete set in $H_0^1(0,1)$, and remain linearly independent since $\deg(\phi_k) = k+2$. Moreover, they satisfy the computationally efficient recurrence relation

$$\phi_k(x) = (2x-1)\phi_{k-1}(x) + \phi_{k-2}(x), \quad k \geq 2, \tag{17}$$

with initial conditions $\phi_0(x) = 2x(1-x)$ and $\phi_1(x) = x(1-x)(2x-1)$.

For the two-dimensional problem, we employ a tensor-product approach, defining the finite-dimensional approximation space

$$V_N = \operatorname{span}\{\Phi_{ij}(x,y) = \phi_i(x)\phi_j(y) : i,j = 0,1,\dots,N\} \subset H_0^1(\Omega). \tag{18}$$

This space has dimension $(N+1)^2$, and each basis function Φ_{ij} automatically vanishes on the entire boundary $\partial\Omega$ due to the properties of the one-dimensional basis functions. The approximate solution is then sought in the form

$$u_N(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\, \phi_i(x)\phi_j(y), \tag{19}$$

where the coefficients $\{c_{ij}\}$ are to be determined through the Galerkin procedure described in the following subsection.

3.2. Weak Formulation and Galerkin Method

To develop our spectral Galerkin scheme, we first derive the weak formulation of the Poisson equation. Rather than working directly with the second-order differential operator in the strong form, we employ Green's identity to reduce the regularity requirements and obtain a symmetric variational problem [25, 26]. This approach not only relaxes the smoothness demands on the solution but also leads to a symmetric stiffness matrix, enabling the use of efficient linear solvers optimized for symmetric positive-definite systems [27, 28]. The variational formulation is fundamental to modern numerical methods for PDEs [29], providing both theoretical insights into well-posedness and practical advantages in implementation [30].

Theorem 1. For $u \in H_0^1(\Omega)$:

$$\int_\Omega \nabla u \cdot \nabla v d\Omega = \int_\Omega f\, v d\Omega \quad \forall v \in H_0^1(\Omega). \tag{20}$$

Proof. Multiply (15) by v:

$$-(\nabla^2 u)v = fv \tag{21}$$

By Integrating over Ω, to get the weak form,

$$-\iint_{\Omega} (\nabla^2 u) v d\Omega = \iint_{\Omega} f\, v d\Omega. \tag{22}$$

From the vector identity:

$$\nabla \cdot (v \nabla u) = \nabla v \cdot \nabla u + v \nabla^2 u. \tag{23}$$

Integrating over Ω:

$$\int_{\Omega} \nabla \cdot (v \nabla u) d\Omega = \int_{\Omega} \nabla v \cdot \nabla u d\Omega + \int_{\Omega} v\, \nabla^2 u d\Omega. \tag{24}$$

Applying the divergence theorem:

$$\int_{\Omega} \nabla \cdot (v \nabla u) d\Omega = \int_{\partial\Omega} (v \nabla u) \cdot \mathbf{n} ds = \int_{\partial\Omega} v\, (\nabla u \cdot \mathbf{n}) ds. \tag{25}$$

where **n** is the outward unit normal vector to $\partial\Omega$.

Rearranging terms:

$$\int_{\Omega} v\, \nabla^2 u d\Omega = \int_{\partial\Omega} v\, (\nabla u \cdot \mathbf{n}) ds - \int_{\Omega} \nabla u \cdot \nabla v d\Omega. \tag{26}$$

By substituting into the weak form:

$$-\left[\iint_{\partial\Omega} v\, (\nabla u \cdot \mathbf{n}) ds - \int_{\Omega} \nabla u \cdot \nabla v d\Omega\right] = \int_{\Omega} f\, v d\Omega. \tag{27}$$

Since $v \in H_0^1(\Omega)$, the trace operator satisfies $\mathrm{Tr}(v) = 0$ on $\partial\Omega$, thus we can impose the boundary conditions:

$$\int_{\partial\Omega} v\, (\nabla u \cdot \mathbf{n}) ds = 0. \tag{28}$$

The equation simplifies to:

$$\int_{\Omega} \nabla u \cdot \nabla v d\Omega = \int_{\Omega} f\, v d\Omega. \tag{29}$$

□

Having established the weak formulation in theorem 1, we now derive the discrete linear system arising from the Galerkin method. The approximate solution $u_N \in V_N$ must satisfy

$$\int_{\Omega} \nabla u_N \cdot \nabla v\, d\Omega = \int_{\Omega} f\, v\, d\Omega \quad \forall v \in V_N. \tag{30}$$

Substituting $u_N = \sum_{i=0}^{N} \sum_{j=0}^{N} c_{ij}\, \Phi_{ij}$ and testing against basis functions Φ_{kl}:

$$\int_{\Omega} \nabla \left(\sum_{i=0}^{N} \sum_{j=0}^{N} c_{ij}\, \Phi_{ij} \right) \cdot \nabla \Phi_{kl} d\Omega = \int_{\Omega} f\, \Phi_{kl} d\Omega \quad \forall k, l = 0, \ldots, N. \tag{31}$$

Becomes:

$$\sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij} \int_{\Omega} \nabla \Phi_{ij} \cdot \nabla\Phi_{kl} d\Omega = \int_{\Omega} f\, \phi_k(x)\phi_l(y) d\Omega \quad \forall k,l = 0,\dots,N. \tag{32}$$

To compute the left-hand side of (32) explicitly, we are going to expand the following:

The gradients are:

$$\begin{aligned} \nabla\Phi_{ij} &= \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x}\left[\phi_i(x)\phi_j(y)\right] \\ \frac{\mathrm{d}}{\mathrm{d}y}\left[\phi_i(x)\phi_j(y)\right] \end{pmatrix} = \begin{pmatrix} \phi'_i(x)\phi_j(y) \\ \phi_i(x)\phi'_j(y) \end{pmatrix}, \\ \nabla\Phi_{kl} &= \begin{pmatrix} \phi_{k'}(x)\phi_l(y) \\ \phi_k(x)\phi_{l'}(y) \end{pmatrix}. \end{aligned} \tag{33}$$

The dot product is:

$$\begin{aligned} \nabla\Phi_{ij} \cdot \nabla\Phi_{kl} &= \left[\phi'_i(x)\phi_j(y)\right]\left[\phi_{k'}(x)\phi_l(y)\right] + \left[\phi_i(x)\phi'_j(y)\right]\left[\phi_k(x)\phi_{l'}(y)\right] \\ &= \phi'_i(x)\phi_{k'}(x)\phi_j(y)\phi_l(y) + \phi_i(x)\phi_k(x)\phi'_j(y)\phi_{l'}(y). \end{aligned} \tag{34}$$

Thus:

$$\int_{\Omega} \nabla \Phi_{ij} \cdot \nabla\Phi_{kl} d\Omega = \int_{\Omega} \left[\phi'_i(x)\phi_{k'}(x)\phi_j(y)\phi_l(y) + \phi_i(x)\phi_k(x)\phi'_j(y)\phi_{l'}(y)\right] \mathrm{d}\Omega \tag{35}$$

In the following steps, we are separating the variables more explicitly, which will be more efficient in forming the matrix system of the problem.
By Fubini's Theorem and the rectangular domain $\Omega = [0,1] \times [0,1]$, from (35):

$$\begin{aligned} \int_{\Omega} \nabla \Phi_{ij} \cdot \nabla\Phi_{kl} d\Omega &= \int_0^1\int_0^1 \phi'_i(x)\phi_{k'}(x)\phi_j(y)\phi_l(y)\mathrm{d}x\mathrm{d}y + \int_0^1\int_0^1 \phi_i(x)\phi_k(x)\phi'_j(y)\phi_{l'}(y)\mathrm{d}x\mathrm{d}y \\ &= \left(\int_0^1 \phi'_i(x)\phi_{k'}(x)dx\right)\left(\int_0^1 \phi_j(y)\phi_l(y)dy\right) \\ &\quad + \left(\int_0^1 \phi_i(x)\phi_k(x)dx\right)\left(\int_0^1 \phi'_j(y)\phi_{l'}(y)dy\right). \end{aligned} \tag{36}$$

Defining the 1D matrices:

$$\begin{aligned} K_{ik} &= \int_0^1 \phi'_i(x)\phi_{k'}(x)dx, \quad \text{we call it stiffness matrix.} \\ M_{ik} &= \int_0^1 \phi_i(x)\phi_k(x)dx, \quad \text{we call it mass matrix.} \end{aligned} \tag{37}$$

The left-hand side simplifies to:

$$\int_{\Omega} \nabla \Phi_{ij} \cdot \nabla\Phi_{kl} d\Omega = K_{ik}M_{jl} + M_{ik}K_{jl} \tag{38}$$

Similarly to the left-hand side expansion, we expand the right-hand side of equation (32) as follows:

$$\int_{\Omega} f\, \Phi_{kl} d\Omega = \int_{\Omega} f(x,y)\Phi_{kl}(x,y)\mathrm{d}\Omega = \int_0^1\int_0^1 f(x,y)\phi_k(x)\phi_l(y)\mathrm{d}x\mathrm{d}y \tag{39}$$

The discrete system

$$\sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\left(K_{ik}M_{jl}+M_{ik}K_{jl}\right)=\int_{\Omega} f\,\Phi_{kl}\,d\Omega,\quad \forall k,l=0,\dots,N. \tag{40}$$

can be expressed as the linear system $\mathbf{Ac}=\mathbf{b}$ through the following constructions. The solution coefficients $\{c_{ij}\}$ are vectorized into $\mathbf{c}\in\mathbb{R}^{(N+1)^2}$ using column-major ordering:

$$\mathbf{c}=[c_{00},c_{10},\dots,c_{N0},c_{01},c_{11},\dots,c_{N1},\dots,c_{0N},\dots,c_{NN}]^T. \tag{41}$$

The system matrix $\mathbf{A}\in\mathbb{R}^{(N+1)^2\times(N+1)^2}$ is given by

$$\mathbf{A}_{(k,l),(i,j)}=K_{ki}M_{lj}+M_{ki}K_{lj}, \tag{42}$$

which, by symmetry of the stiffness matrix K and mass matrix M ($K_{ki}=K_{ik}$, $M_{lj}=M_{jl}$), corresponds to the Kronecker product form

$$\mathbf{A}=M\otimes K+K\otimes M, \tag{43}$$

where $K,M\in\mathbb{R}^{(N+1)\times(N+1)}$.

The right-hand side vector $\mathbf{b}\in\mathbb{R}^{(N+1)^2}$ uses identical column-major ordering with components:

$$\mathbf{b}=\left[\int_{\Omega} f\,\Phi_{00}d\Omega,\ \int_{\Omega} f\,\Phi_{10}d\Omega,\ \dots,\ \int_{\Omega} f\,\Phi_{N0}d\Omega,\ \int_{\Omega} f\,\Phi_{01}d\Omega,\ \dots,\ \int_{\Omega} f\,\Phi_{NN}d\Omega\right]^T. \tag{44}$$

4. Numerical Solution Scheme

Lemma 1. The basis functions $\phi_k(x)=x(1-x)L_k^s(x)$ satisfy:

1. $\phi_k(x)$ is symmetric about $x=1/2$ if k is even: $\phi_k(x)=\phi_k(1-x)$
2. $\phi_k(x)$ is antisymmetric about $x=1/2$ if k is odd: $\phi_k(x)=-\phi_k(1-x)$
3. $\phi_k(1/2)=0$ if k is odd, and $\phi_k(1/2)=1/2$ if k is even

Proof. We establish the symmetry properties of $L_k^s(x)$, then extend these to $\phi_k(x)$.

For the Symmetry of $L_k^s(x)$ about $x=1/2$:

We prove by induction that for all $k\in\mathbb{N}_0$:

$$L_k^s(1-x)=\begin{cases}L_k^s(x) & \text{if } k \text{ is even}\\ -L_k^s(x) & \text{if } k \text{ is odd}\end{cases} \tag{45}$$

For the base cases, we have $L_0^s(x)=2$, which gives

$$L_0^s(1-x)=2=L_0^s(x) \tag{46}$$

and $L_1^s(x)=2x-1$, which gives

$$L_1^s(1-x)=2(1-x)-1=1-2x=-(2x-1)=-L_1^s(x) \tag{47}$$

Assume the property holds for all $m<k$ where $k\geq 2$. The recurrence relation $L_k^s(x)=(2x-1)L_{k-1}^s(x)+L_{k-2}^s(x)$ evaluated at $1-x$ yields

$$L_k^s(1-x) = (2(1-x)-1)L_{k-1}^s(1-x) + L_{k-2}^s(1-x) = (1-2x)L_{k-1}^s(1-x) + L_{k-2}^s(1-x) \tag{48}$$

When k is even, $k-1$ is odd and $k-2$ is even. By the inductive hypothesis, $L_{k-1}^s(1-x) = -L_{k-1}^s(x)$ and $L_{k-2}^s(1-x) = L_{k-2}^s(x)$. Therefore

$$\begin{aligned} L_k^s(1-x) &= (1-2x)(-L_{k-1}^s(x)) + L_{k-2}^s(x) \\ &= (2x-1)L_{k-1}^s(x) + L_{k-2}^s(x) \\ &= L_k^s(x) \end{aligned} \tag{49}$$

When k is odd, $k-1$ is even and $k-2$ is odd. By the inductive hypothesis,

$L_{k-1}^s(1-x) = L_{k-1}^s(x)$ and $L_{k-2}^s(1-x) = -L_{k-2}^s(x)$. Therefore

$$\begin{aligned} L_k^s(1-x) &= (1-2x)L_{k-1}^s(x) + (-L_{k-2}^s(x)) \\ &= -[(2x-1)L_{k-1}^s(x) + L_{k-2}^s(x)] \\ &= -L_k^s(x) \end{aligned} \tag{50}$$

For the Symmetry of $\phi_k(x)$ about $x = 1/2$:

The factor $g(x) = x(1-x)$ satisfies

$$g(1-x) = (1-x)[1-(1-x)] = (1-x)x = x(1-x) = g(x) \tag{51}$$

Hence $g(x)$ is symmetric about $x = 1/2$. For the basis functions, we have

$$\phi_k(1-x) = (1-x)xL_k^s(1-x) = x(1-x)L_k^s(1-x) \tag{52}$$

When k is even, $L_k^s(1-x) = L_k^s(x)$, yielding

$$\phi_k(1-x) = x(1-x)L_k^s(x) = \phi_k(x) \tag{53}$$

When k is odd, $L_k^s(1-x) = -L_k^s(x)$, yielding

$$\phi_k(1-x) = x(1-x)(-L_k^s(x)) = -\phi_k(x) \tag{54}$$

This establishes parts (i) and (ii) of the lemma.

For the Values at $x = 1/2$:

We prove by induction that for all $k \in \mathbb{N}_0$:

$$L_k^s\left(\frac{1}{2}\right) = \begin{cases} 2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases} \tag{55}$$

For the base cases, $L_0^s(1/2) = 2$ and $L_1^s(1/2) = 2 \cdot (1/2) - 1 = 0$.

Assume the property holds for all $m < k$ where $k \geq 2$. Evaluating the recurrence relation at $x = 1/2$:

$$L_k^s\left(\frac{1}{2}\right) = \left(2 \cdot \frac{1}{2} - 1\right) L_{k-1}^s\left(\frac{1}{2}\right) + L_{k-2}^s\left(\frac{1}{2}\right) = 0 \cdot L_{k-1}^s\left(\frac{1}{2}\right) + L_{k-2}^s\left(\frac{1}{2}\right) = L_{k-2}^s\left(\frac{1}{2}\right) \tag{56}$$

When k is even, $k-2$ is even, so $L_{k-2}^s(1/2) = 2$ by the inductive hypothesis. When k is odd, $k-2$ is odd, so $L_{k-2}^s(1/2) = 0$ by the inductive hypothesis.

For the basis functions at $x = 1/2$:

$$\phi_k\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \frac{1}{2} \cdot L_k^s\left(\frac{1}{2}\right) = \frac{1}{4} L_k^s\left(\frac{1}{2}\right) \tag{57}$$

When k is even, $\phi_k(1/2) = (1/4) \cdot 2 = 1/2$. When k is odd, $\phi_k(1/2) = (1/4) \cdot 0 = 0$.

This establishes part (iii) of the lemma, completing the proof. □

Theorem 2. For the mass matrix $M_{ij} = \int_0^1 \phi_i(x)\phi_j(x)dx$ and stiffness matrix $K_{ij} = \int_0^1 \phi'_i(x)\phi'_j(x)dx$ with boundary-adjusted basis functions (modified shifted Lucas polynomials) $\phi_k(x) = x(1-x)L_k^S(x)$, we have:

$$M_{ij} = 0 \quad \text{and} \quad K_{ij} = 0 \quad \text{whenever} \quad i+j \quad \text{is odd.} \tag{58}$$

Proof. The proof consists of three parts:

1. Establishing symmetry properties of the derivatives.
2. proving antisymmetry of the integrands when $i + j$ is odd.
3. showing the integrals vanish over the symmetric interval [0,1].

Part 1: Symmetry Properties of $\phi_{k'}(x)$

Proposition 1. The derivatives satisfy:

1. For even k: $\phi_{k'}(x)$ is antisymmetric about $x = \frac{1}{2}$: $\phi_{k'}(1-x) = -\phi_{k'}(x)$
2. For odd k: $\phi_{k'}(x)$ is symmetric about $x = \frac{1}{2}$: $\phi_{k'}(1-x) = \phi_{k'}(x)$

Proof. Differentiate the symmetry relation of $\phi_k(x)$ with respect to x:

$$\frac{d}{dx}[\phi_k(1-x)] = \frac{d}{dx}[\sigma_k\phi_k(x)] \tag{59}$$

where $\sigma_k = 1$ for even k and $\sigma_k = -1$ for odd k.

Applying the chain rule to the left side:

$$\phi_{k'}(1-x)\cdot(-1) = \sigma_k\phi_{k'}(x) \tag{60}$$

Case 1: k even ($\sigma_k = 1$)

$$-\phi_{k'}(1-x) = \phi_{k'}(x) \Longrightarrow \phi_{k'}(1-x) = -\phi_{k'}(x) \tag{61}$$

Case 2: k odd ($\sigma_k = -1$)

$$-\phi_{k'}(1-x) = -\phi_{k'}(x) \Longrightarrow \phi_{k'}(1-x) = \phi_{k'}(x) \tag{62}$$

Thus, $\phi_{k'}(x)$ inherits the complementary symmetry to $\phi_k(x)$. □

Part 2: Antisymmetry of Integrands

Proposition 2. When $i + j$ is odd, the product $\phi_i(x)\phi_j(x)$ is antisymmetric about $x = \frac{1}{2}$:

$$[\phi_i\phi_j](1-x) = -\phi_i(x)\phi_j(x) \tag{63}$$

Proof. Without loss of generality, assume i even and j odd (since $i + j$ odd implies opposite parities). From the basis function symmetries:

$$\phi_i(1-x) = \phi_i(x), \quad \phi_j(1-x) = -\phi_j(x) \tag{64}$$

Thus:

$$[\phi_i\phi_j](1-x) = \phi_i(1-x)\phi_j(1-x) = \phi_i(x)\cdot(-\phi_j(x)) = -\phi_i(x)\phi_j(x) \tag{65}$$

The case i odd, j even follows identically by commutativity. □

Proposition 3. When $i+j$ is odd, the product $\phi'_i(x)\phi'_j(x)$ is antisymmetric about $x=\frac{1}{2}$:

$$[\phi'_i\phi'_j](1-x) = -\phi'_i(x)\phi'_j(x) \tag{66}$$

Proof. Assume i even, j odd. From Proposition 1 :

$$\phi'_i(1-x) = -\phi'_i(x), \quad \phi'_j(1-x) = \phi'_j(x) \tag{67}$$

Thus:

$$[\phi'_i\phi'_j](1-x) = \phi'_i(1-x)\phi'_j(1-x) = (-\phi'_i(x))\cdot\phi'_j(x) = -\phi'_i(x)\phi'_j(x) \tag{68}$$

The case i odd, j even follows similarly. □

Part 3: Vanishing of the Integrals

Proposition 4. For any function $h(x)$ antisymmetric about $x=\frac{1}{2}$ (i.e., $h(1-x)=-h(x)$),

$$\int_0^1 h\,(x)dx = 0. \tag{69}$$

Proof. Decompose the integral and apply the substitution $x=1-u$ in the second term:

$$\begin{aligned}\int_0^1 h\,(x)dx &= \int_0^{1/2} h\,(x)dx + \int_{1/2}^1 h\,(x)dx \\ &= \int_0^{1/2} h\,(x)dx + \int_{1/2}^0 h\,(1-u)(-du) \\ &= \int_0^{1/2} h\,(x)dx + \int_0^{1/2} h\,(1-u)du.\end{aligned} \tag{70}$$

Using antisymmetry $h(1-u)=-h(u)$:

$$\begin{aligned}\int_0^{1/2} h\,(1-u)du &= \int_0^{1/2} -h(u)du \\ &= -\int_0^{1/2} h\,(u)du.\end{aligned} \tag{71}$$

Thus:

$$\begin{aligned}\int_0^1 h\,(x)dx &= \int_0^{1/2} h\,(x)dx + \left(-\int_0^{1/2} h\,(u)du\right) \\ &= \int_0^{1/2} h\,(x)dx - \int_0^{1/2} h\,(x)dx \\ &= 0.\end{aligned} \tag{72}$$

□

From parts 1, 2 and 3, we finally conclude that when $i+j$ is odd: $\phi_i(x)\phi_j(x)$ is antisymmetric (Propostion 2) and $\phi'_i(x)\phi'_j(x)$ is antisymmetric (Proposition 3).

Thus, from proposition 4,

$$M_{ij} = \int_0^1 \phi_i \phi_j dx = 0 \quad \text{and} \quad K_{ij} = \int_0^1 \phi'_i \phi'_j dx = 0 \tag{73}$$

This completes the proof. □

Theorem 3. For all non-negative integers i and j, and for all $t \in \mathbb{R}$, the Lucas polynomials $L_n(t)$ satisfy:

$$L_i(t)L_j(t) = L_{i+j}(t) + (-1)^{\min(i,j)} L_{|i-j|}(t). \tag{74}$$

Proof. To establish this identity, we utilize Binet's closed-form formula for the Lucas polynomials. The Lucas polynomials admit the representation.

$$L_n(t) = \alpha^n + \beta^n, \tag{75}$$

where α and β are the roots of the characteristic equation $r^2 - tr - 1 = 0$, explicitly given by

$$\alpha = \frac{t + \sqrt{t^2 + 4}}{2}, \quad \beta = \frac{t - \sqrt{t^2 + 4}}{2}. \tag{76}$$

These roots satisfy the fundamental relations.

$$\alpha + \beta = t, \quad \alpha\beta = -1, \quad \text{and} \quad \alpha - \beta = \sqrt{t^2 + 4}. \tag{77}$$

From $\alpha\beta = -1$, it follows that $\beta = -\alpha^{-1}$ and $\alpha = -\beta^{-1}$.

Consider the product $L_i(t)L_j(t)$. Applying Binet's formula, we have

$$L_i(t)L_j(t) = (\alpha^i + \beta^i)(\alpha^j + \beta^j). \tag{78}$$

Expanding the right-hand side yields

$$L_i(t)L_j(t) = \alpha^i\alpha^j + \alpha^i\beta^j + \beta^i\alpha^j + \beta^i\beta^j = \alpha^{i+j} + \beta^{i+j} + \alpha^i\beta^j + \beta^i\alpha^j. \tag{79}$$

We now simplify the cross terms $\alpha^i\beta^j$ and $\beta^i\alpha^j$ using the relation $\beta = -\alpha^{-1}$. First,

$$\alpha^i\beta^j = \alpha^i(-\alpha^{-1})^j = (-1)^j\alpha^i\alpha^{-j} = (-1)^j\alpha^{i-j}. \tag{80}$$

Similarly,

$$\beta^i\alpha^j = (-\alpha^{-1})^i\alpha^j = (-1)^i\alpha^{-i}\alpha^j = (-1)^i\alpha^{j-i}. \tag{81}$$

To express both cross terms consistently, we observe that

$$(-1)^i\alpha^{j-i} = (-1)^i[(-1)^{j-i}\beta^{i-j}] = (-1)^{i+j-i}\beta^{i-j} = (-1)^j\beta^{i-j}. \tag{82}$$

Thus, the cross terms can be written uniformly as

$$\alpha^i\beta^j = (-1)^j\alpha^{i-j}, \quad \beta^i\alpha^j = (-1)^j\beta^{i-j}. \tag{83}$$

The sum of the cross terms is therefore

$$\alpha^i\beta^j + \beta^i\alpha^j = (-1)^j\alpha^{i-j} + (-1)^j\beta^{i-j} = (-1)^j(\alpha^{i-j} + \beta^{i-j}). \tag{84}$$

Substituting this back into the expression for the product gives

$$L_i(t)L_j(t) = \alpha^{i+j} + \beta^{i+j} + (-1)^j(\alpha^{i-j} + \beta^{i-j}). \tag{85}$$

By Binet's formula, $\alpha^{i+j} + \beta^{i+j} = L_{i+j}(t)$. The term $\alpha^{i-j} + \beta^{i-j}$ requires careful handling due to the exponent $i - j$, which may be negative. We analyze this by considering two cases based on the relative values of i and j.

- **When i ≥ j:** Here, $i - j \geq 0$, so $\alpha^{i-j} + \beta^{i-j} = L_{i-j}(t)$. Since $|i - j| = i - j$ and $\min(i,j) = j$, we have

$$(-1)^j(\alpha^{i-j} + \beta^{i-j}) = (-1)^j L_{i-j}(t) = (-1)^{\min(i,j)} L_{|i-j|}(t). \tag{86}$$

- **When i < j:** Here, $i - j < 0$, and we use the property that for negative integers k, $\alpha^k + \beta^k = (-1)^k L_{-k}(t)$, which follows from $\alpha\beta = -1$ and Binet's formula applied to positive indices. Specifically, setting $k = i - j$, so $k < 0$, and letting $m = -k = j - i > 0$, we have

$$\begin{aligned} \alpha^{i-j} + \beta^{i-j} &= \alpha^{-m} + \beta^{-m} = (-\beta)^m + (-\alpha)^m = (-1)^m(\beta^m + \alpha^m) = (-1)^m(\alpha^m + \beta^m) \\ &= (-1)^m L_m(t), \end{aligned} \tag{87}$$

where $m = j - i$. Since $|i - j| = j - i = m$, this becomes

$$\alpha^{i-j} + \beta^{i-j} = (-1)^{|i-j|} L_{|i-j|}(t). \tag{88}$$

Thus, the cross-term contribution is

$$(-1)^j(\alpha^{i-j} + \beta^{i-j}) = (-1)^j(-1)^{|i-j|} L_{|i-j|}(t) = (-1)^j(-1)^{j-i} L_{|i-j|}(t). \tag{89}$$

Simplifying the exponent,

$$(-1)^{j+j-i} = (-1)^{2j-i} = (-1)^{-i} = (-1)^i, \tag{90}$$

because $(-1)^{2j} = 1$ and $(-1)^{-i} = (-1)^i$ (as $(-1)^i$ is its own inverse). Since $\min(i,j) = i$ when $i < j$, we conclude

$$(-1)^i L_{|i-j|}(t) = (-1)^{\min(i,j)} L_{|i-j|}(t). \tag{91}$$

In both cases, the expression simplifies to

$$(-1)^j(\alpha^{i-j} + \beta^{i-j}) = (-1)^{\min(i,j)} L_{|i-j|}(t). \tag{92}$$

Therefore,

$$L_i(t)L_j(t) = L_{i+j}(t) + (-1)^{\min(i,j)} L_{|i-j|}(t), \tag{93}$$

Completing the proof. □

Theorem 4. For non-negative integers i and j such that $i + j$ is even, the integral of the product of shifted Lucas polynomials $L_n^s(x)$against the weight $[x(1-x)]^2$ over $[0,1]$ is given by:

$$\begin{aligned} \int_0^1 L_i^s(x)L_j^s(x)[x(1-x)]^2 dx &= \frac{1}{16}\sum_{k=0}^{\lfloor\frac{i+j}{2}\rfloor} c_{i+j,k}\left(\frac{1}{i+j-2k+1} - \frac{2}{i+j-2k+3} + \frac{1}{i+j-2k+5}\right) \\ &+ \frac{(-1)^{\min(i,j)}}{16}\sum_{k=0}^{\lfloor\frac{|i-j|}{2}\rfloor} c_{|i-j|,k}\left(\frac{1}{|i-j|-2k+1} - \frac{2}{|i-j|-2k+3} + \frac{1}{|i-j|-2k+5}\right), \end{aligned} \tag{94}$$

where the coefficients $c_{n,k}$ are defined as:

$$c_{n,k} = \begin{cases} 2 & \text{if } n = 0 \text{ and } k = 0, \\ \dfrac{n}{n-k}\dbinom{n-k}{k} & \text{otherwise.} \end{cases} \tag{95}$$

Proof. Applying the substitution $t = 2x - 1$, which implies:

$$x = \frac{t+1}{2}, \quad dx = \frac{dt}{2}. \tag{96}$$

The limits $x \in [0,1]$ transform to $t \in [-1,1]$. The weight function simplifies to:

$$\begin{gathered} x(1-x) = \frac{t+1}{2} \cdot \frac{1-t}{2} = \frac{1-t^2}{4}, \\ [x(1-x)]^2 = \left(\frac{1-t^2}{4}\right)^2 = \frac{(1-t^2)^2}{16}. \end{gathered} \tag{97}$$

The integral transforms to:

$$\int_0^1 L_i^s(x) L_j^s(x) [x(1-x)]^2 dx = \frac{1}{32} \int_{-1}^1 L_i(t) L_j(t) (1-t^2)^2 dt. \tag{98}$$

By Theorem 3:

$$L_i(t) L_j(t) = L_{i+j}(t) + (-1)^{\min(i,j)} L_{|i-j|}(t). \tag{99}$$

Substitute into (98):

$$\begin{aligned} &\frac{1}{32} \int_{-1}^1 \left[L_{i+j}(t) + (-1)^{\min(i,j)} L_{|i-j|}(t)\right](1-t^2)^2 dt \\ &\quad = \frac{1}{32}\left[\int_{-1}^1 L_{i+j}(t)(1-t^2)^2 dt + (-1)^{\min(i,j)} \int_{-1}^1 L_{|i-j|}(t)(1-t^2)^2 dt\right]. \end{aligned} \tag{100}$$

We define the auxiliary integral for even n (since $i+j$ even implies $i+j$ and $|i-j|$ are even):

$$\mathcal{I}(n) = \int_{-1}^1 L_n(t)(1-t^2)^2 dt, \quad n \in 2\mathbb{Z}_{\geq 0}. \tag{101}$$

Expressing $L_n(t)$ via its explicit form:

$$L_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,k}\, t^{n-2k}, \quad c_{n,k} = \begin{cases} 2 & n=0,\ k=0 \\ \dfrac{n}{n-k}\dbinom{n-k}{k} & \text{otherwise} \end{cases}. \tag{102}$$

Thus:

$$\mathcal{I}(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,k} \int_{-1}^1 t^{n-2k}(1-t^2)^2 dt. \tag{103}$$

Expanding $(1-t^2)^2 = 1 - 2t^2 + t^4$. Since n is even and $n-2k$ is even for all k, the integrand $t^{n-2k}(1-t^2)^2$ is even. Therefore:

$$\int_{-1}^1 t^m (1-2t^2+t^4) dt = 2\int_0^1 t^m (1-2t^2+t^4) dt, \quad m = n-2k. \tag{104}$$

The antiderivative is:

$$\int t^m (1-2t^2+t^4) dt = \frac{t^{m+1}}{m+1} - \frac{2t^{m+3}}{m+3} + \frac{t^{m+5}}{m+5}. \tag{105}$$

Evaluating from 0 to 1 gives:

$$2\left[\frac{1}{m+1}-\frac{2}{m+3}+\frac{1}{m+5}\right]. \tag{106}$$

Substituting $m = n - 2k$ into (103):

$$\mathcal{J}(n) = 2\sum_{k=0}^{\lfloor n/2\rfloor} c_{n,k}\left(\frac{1}{n-2k+1}-\frac{2}{n-2k+3}+\frac{1}{n-2k+5}\right). \tag{107}$$

Substituting (107) into the product identity:

$$\begin{aligned}
&\int_0^1 L_i^s(x)L_j^s(x)[x(1-x)]^2dx \\
&= \frac{1}{32}\left[\mathcal{J}(i+j)+(-1)^{\min(i,j)}\mathcal{J}(|i-j|)\right] \\
&= \frac{1}{32}[2\sum_{k=0}^{\lfloor\frac{i+j}{2}\rfloor} c_{i+j,k}\left(\frac{1}{i+j-2k+1}-\frac{2}{i+j-2k+3}+\frac{1}{i+j-2k+5}\right) \\
&+(-1)^{\min(i,j)}\cdot 2\sum_{k=0}^{\lfloor\frac{|i-j|}{2}\rfloor} c_{|i-j|,k}\left(\frac{1}{|i-j|-2k+1}-\frac{2}{|i-j|-2k+3}+\frac{1}{|i-j|-2k+5}\right)] \\
&= \frac{1}{16}\sum_{k=0}^{\lfloor\frac{i+j}{2}\rfloor} c_{i+j,k}\left(\frac{1}{i+j-2k+1}-\frac{2}{i+j-2k+3}+\frac{1}{i+j-2k+5}\right) \\
&+\frac{(-1)^{\min(i,j)}}{16}\sum_{k=0}^{\lfloor\frac{|i-j|}{2}\rfloor} c_{|i-j|,k}\left(\frac{1}{|i-j|-2k+1}-\frac{2}{|i-j|-2k+3}+\frac{1}{|i-j|-2k+5}\right).
\end{aligned} \tag{108}$$

Where,

$$c_{n,k} = \begin{cases} 2 & \text{if } n=0 \text{ and } k=0, \\ \dfrac{n}{n-k}\dbinom{n-k}{k} & \text{otherwise.} \end{cases} \tag{109}$$

The binomial coefficient $\binom{n-k}{k}$ is well-defined for $0 \le k \le \lfloor n/2 \rfloor$, ensuring all terms are valid. This completes the proof. □

Theorem 5. Let $L_n^s(x) = L_n(2x-1)$ denote the shifted Lucas polynomial, where $L_n(u)$ is the n-th Lucas polynomial. For non-negative integers $i,j \ge 0$ with $i+j$ even, the integral

$$I_{i,j} = \int_0^1 \left[(1-2x)L_i^s(x)+x(1-x)\frac{dL_i^s}{dx}\right]\left[(1-2x)L_j^s(x)+x(1-x)\frac{dL_j^s}{dx}\right]dx \tag{110}$$

admits the explicit algebraic expression:

$$I_{i,j} = \frac{1}{8}\sum_{p=0}^{i+1}\sum_{q=0}^{j+1} b_p^{(i)}\, b_q^{(j)}\,\frac{1+(-1)^{p+q}}{p+q+1}, \tag{111}$$

where the coefficients $b_k^{(n)}$ for $n \ge 0$, $0 \le k \le n+1$ are defined by:

$$b_k^{(n)} = \begin{cases} \dfrac{2n}{n+1}\dbinom{\frac{n+1}{2}}{\frac{n-1}{2}} & \text{if } k=0 \text{ and } n \text{ is odd,} \\ 2\left(\dfrac{2n}{n+2}\dbinom{\frac{n}{2}+1}{\frac{n}{2}-1} - 2\right) & \text{if } k=1 \text{ and } n \text{ is even,} \\ (k+1)\left(\dfrac{2n}{n+k+1}\dbinom{\frac{n+k+1}{2}}{\frac{n-k-1}{2}} - \dfrac{2n}{n+k-1}\dbinom{\frac{n+k-1}{2}}{\frac{n-k+1}{2}}\right) & \text{if } k\geq 2 \text{ and } n-k \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{112}$$

and the binomial coefficients are zero if arguments are non-integer, the lower index is negative, or the lower index exceeds the upper index.

Proof. Apply the substitution $u = 2x - 1$, mapping $x \in [0,1]$ to $u \in [-1,1]$. This yields:

$$x = \frac{u+1}{2}, \quad dx = \frac{du}{2}, \quad 1-2x = -u, \quad x(1-x) = \frac{1-u^2}{4}. \tag{113}$$

The shifted Lucas polynomial and its derivative transform as:

$$L_i^s(x) = L_i(u), \quad \frac{dL_i^s}{dx} = \frac{dL_i}{du}\cdot\frac{du}{dx} = 2\frac{dL_i}{du}. \tag{114}$$

Define the kernel for index i:

$$A_i(x) = (1-2x)L_i^s(x) + x(1-x)\frac{dL_i^s}{dx} = -uL_i(u) + \frac{1-u^2}{4}\cdot 2\frac{dL_i}{du} = -uL_i(u) + \frac{1-u^2}{2}\frac{dL_i}{du}. \tag{115}$$

The integrand becomes $A_i(x)A_j(x)$, and the integral transforms to:

$$I_{i,j} = \int_{-1}^{1}\left[-uL_i(u) + \frac{1-u^2}{2}\frac{dL_i}{du}\right]\left[-uL_j(u) + \frac{1-u^2}{2}\frac{dL_j}{du}\right]\frac{du}{2}. \tag{116}$$

Defining the auxiliary polynomial:

$$D_n(u) = (1-u^2)\frac{dL_n}{du} - 2uL_n(u). \tag{117}$$

We observe that:

$$-uL_n(u) + \frac{1-u^2}{2}\frac{dL_n}{du} = \frac{1}{2}\left[(1-u^2)\frac{dL_n}{du} - 2uL_n(u)\right] = \frac{1}{2}D_n(u). \tag{118}$$

Thus, the integrand simplifies to:

$$A_i(x)A_j(x) = \left(\frac{1}{2}D_i(u)\right)\left(\frac{1}{2}D_j(u)\right) = \frac{1}{4}D_i(u)D_j(u). \tag{119}$$

The integral is now:

$$I_{i,j} = \int_{-1}^{1}\frac{1}{4}D_i(u)D_j(u)\cdot\frac{du}{2} = \frac{1}{8}\int_{-1}^{1}D_i(u)D_j(u)du. \tag{120}$$

The Lucas polynomial $L_n(u)$ has degree n and expansion:

$$L_n(u) = \sum_{m=0}^{n} c_m^{(n)} u^m, \quad c_m^{(n)} = \begin{cases} \dfrac{n}{n-\ell}\dbinom{n-\ell}{\ell} & \text{if } m = n-2\ell, \quad \ell = 0,1,\dots,\left\lfloor \dfrac{n}{2} \right\rfloor, \\ 0 & \text{otherwise.} \end{cases} \tag{121}$$

Its derivative is:

$$\frac{dL_n}{du} = \sum_{m=1}^{n} m\, c_m^{(n)} u^{m-1} = \sum_{m=0}^{n-1} (m+1) c_{m+1}^{(n)} u^m. \tag{122}$$

By Substituting into $D_n(u)$:

$$D_n(u) = (1-u^2) \sum_{m=0}^{n-1} (m+1) c_{m+1}^{(n)} u^m - 2u \sum_{m=0}^{n} c_m^{(n)} u^m. \tag{123}$$

Then, by expanding both terms:

$$\begin{aligned} (1-u^2) \sum_{m=0}^{n-1} (m+1) c_{m+1}^{(n)} u^m &= \sum_{m=0}^{n-1} (m+1) c_{m+1}^{(n)} u^m - \sum_{m=0}^{n-1} (m+1) c_{m+1}^{(n)} u^{m+2} \\ &= \sum_{k=0}^{n-1} (k+1) c_{k+1}^{(n)} u^k - \sum_{k=2}^{n+1} (k-1) c_{k-1}^{(n)} u^k, \\ -2u \sum_{m=0}^{n} c_m^{(n)} u^m &= \sum_{k=1}^{n+1} -2 c_{k-1}^{(n)} u^k. \end{aligned} \tag{124}$$

After, Combining and extracting coefficients of u^k, we get:

- For $k = 0$: $1 \cdot c_1^{(n)} = c_1^{(n)}$
- For $k = 1$: $2c_2^{(n)} - 2c_0^{(n)}$
- For $k \geq 2$: $(k+1)c_{k+1}^{(n)} - (k-1)c_{k-1}^{(n)} - 2c_{k-1}^{(n)} = (k+1)\left(c_{k+1}^{(n)} - c_{k-1}^{(n)}\right)$

Thus, $D_n(u) = \sum_{k=0}^{n+1} b_k^{(n)} u^k$ with:

$$b_k^{(n)} = \begin{cases} c_1^{(n)} & k = 0, \\ 2c_2^{(n)} - 2c_0^{(n)} & k = 1, \\ (k+1)\left(c_{k+1}^{(n)} - c_{k-1}^{(n)}\right) & k \geq 2. \end{cases} \tag{125}$$

In the following, we derive closed forms for $b_k^{(n)}$ using the expression for $c_m^{(n)}$;
Case 1: $k = 0$ $b_0^{(n)} = c_1^{(n)}$ is non-zero only if n odd. Set $m = 1 = n - 2\ell$, so $\ell = (n-1)/2$:

$$c_1^{(n)} = \frac{n}{n - \frac{n-1}{2}} \binom{n - \frac{n-1}{2}}{\frac{n-1}{2}} = \frac{n}{\frac{n+1}{2}} \binom{\frac{n+1}{2}}{\frac{n-1}{2}} = \frac{2n}{n+1} \binom{\frac{n+1}{2}}{\frac{n-1}{2}}. \tag{126}$$

Thus:

$$b_0^{(n)} = \begin{cases} \dfrac{2n}{n+1}\dbinom{\frac{n+1}{2}}{\frac{n-1}{2}} & n \text{ odd}, \\ 0 & \text{otherwise}. \end{cases} \tag{127}$$

Case 2: $k = 1$ $b_1^{(n)} = 2c_2^{(n)} - 2c_0^{(n)}$ is non-zero only if n even.

1. For $c_0^{(n)}$: set $m = 0 = n - 2\ell$, so $\ell = n/2$:

$$c_0^{(n)} = \frac{n}{n-\frac{n}{2}}\binom{n-\frac{n}{2}}{\frac{n}{2}} = \frac{n}{\frac{n}{2}}\binom{\frac{n}{2}}{\frac{n}{2}} = 2 \cdot 1 = 2. \tag{128}$$

2. For $c_2^{(n)}$: set $m = 2 = n - 2\ell$, so $\ell = (n-2)/2$:

$$c_2^{(n)} = \frac{n}{n-\frac{n-2}{2}}\binom{n-\frac{n-2}{2}}{\frac{n-2}{2}} = \frac{n}{\frac{n+2}{2}}\binom{\frac{n+2}{2}}{\frac{n-2}{2}} = \frac{2n}{n+2}\binom{\frac{n}{2}+1}{\frac{n}{2}-1}. \tag{129}$$

Thus:

$$b_1^{(n)} = 2\left(\frac{2n}{n+2}\binom{\frac{n}{2}+1}{\frac{n}{2}-1}\right) - 2 \cdot 2 = 2\left(\frac{2n}{n+2}\binom{\frac{n}{2}+1}{\frac{n}{2}-1} - 2\right). \tag{130}$$

Case 3: $k \geq 2$ $b_k^{(n)} = (k+1)\left(c_{k+1}^{(n)} - c_{k-1}^{(n)}\right)$ is non-zero only if $n-k$ odd (ensuring $c_{k+1}^{(n)}, c_{k-1}^{(n)}$ exist).

1. For $c_{k+1}^{(n)}$: set $m = k+1 = n - 2\ell_1$, so $\ell_1 = (n-k-1)/2$:

$$c_{k+1}^{(n)} = \frac{n}{n-\frac{n-k-1}{2}}\binom{n-\frac{n-k-1}{2}}{\frac{n-k-1}{2}} = \frac{2n}{n+k+1}\binom{\frac{n+k+1}{2}}{\frac{n-k-1}{2}}. \tag{131}$$

2. For $c_{k-1}^{(n)}$: set $m = k-1 = n - 2\ell_2$, so $\ell_2 = (n-k+1)/2$:

$$c_{k-1}^{(n)} = \frac{n}{n-\frac{n-k+1}{2}}\binom{n-\frac{n-k+1}{2}}{\frac{n-k+1}{2}} = \frac{2n}{n+k-1}\binom{\frac{n+k-1}{2}}{\frac{n-k+1}{2}}. \tag{132}$$

Thus:

$$b_k^{(n)} = (k+1)\left(\frac{2n}{n+k+1}\binom{\frac{n+k+1}{2}}{\frac{n-k-1}{2}} - \frac{2n}{n+k-1}\binom{\frac{n+k-1}{2}}{\frac{n-k+1}{2}}\right). \tag{133}$$

In the following step, we evaluate the integral in its final form:
Expanding $D_i(u)D_j(u)$:

$$\int_{-1}^{1} D_i(u)D_j(u)du = \sum_{p=0}^{i+1}\sum_{q=0}^{j+1} b_p^{(i)} b_q^{(j)} \int_{-1}^{1} u^{p+q}\, du. \tag{134}$$

Evaluating the monomial integral:

$$\int_{-1}^{1} u^m \, du = \left[\frac{u^{m+1}}{m+1}\right]_{-1}^{1} = \frac{1-(-1)^{m+1}}{m+1} = \frac{1+(-1)^m}{m+1}. \tag{135}$$

Setting $m = p + q$:

$$\int_{-1}^{1} u^{p+q} \, du = \frac{1+(-1)^{p+q}}{p+q+1}. \tag{136}$$

Thus:

$$I_{i,j} = \frac{1}{8}\sum_{p=0}^{i+1}\sum_{q=0}^{j+1} b_p^{(i)} \, b_q^{(j)} \, \frac{1+(-1)^{p+q}}{p+q+1}. \tag{137}$$

The term $\frac{1+(-1)^{p+q}}{p+q+1}$ vanishes when $p+q$ is odd and equals $\frac{2}{p+q+1}$ when even. The condition $i+j$ even ensures consistency with the original integral's symmetry. Binomial coefficients are zero for invalid indices, confirming all terms are well-defined.
This completes the derivation. □

Remark 1. The formula is computationally efficient, requiring only evaluations of binomial coefficients and a double sum over bounded indices. Symmetry $I_{i,j} = I_{j,i}$ is evident from the expression.

Remark 2. The formula (111) and is consistent with the result we got before in Theorem 2, when the parities $i+j$ is odd the integral vanishes.

Corollary 1. Based on theorem 5, the integral is expressed as a double sum depending only on i and j, with all coefficients defined explicitly as follows:

$$I_{i,j} = \frac{1}{8}\sum_{p=0}^{i+1}\sum_{q=0}^{j+1} \left(\begin{cases} \frac{2i}{i+1}\binom{\frac{i+1}{2}}{\frac{i-1}{2}} & \text{if } p = 0 \text{ and } i \text{ odd} \\ 2\left(\frac{2i}{i+2}\binom{\frac{i}{2}+1}{\frac{i}{2}-1} - 2\right) & \text{if } p = 1 \text{ and } i \text{ even} \\ (p+1)\left(\frac{2i}{i+p+1}\binom{\frac{i+p+1}{2}}{\frac{i-p-1}{2}} - \frac{2i}{i+p-1}\binom{\frac{i+p-1}{2}}{\frac{i-p+1}{2}}\right) & \text{if } p \geq 2 \text{ and } i - p \text{ o} \\ 0 & \text{otherwise} \end{cases} \right)$$
$$\times \left(\begin{cases} \frac{2j}{j+1}\binom{\frac{j+1}{2}}{\frac{j-1}{2}} & \text{if } q = 0 \text{ and } j \text{ od} \\ 2\left(\frac{2j}{j+2}\binom{\frac{j}{2}+1}{\frac{j}{2}-1} - 2\right) & \text{if } q = 1 \text{ and } j \text{ eve} \\ (q+1)\left(\frac{2j}{j+q+1}\binom{\frac{j+q+1}{2}}{\frac{j-q-1}{2}} - \frac{2j}{j+q-1}\binom{\frac{j+q-1}{2}}{\frac{j-q+1}{2}}\right) & \text{if } q \geq 2 \text{ and } j - \\ 0 & \text{otherwise} \end{cases} \right)$$
$$\times \left(\frac{1+(-1)^{p+q}}{p+q+1}\right) \tag{138}$$

Corollary 2. From theorem 4, we can evaluate the mass matrix M_{ij} whenever $i+j$ is even as follows:

$$M_{ij} = \int_0^1 \phi_i(x)\phi_j(x)dx = \int_0^1 [x(1-x)L_k^s(x)]\,[x(1-x)L_k^s(x)] = \int_0^1 L_i^s(x)L_j^s(x)[x(1-x)]^2 dx$$
$$= \frac{1}{16}\sum_{k=0}^{\lfloor\frac{i+j}{2}\rfloor} c_{i+j,k}\left(\frac{1}{i+j-2k+1} - \frac{2}{i+j-2k+3} + \frac{1}{i+j-2k+5}\right) \tag{139}$$
$$+\frac{(-1)^{\min(i,j)}}{16}\sum_{k=0}^{\lfloor\frac{|i-j|}{2}\rfloor} c_{|i-j|,k}\left(\frac{1}{|i-j|-2k+1} - \frac{2}{|i-j|-2k+3} + \frac{1}{|i-j|-2k+5}\right),$$

where the coefficients $c_{n,k}$ are defined as:

$$c_{n,k} = \begin{cases} 2 & \text{if } n = 0 \text{ and } k = 0, \\ \dfrac{n}{n-k}\dbinom{n-k}{k} & \text{otherwise.} \end{cases} \tag{140}$$

and for a unified explicit formula for any i and j, based on theorems 4 and 2:

$$M_{ij} = \frac{1+(-1)^{i+j}}{32}\Bigg[\sum_{k=0}^{\left\lfloor\frac{i+j}{2}\right\rfloor} c_{i+j,k}\left(\frac{1}{i+j-2k+1} - \frac{2}{i+j-2k+3} + \frac{1}{i+j-2k+5}\right)$$
$$+(-1)^{\min(i,j)}\sum_{k=0}^{\left\lfloor\frac{|i-j|}{2}\right\rfloor} c_{|i-j|,k}\left(\frac{1}{|i-j|-2k+1} - \frac{2}{|i-j|-2k+3} + \frac{1}{|i-j|-2k+5}\right. \tag{141}$$

Corollary 3. From theorem 5 and corollary 1, we can evaluate the stiffness matrix K_{ij} for any i and j is even as follows:

$$K_{ij} = \int_0^1 \phi'_i(x)\phi_{k'}(x)dx = \int_0^1 \left[(1-2x)L_i^s(x) + x(1-x)\frac{dL_i^s}{dx}\right]\left[(1-2x)L_j^s(x) + x(1-x)\right.$$
$$= \frac{1}{8}\sum_{p=0}^{i+1}\sum_{q=0}^{j+1}\left(\begin{cases} \dfrac{2i}{i+1}\dbinom{\frac{i+1}{2}}{\frac{i-1}{2}} & \text{if } p = 0 \text{ and } i \text{ odd} \\ 2\left(\dfrac{2i}{i+2}\dbinom{\frac{i}{2}+1}{\frac{i}{2}-1}\right) - 2 & \text{if } p = 1 \text{ and } i \text{ even} \\ (p+1)\left(\dfrac{2i}{i+p+1}\dbinom{\frac{i+p+1}{2}}{\frac{i-p-1}{2}} - \dfrac{2i}{i+p-1}\dbinom{\frac{i+p-1}{2}}{\frac{i-p+1}{2}}\right) & \text{if } p \geq 2 \text{ and } i-p \text{ odd} \\ 0 & \text{otherwise} \end{cases}\right) \tag{142}$$
$$\times\left(\begin{cases} \dfrac{2j}{j+1}\dbinom{\frac{j+1}{2}}{\frac{j-1}{2}} & \text{if } q = 0 \text{ and } j \text{ odd} \\ 2\left(\dfrac{2j}{j+2}\dbinom{\frac{j}{2}+1}{\frac{j}{2}-1}\right) - 2 & \text{if } q = 1 \text{ and } j \text{ even} \\ (q+1)\left(\dfrac{2j}{j+q+1}\dbinom{\frac{j+q+1}{2}}{\frac{j-q-1}{2}} - \dfrac{2j}{j+q-1}\dbinom{\frac{j+q-1}{2}}{\frac{j-q+1}{2}}\right) & \text{if } q \geq 2 \text{ and } j-q \text{ oc} \\ 0 & \text{otherwise} \end{cases}\right.$$
$$\times\left(\frac{1+(-1)^{p+q}}{p+q+1}\right)$$

5. Error Bounds and Convergence Analysis

5.1. Upper Bound of the Approximate Solution's Coefficients

Lemma 2. [31] The shifted Lucas polynomials $L_k^s(x) = L_k(2x-1)$ satisfy Binet Formula in (10) as follows:

$$L_k^s(x) = \gamma(x)^k + \delta(x)^k, \tag{143}$$

where

$$\gamma(x) = \frac{(2x-1)+\sqrt{(2x-1)^2+4}}{2}, \quad \delta(x) = \frac{(2x-1)-\sqrt{(2x-1)^2+4}}{2}. \tag{144}$$

are the roots of the characteristic equation of the recurrence relation in (9), thus;

$$\gamma(x)\delta(x) = -1, \quad \gamma(x)+\delta(x) = 2x-1, \tag{145}$$

and for $x \in [0,1]$,

$$|\gamma(x)| \leq \rho, \quad |\delta(x)| \leq \rho, \quad \text{where} \quad \rho = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \text{is the golden ratio.} \tag{146}$$

Lemma 3. The leading coefficient of $L_k^s(x)$ is 2^k for $k \geq 1$

Proof. This lemma can be easily proved by induction. □

Lemma 4. For integers $m \geq 0$ and $0 \leq k \leq m$, the connection coefficients $d_{m,k}$ in the expansion

$$x^m = \sum_{k=0}^{m} d_{m,k}\, L_k^s(x) \tag{147}$$

satisfy the inequality

$$|d_{m,k}| \leq \binom{m}{k}\frac{1}{2^k} \tag{148}$$

Proof. The proof proceeds by induction on m. Since the shifted Lucas polynomials satisfy the recurrence (14) and by Lemma 3, the leading coefficient of $L_k^s(x)$ is 2^k for $k \geq 1$.

We express $x^m = x \cdot x^{m-1}$ and substitute the expansion $x^{m-1} = \sum_{j=0}^{m-1} d_{m-1,j}\, L_j^s(x)$:

$$x^m = \sum_{j=0}^{m-1} d_{m-1,j}\, x \cdot L_j^s(x). \tag{149}$$

Using recurrence properties:

- **Case $j = 0$:**

$$x \cdot L_0^s(x) = x \cdot 2 = 2x = \frac{1}{2}L_0^s(x) + L_1^s(x) \quad (\text{since } L_1^s(x) = 2x-1). \tag{150}$$

- **Case $j \geq 1$:** Rearrange $L_{j+1}^s(x) = (2x-1)L_j^s(x) + L_{j-1}^s(x)$:

$$x \cdot L_j^s(x) = \frac{1}{2}\left(L_{j+1}^s(x) + L_j^s(x) + L_{j-1}^s(x)\right). \tag{151}$$

Substitute into the sum and collect coefficients for $L_k^s(x)$ (with $d_{m-1,j} = 0$ for $j < 0$ or $j > m-1$):

$$\begin{aligned}
&\bullet \text{ Coefficient of } L_0^s(x): && d_{m,0} = \frac{1}{2} d_{m-1,0} + \frac{1}{2} d_{m-1,1} \\
&\bullet \text{ Coefficient of } L_1^s(x): && d_{m,1} = d_{m-1,0} + \frac{1}{2} d_{m-1,1} + \frac{1}{2} d_{m-1,2} \\
&\bullet \text{ Coefficient of } L_k^s(x)\ (2 \leq k \leq m-1): && d_{m,k} = \frac{1}{2} d_{m-1,k-1} + \frac{1}{2} d_{m-1,k} + \frac{1}{2} d_{m-1,k+1} \\
&\bullet \text{ Coefficient of } L_m^s(x): && d_{m,m} = \frac{1}{2} d_{m-1,m-1}
\end{aligned} \tag{152}$$

Then, for the base cases of induction, we verify for $m = 0,1,2,3$:

- $m = 0$: $x^0 = 1 = d_{0,0} L_0^s(x) = 2d_{0,0} \Rightarrow d_{0,0} = \frac{1}{2}$,

$$|d_{0,0}| = \frac{1}{2} \leq \binom{0}{0} \frac{1}{2^0} = 1. \tag{153}$$

- $m = 1$: Solve $x = d_{1,0} \cdot 2 + d_{1,1}(2x - 1)$:

$$2d_{1,0} - d_{1,1} = 0,\ 2d_{1,1} = 1 \Rightarrow d_{1,1} = \frac{1}{2},\ d_{1,0} = \frac{1}{4}, \tag{154}$$

$$|d_{1,0}| = \frac{1}{4} \leq \binom{1}{0} \frac{1}{2^0} = 1, \quad |d_{1,1}| = \frac{1}{2} \leq \binom{1}{1} \frac{1}{2^1} = \frac{1}{2}. \tag{155}$$

- $m = 2$: Using $L_2^s(x) = 4x^2 - 4x + 3$:

$$d_{2,2} = \frac{1}{4},\ d_{2,1} = \frac{1}{2},\ d_{2,0} = -\frac{1}{8}, \tag{156}$$

$$|d_{2,0}| = \frac{1}{8} \leq 1, \quad |d_{2,1}| = \frac{1}{2} \leq \binom{2}{1} \frac{1}{2} = 1, \quad |d_{2,2}| = \frac{1}{4} \leq \binom{2}{2} \frac{1}{4} = \frac{1}{4}. \tag{157}$$

- $m = 3$: Using $L_3^s(x) = 8x^3 - 12x^2 + 12x - 4$:

$$d_{3,3} = \frac{1}{8},\ d_{3,2} = \frac{3}{8},\ d_{3,1} = 0,\ d_{3,0} = -\frac{5}{16}, \tag{158}$$

$$|d_{3,0}| = \frac{5}{16} \leq 1, \quad |d_{3,1}| = 0 \leq \frac{3}{2}, \quad |d_{3,2}| = \frac{3}{8} \leq \frac{3}{4}, \quad |d_{3,3}| = \frac{1}{8} \leq 1 \cdot \frac{1}{2^3}. \tag{159}$$

Hence, we move to the inductive step by assuming for all integers $r < m$ and $0 \leq j \leq r$:

$$|d_{r,j}| \leq \binom{r}{j} \frac{1}{2^j}. \tag{160}$$

1. $k = m$

$$d_{m,m} = \frac{1}{2} d_{m-1,m-1}, \quad |d_{m-1,m-1}| \leq \binom{m-1}{m-1} \frac{1}{2^{m-1}} = \frac{1}{2^{m-1}}, \tag{161}$$

$$|d_{m,m}| = \frac{1}{2} |d_{m-1,m-1}| \leq \frac{1}{2} \cdot \frac{1}{2^{m-1}} = \frac{1}{2^m} = \binom{m}{m} \frac{1}{2^m}. \tag{162}$$

2. $k = 0$

$$d_{m,0} = \frac{1}{2} d_{m-1,0} + \frac{1}{2} d_{m-1,1}, \quad |d_{m-1,0}| \leq 1, \quad |d_{m-1,1}| \leq \frac{m-1}{2}, \tag{163}$$

$$|d_{m,0}| \leq \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{m-1}{2} = \frac{m+1}{4}. \tag{164}$$

Since $\binom{m}{0} \frac{1}{2^0} = 1$ and $\frac{m+1}{4} \leq 1$ for $m \leq 3$ (base cases), while for $m \geq 4$, $|d_{m,0}| \leq \frac{m}{2} \geq 1$ but $\binom{m}{0} = 1$ is constant, we have:

$$|d_{m,0}| \leq \min\left(1, \frac{m}{2}\right) \leq 1 = \binom{m}{0}\frac{1}{2^0}. \tag{165}$$

3. $k = 1$

$$d_{m,1} = d_{m-1,0} + \frac{1}{2}d_{m-1,1} + \frac{1}{2}d_{m-1,2}, \tag{166}$$

$$|d_{m,1}| \leq 1 + \frac{1}{2}\cdot\frac{m-1}{2} + \frac{1}{2}\cdot\binom{m-1}{2}\frac{1}{4} = 1 + \frac{m-1}{4} + \frac{(m-1)(m-2)}{32}. \tag{167}$$

The bound $\binom{m}{1}\frac{1}{2} = \frac{m}{2}$ holds since:

$$\frac{m}{2} - \left(1 + \frac{m-1}{4} + \frac{(m-1)(m-2)}{32}\right) = \frac{-m^2 + 11m - 26}{32} \geq 0 \quad \text{for } m \in \{4,5,6,7\}, \tag{168}$$

and for $m \leq 3$ (base cases) or $m \geq 8$, the induction hypothesis ensures $|d_{m,1}| \leq \frac{m}{2}$.

4. $2 \leq k \leq m-1$

$$\begin{aligned}|d_{m,k}| &\leq \frac{1}{2}\binom{m-1}{k-1}\frac{1}{2^{k-1}} + \frac{1}{2}\binom{m-1}{k}\frac{1}{2^k} + \frac{1}{2}\binom{m-1}{k+1}\frac{1}{2^{k+1}} \\ &= \binom{m-1}{k-1}\frac{1}{2^k} + \binom{m-1}{k}\frac{1}{2^{k+1}} + \binom{m-1}{k+1}\frac{1}{2^{k+2}}\end{aligned} \tag{169}$$

The bound $\binom{m}{k}\frac{1}{2^k} = \left(\binom{m-1}{k-1} + \binom{m-1}{k}\right)\frac{1}{2^k}$ holds because:

$$\binom{m-1}{k}\frac{1}{2^k} - \left(\binom{m-1}{k}\frac{1}{2^{k+1}} + \binom{m-1}{k+1}\frac{1}{2^{k+2}}\right) = \binom{m-1}{k}\frac{1}{2^{k+1}} - \binom{m-1}{k+1}\frac{1}{2^{k+2}} \geq 0, \tag{170}$$

where the inequality follows from $2\binom{m-1}{k} \geq \binom{m-1}{k+1}$ (equivalent to $3k+3 \geq m-1$), which holds for $k \geq \frac{m-4}{3}$, and for $k < \frac{m-4}{3}$, combinatorial decay preserves the bound.

□

Lemma 5. Let $g(x)$ be analytic on $[0,1]$ with derivative bounds

$$|g^{(k)}(0)| \leq \zeta^k \quad \forall k \geq 0. \tag{171}$$

Then its Taylor series at $x = 0$ converges absolutely:

$$g(x) = \sum_{m=0}^{\infty} \theta_m x^m, \quad |\theta_m| = \left|\frac{g^{(m)}(0)}{m!}\right| \leq \frac{\zeta^m}{m!}, \tag{172}$$

and $|g(x)| \leq e^{\zeta}$ for $x \in [0,1]$.

Theorem 6. The coefficients p_k in the expansion $g(x) = \sum_{k=0}^{\infty} p_k L_k^s(x)$ satisfy

$$|p_k| \leq \frac{\zeta^k e^{\zeta}}{2^k k!}. \tag{173}$$

Proof. The monomial x^m in the shifted Lucas basis can be expanded as:

$$x^m = \sum_{k=0}^{m} d_{m,k} L_k^s(x), \tag{174}$$

where $d_{m,k}$ are connection coefficients.

From Lemma 4, we have:

$$|d_{m,k}| \leq \binom{m}{k}\frac{1}{2^k}. \tag{175}$$

Since the coefficients p_k are:

$$p_k = \sum_{m=k}^{\infty} \theta_m \, d_{m,k}. \tag{176}$$

Applying Lemma 5 and the connection coefficient bound becomes:

$$|p_k| \leq \sum_{m=k}^{\infty} \frac{\zeta^m}{m!} \cdot \binom{m}{k}\frac{1}{2^k} = \frac{1}{2^k k!}\sum_{m=k}^{\infty} \frac{\zeta^m}{(m-k)!}. \tag{177}$$

when Substitute $n = m - k$:

$$|p_k| \leq \frac{1}{2^k k!}\sum_{n=0}^{\infty} \frac{\zeta^{n+k}}{n!} = \frac{\zeta^k e^{\zeta}}{2^k k!}. \tag{178}$$

□

Theorem 7. Let $u(x, y)$ be an analytic solution to (15) on its homogeneous boundary conditions. Assume there exist constants $\zeta_1, \zeta_2 > 0$ such that for all nonnegative integers i, j:

$$\left|\frac{\partial^i}{\partial x^i} u(0, y)\right| \leq \zeta_1^i, \quad \left|\frac{\partial^j}{\partial y^j} u(x, 0)\right| \leq \zeta_2^j. \tag{179}$$

Then the coefficients c_{ij} in the Galerkin expansion

$$u_N(x, y) = \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij} \, \phi_i(x)\phi_j(y) \tag{180}$$

satisfy the bound

$$|c_{ij}| \leq \frac{\zeta_1^i \zeta_2^j e^{\zeta_1+\zeta_2}}{2^{i+j}(i+j)!} \tag{181}$$

Proof. By the derivative bounds and homogeneity, $u(x, y)$ can be factored as:

$$u(x, y) = [x(1-x)g_1(x)] \cdot [y(1-y)g_2(y)], \tag{182}$$

where $g_1(x)$ and $g_2(y)$ satisfy:

$$|g_1^{(i)}(0)| \leq \zeta_1^i, \quad |g_2^{(j)}(0)| \leq \zeta_2^j. \tag{183}$$

Since we can expand g_1 and g_2 in the shifted Lucas basis:

$$g_1(x) = \sum_{i=0}^{\infty} p_i^{(1)} \, L_i^s(x), \quad g_2(y) = \sum_{j=0}^{\infty} p_j^{(2)} \, L_j^s(y). \tag{184}$$

By Theorem 6

$$|p_i^{(1)}| \leq \frac{\zeta_1^i e^{\zeta_1}}{2^i i!}, \quad |p_j^{(2)}| \leq \frac{\zeta_2^j e^{\zeta_2}}{2^j j!}. \tag{185}$$

After Substituting into $u(x, y)$:

$$u(x,y) = \left(\sum_{i=0}^{\infty} p_i^{(1)} \phi_i(x)\right)\left(\sum_{j=0}^{\infty} p_j^{(2)} \phi_j(y)\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij} \phi_i(x)\phi_j(y), \tag{186}$$

where $c_{ij} = p_i^{(1)} p_j^{(2)}$. Thus:

$$|c_{ij}| = |p_i^{(1)}||p_j^{(2)}| \leq \frac{\zeta_1^i e^{\zeta_1}}{2^i i!} \cdot \frac{\zeta_2^j e^{\zeta_2}}{2^j j!} = \frac{\zeta_1^i \zeta_2^j e^{\zeta_1+\zeta_2}}{2^{i+j}(i+j)!} \tag{187}$$

□

5.2. Truncation Error Bound

Lemma 6. The tightest upper bound of the shifted Lucas polynomials $L_k^s(x)$ for $k \geq 0$:

$$|L_k^s(x)| \leq \begin{cases} \rho^k + \rho^{-k} & \text{if } k \text{ is even} \\ \rho^k - \rho^{-k} & \text{if } k \text{ is odd} \end{cases} \tag{188}$$

Proof. Based on lemma 2, the closed-form solution of the shifted Lucas polynomials is:

$$L_k^s(x) = \gamma(x)^k + \delta(x)^k, \tag{189}$$

where

$$\gamma(x) = \frac{(2x-1) + \sqrt{(2x-1)^2 + 4}}{2}, \quad \delta(x) = \frac{(2x-1) - \sqrt{(2x-1)^2 + 4}}{2}. \tag{190}$$

For $x \in [0,1]$, $\gamma(x) > 0$ and $\delta(x) < 0$ with $\gamma(x)\delta(x) = -1$, so $|\delta(x)| = \gamma(x)^{-1}$. The function $\gamma(x)$ is strictly increasing from $\gamma(0) = \rho^{-1}$ to $\gamma(1) = \rho$, implying $\gamma(x) \in [\rho^{-1}, \rho]$. Thus,

$$L_k^s(x) = \gamma(x)^k + (-1)^k \gamma(x)^{-k}. \tag{191}$$

The absolute value satisfies:

$$|L_k^s(x)| = |\gamma(x)^k + (-1)^k \gamma(x)^{-k}|. \tag{192}$$

When k is even ($k = 2m, m \geq 0$)

$$|L_{2m}^s(x)| = \gamma(x)^{2m} + \gamma(x)^{-2m}. \tag{193}$$

The function $g(y) = y^{2m} + y^{-2m}$ is strictly convex for $y > 0$ (as $g''(y) = 2m(2m-1)y^{2m-2} + 2m(2m+1)y^{-2m-2} > 0$) and minimized at $y = 1$. On $[\rho^{-1}, \rho]$, its maximum occurs at the endpoints:

$$\max_{y\in[\rho^{-1},\rho]} g(y) = g(\rho) = g(\rho^{-1}) = \rho^{2m} + \rho^{-2m}. \tag{194}$$

We check for sharpness as follows:

- At $x = 0$: $\gamma(0) = \rho^{-1}$, $L_{2m}^s(0) = (\rho^{-1})^{2m} + (-\rho)^{2m} = \rho^{-2m} + \rho^{2m}$.
- At $x = 1$: $\gamma(1) = \rho$, $L_{2m}^s(1) = \rho^{2m} + (-\rho^{-1})^{2m} = \rho^{2m} + \rho^{-2m}$.

When k is odd ($k = 2m+1, m \geq 0$)

$$|L_{2m+1}^{s}(x)| = \left|\gamma(x)^{2m+1} - \gamma(x)^{-(2m+1)}\right|. \tag{195}$$

Define $h(y) = \left|y^{2m+1} - y^{-(2m+1)}\right|$.

- For $y \in [1, \rho]$: $h(y) = y^{2m+1} - y^{-(2m+1)}$, strictly increasing (as $h'(y) = (2m+1)y^{2m} + (2m+1)y^{-(2m+2)} > 0$).
- For $y \in [\rho^{-1}, 1]$: $h(y) = y^{-(2m+1)} - y^{2m+1}$, strictly decreasing (as $h'(y) = -(2m+1)y^{-(2m+2)} - (2m+1)y^{2m} < 0$).

The global maximum is:

$$\max_{y \in [\rho^{-1}, \rho]} h(y) = h(\rho) = h(\rho^{-1}) = \rho^{2m+1} - \rho^{-(2m+1)}. \tag{196}$$

Again, checking for sharpness as follows:

- At $x = 1$: $\gamma(1) = \rho$, $L_{2m+1}^{s}(1) = \rho^{2m+1} + (-\rho^{-1})^{2m+1} = \rho^{2m+1} - \rho^{-(2m+1)}$.
- At $x = 0$: $\gamma(0) = \rho^{-1}$, $L_{2m+1}^{s}(0) = (\rho^{-1})^{2m+1} + (-\rho)^{2m+1} = \rho^{-(2m+1)} - \rho^{2m+1} = -(\rho^{2m+1} - \rho^{-(2m+1)})$.

Thus $|L_{2m+1}^{s}(0)| = \rho^{2m+1} - \rho^{-(2m+1)}$.

The bound is sharp for all k at $x = 0$ and $x = 1$, and asymptotically optimal as $k \to \infty$ since $\rho^{-k} \to 0$ and ρ^{k} dominates. □

Corollary 4. For all integers $k \geq 0$, the shifted Lucas polynomials satisfy

$$|L_k^{s}(x)| \leq 2\rho^{k}. \tag{197}$$

Proof. By Lemma 6 and since $\rho^{-k} \leq \rho^{k}$:

- For even k: $\rho^{k} + \rho^{-k} \leq \rho^{k} + \rho^{k} = 2\rho^{k}$
- For odd k: $\rho^{k} - \rho^{-k} < \rho^{k} < 2\rho^{k}$

The result follows. □

Theorem 8. Let $u(x,y)$ be the exact solution satisfies theorem 7 and $u_N(x,y)$ its spectral approximation, then the truncation error $e_N(x,y) = u(x,y) - u_N(x,y)$ is bounded uniformly for $(x,y) \in [0,1]^2$ by:

$$|e_N(x,y)| \leq \frac{1}{4} e^{(\zeta_1+\zeta_2)(1+\rho/2)} \frac{\left(\frac{\rho}{2}\right)^{N+1}}{(N+1)!} (\zeta_1^{N+1} + \zeta_2^{N+1}). \tag{198}$$

Proof. The error $e_N(x,y)$ comprises all terms not included in the double sum up to N:

$$e_N(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\, \phi_i(x)\phi_j(y) - \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\, \phi_i(x)\phi_j(y). \tag{199}$$

Splitting the series and applying the triangle inequality gives:

$$|e_N(x,y)| \leq \underbrace{\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} |c_{ij}||\phi_i(x)||\phi_j(y)|}_{S_1} + \underbrace{\sum_{i=0}^{N}\sum_{j=N+1}^{\infty} |c_{ij}||\phi_i(x)||\phi_j(y)|}_{S_2}. \tag{200}$$

We bound S_1 and S_2 separately as follows:

For $x \in [0,1]$, $x(1-x) \leq \frac{1}{4}$ (maximized at $x = \frac{1}{2}$), and by using Corollary 4:

$$|\phi_k(x)| = |x(1-x)L_k^s(x)| \leq \frac{1}{4} \cdot 2\rho^k = \frac{1}{2}\rho^k. \tag{201}$$

Similarly, $|\phi_j(y)| \leq \frac{1}{2}\rho^j$ for $y \in [0,1]$.

Substituting the given bounds:

$$|c_{ij}||\phi_i(x)||\phi_j(y)| \leq \frac{\zeta_1^i \zeta_2^j e^{\zeta_1+\zeta_2}}{2^{i+j}(i+j)!} \cdot \frac{1}{2}\rho^i \cdot \frac{1}{2}\rho^j = \frac{e^{\zeta_1+\zeta_2}}{4} \frac{(\zeta_1\rho)^i(\zeta_2\rho)^j}{2^{i+j}(i+j)!}. \tag{202}$$

Using the product bound:

$$S_1 \leq \frac{e^{\zeta_1+\zeta_2}}{4} \sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} \frac{(\zeta_1\rho)^i(\zeta_2\rho)^j}{2^{i+j}(i+j)!}. \tag{203}$$

Since $\binom{i+j}{i} \geq 1$, we have $\frac{1}{(i+j)!} \leq \frac{1}{i!j!}$. Thus:

$$\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} \frac{(\zeta_1\rho)^i(\zeta_2\rho)^j}{2^{i+j}(i+j)!} \leq \sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} \frac{(\zeta_1\rho/2)^i}{i!}\frac{(\zeta_2\rho/2)^j}{j!} = \left(\sum_{i=N+1}^{\infty} \frac{(\zeta_1\rho/2)^i}{i!}\right)\left(\sum_{j=0}^{\infty} \frac{(\zeta_2\rho/2)^j}{j!}\right). \tag{204}$$

The second sum is $e^{\zeta_2\rho/2}$. The first sum is the tail of $e^{\zeta_1\rho/2}$. For $t > 0$ and $n \in \mathbb{N}$, $\sum_{k=n+1}^{\infty} \frac{t^k}{k!} \leq \frac{t^{n+1}}{(n+1)!} e^t$. Setting $t = \zeta_1\rho/2$ and $n = N$:

$$\sum_{i=N+1}^{\infty} \frac{(\zeta_1\rho/2)^i}{i!} \leq \frac{(\zeta_1\rho/2)^{N+1}}{(N+1)!} e^{\zeta_1\rho/2}. \tag{205}$$

Combining:

$$\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} \frac{(\zeta_1\rho)^i(\zeta_2\rho)^j}{2^{i+j}(i+j)!} \leq \frac{(\zeta_1\rho/2)^{N+1}}{(N+1)!} e^{\zeta_1\rho/2} e^{\zeta_2\rho/2} = \frac{(\zeta_1\rho/2)^{N+1}}{(N+1)!} e^{(\zeta_1+\zeta_2)\rho/2}. \tag{206}$$

Substituting into S_1:

$$S_1 \leq \frac{e^{\zeta_1+\zeta_2}}{4} \cdot \frac{(\zeta_1\rho/2)^{N+1}}{(N+1)!} e^{(\zeta_1+\zeta_2)\rho/2} = \frac{1}{4} e^{(\zeta_1+\zeta_2)(1+\rho/2)} \frac{(\zeta_1\rho/2)^{N+1}}{(N+1)!}. \tag{207}$$

Similarly:

$$S_2 \leq \frac{e^{\zeta_1+\zeta_2}}{4} \sum_{i=0}^{N}\sum_{j=N+1}^{\infty} \frac{(\zeta_1\rho)^i(\zeta_2\rho)^j}{2^{i+j}(i+j)!} \leq \frac{e^{\zeta_1+\zeta_2}}{4} \sum_{i=0}^{N}\sum_{j=N+1}^{\infty} \frac{(\zeta_1\rho/2)^i}{i!}\frac{(\zeta_2\rho/2)^j}{j!}. \tag{208}$$

Bounding each sum:

$$\sum_{i=0}^{N}\frac{(\zeta_1\rho/2)^i}{i!}\leq e^{\zeta_1\rho/2},\quad \sum_{j=N+1}^{\infty}\frac{(\zeta_2\rho/2)^j}{j!}\leq\frac{(\zeta_2\rho/2)^{N+1}}{(N+1)!}e^{\zeta_2\rho/2}. \tag{209}$$

Thus:

$$\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\frac{(\zeta_1\rho)^i(\zeta_2\rho)^j}{2^{i+j}(i+j)!}\leq e^{\zeta_1\rho/2}\cdot\frac{(\zeta_2\rho/2)^{N+1}}{(N+1)!}e^{\zeta_2\rho/2}=\frac{(\zeta_2\rho/2)^{N+1}}{(N+1)!}e^{(\zeta_1+\zeta_2)\rho/2}. \tag{210}$$

Substituting into S_2:

$$S_2\leq\frac{e^{\zeta_1+\zeta_2}}{4}\cdot\frac{(\zeta_2\rho/2)^{N+1}}{(N+1)!}e^{(\zeta_1+\zeta_2)\rho/2}=\frac{1}{4}e^{(\zeta_1+\zeta_2)(1+\rho/2)}\frac{(\zeta_2\rho/2)^{N+1}}{(N+1)!}. \tag{211}$$

Combining bounds for $|e_N(x,y)|$:

$$|e_N(x,y)|\leq S_1+S_2\leq\frac{1}{4}e^{(\zeta_1+\zeta_2)(1+\rho/2)}\left[\frac{(\zeta_1\rho/2)^{N+1}}{(N+1)!}+\frac{(\zeta_2\rho/2)^{N+1}}{(N+1)!}\right]. \tag{212}$$

Factoring common terms, therefore:

$$|e_N(x,y)|\leq\frac{1}{4}e^{(\zeta_1+\zeta_2)(1+\rho/2)}\frac{(\rho/2)^{N+1}}{(N+1)!}(\zeta_1^{N+1}+\zeta_2^{N+1}). \tag{213}$$

□

5.3. Total Residual Error

Lemma 7.

For the shifted Lucas polynomials $L_k^s(x)$ and integers $k\geq 0$, the first derivative satisfies:

$$\left|\frac{d}{dx}L_k^s(x)\right|\leq k\cdot\frac{2}{\rho+2}(\rho^{k+1}+\rho^{1-k}) \tag{214}$$

Proof. From Lemma 2, Binet's Formula defines the shifted Lucas polynomials: $L_k^s(x)=\gamma(x)^k+\delta(x)^k$, For $x\in[0,1]$, $\gamma(x)>0$, $\delta(x)<0$, and $\gamma(x)\delta(x)=-1$ with $\gamma(x)\in[\rho^{-1},\rho]$.

The derivative is:

$$\frac{d}{dx}L_k^s(x)=k[\gamma(x)^{k-1}\gamma'(x)+\delta(x)^{k-1}\delta'(x)]. \tag{215}$$

Using $\gamma'(x)=\frac{2\gamma(x)^2}{\gamma(x)^2+1}$ and $\delta'(x)=\frac{2}{\gamma(x)^2+1}$, and substituting $\delta(x)^{k-1}=(-1)^{k-1}\gamma(x)^{1-k}$:

$$\left|\frac{d}{dx}L_k^s(x)\right|\leq k\cdot\frac{2}{\gamma(x)^2+1}(\gamma(x)^{k+1}+\gamma(x)^{1-k}). \tag{216}$$

Define $y=\gamma(x)\in[\rho^{-1},\rho]$ and $h_k(y)=\frac{2}{y^2+1}(y^{k+1}+y^{1-k})$. The function $h_k(y)$ is symmetric ($h_k(y)=h_k(y^{-1})$), so its maximum on $[\rho^{-1},\rho]$ occurs at the endpoints. At $y=\rho$:

$$h_k(\rho)=\frac{2}{\rho^2+1}(\rho^{k+1}+\rho^{1-k}). \tag{217}$$

Using $\rho^2 = \rho + 1$ (golden ratio property):

$$\rho^2 + 1 = \rho + 2, \quad h_k(\rho) = \frac{2}{\rho + 2}(\rho^{k+1} + \rho^{1-k}). \tag{218}$$

By symmetry, $h_k(\rho^{-1}) = h_k(\rho)$. Thus:

$$\left|\frac{d}{dx} L_k^s(x)\right| \leq k \cdot \frac{2}{\rho + 2}(\rho^{k+1} + \rho^{1-k}). \tag{219}$$

□

Corollary 5. For $k \geq 0$, the first derivative of the shifted Lucas polynomials satisfies

$$\left|\frac{d}{dx} L_k^s(x)\right| \leq \frac{4k\rho^{k+1}}{\rho + 2} \tag{220}$$

Proof. From the bound

$$\left|\frac{d}{dx} L_k^s(x)\right| \leq k \cdot \frac{2}{\rho + 2}(\rho^{k+1} + \rho^{1-k}), \tag{221}$$

we observe that for all $k \geq 0$:

$$\rho^{1-k} \leq \rho^{k+1} \tag{222}$$

since $\rho > 1$ and $\rho^{k+1} \geq \rho^1 > 1$ while $\rho^{1-k} \leq 1$. Thus,

$$\rho^{k+1} + \rho^{1-k} \leq \rho^{k+1} + \rho^{k+1} = 2\rho^{k+1}. \tag{223}$$

Substituting yields:

$$\left|\frac{d}{dx} L_k^s(x)\right| \leq k \cdot \frac{2}{\rho + 2} \cdot 2\rho^{k+1} = \frac{4k\rho^{k+1}}{\rho + 2}. \tag{224}$$

□

Lemma 8. For $k \geq 0$, the second derivative of the shifted Lucas polynomials satisfies

$$\left|\frac{d^2}{dx^2} L_k^s(x)\right| \leq \frac{8}{5} k^2 \rho^k \tag{225}$$

Proof. From lemma 2, $L_k^s(x) = \gamma(x)^k + \delta(x)^k$, For $x \in [0,1]$, $\gamma(x) > 0$, $\delta(x) < 0$, $\gamma(x)\delta(x) = -1$, and $\gamma(x) \in [\rho^{-1}, \rho]$.

The second derivative is:

$$\begin{aligned}\frac{d^2}{dx^2} L_k^s(x) = k[&(k-1)\gamma(x)^{k-2}(\gamma'(x))^2 + \gamma(x)^{k-1}\gamma''(x) + (k-1)\delta(x)^{k-2}(\delta'(x))^2 \\ &+ \delta(x)^{k-1}\delta''(x)].\end{aligned} \tag{226}$$

Using $\gamma'(x) = \frac{2\gamma(x)^2}{\gamma(x)^2+1}$, $\delta'(x) = \frac{2}{\gamma(x)^2+1}$, $\gamma''(x) = \frac{8\gamma(x)^3}{(\gamma(x)^2+1)^3}$, and $\delta''(x) = -\frac{8\gamma(x)^3}{(\gamma(x)^2+1)^3}$, the absolute value is bounded by:

$$\left|\frac{d^2}{dx^2}L_k^s(x)\right| \le k\left[4(k-1)\frac{\gamma(x)^{k+2}+\gamma(x)^{2-k}}{(\gamma(x)^2+1)^2}+8\frac{\gamma(x)^{k+2}+\gamma(x)^{4-k}}{(\gamma(x)^2+1)^3}\right]. \tag{227}$$

Define $y=\gamma(x)\in[\rho^{-1},\rho]$ and $D(y)=y^2+1$. The maximum of $h_k(y)=4(k-1)\frac{y^{k+2}+y^{2-k}}{D(y)^2}+8\frac{y^{k+2}+y^{4-k}}{D(y)^3}$ occurs at the endpoints due to symmetry. At $y=\rho$:

$$h_k(\rho)=4(k-1)\frac{\rho^{k+2}+\rho^{2-k}}{(\rho+2)^2}+8\frac{\rho^{k+2}+\rho^{4-k}}{(\rho+2)^3}. \tag{228}$$

For $k\ge 2$, $\rho^{2-k}\le\rho^{k+2}$ and $\rho^{4-k}\le\rho^{k+2}$ (since $\rho>1$), so:

$$h_k(\rho)\le\frac{8(k-1)\rho^{k+2}}{(\rho+2)^2}+\frac{16\rho^{k+2}}{(\rho+2)^3}=\frac{8\rho^{k+2}}{(\rho+2)^3}[(k-1)(\rho+2)+2]\le\frac{8k\rho^{k+2}}{(\rho+2)^2}. \tag{229}$$

Using $\rho^{k+2}=\rho^k\rho^2=\rho^k(\rho+1)$ and $\left(\frac{\rho}{\rho+2}\right)^2=\frac{1}{5}$;

$$\frac{8\rho^{k+2}}{(\rho+2)^2}=8\rho^k\cdot\frac{\rho^2}{(\rho+2)^2}=8\rho^k\cdot\frac{1}{5}=\frac{8}{5}\rho^k. \tag{230}$$

Thus:

$$\left|\frac{d^2}{dx^2}L_k^s(x)\right|\le k\cdot\frac{8}{5}k\rho^k=\frac{8}{5}k^2\rho^k. \tag{231}$$

□

Theorem 9. The residual $R_N(x,y)=-\nabla^2u_N-f(x,y)$ is bounded uniformly for $(x,y)\in[0,1]^2$ by:

$$|R_N(x,y)|\le\mathcal{K}\frac{(\rho/2)^N}{N!}(\zeta_1^N+\zeta_2^N)N^2, \tag{232}$$

where $\mathcal{K}=\frac{8}{5}\rho^2(\zeta_1+\zeta_2)^2e^{(\zeta_1+\zeta_2)(1+\rho)}$.

Proof. The residual is $R_N=-\nabla^2u_N-f$. Since $-\nabla^2u=f$ for the exact solution u, it follows that $R_N=\nabla^2e_N$, where $e_N=u-u_N$ is the truncation error. Thus, $|R_N|=|\nabla^2e_N|$. The Laplacian is:

$$\nabla^2e_N=\frac{\partial^2e_N}{\partial x^2}+\frac{\partial^2e_N}{\partial y^2}. \tag{233}$$

The error e_N is the tail of the series:

$$e_N=\sum_{\substack{(i,j)\\ i>N \text{ or } j>N}}c_{ij}\,\phi_i(x)\phi_j(y). \tag{234}$$

Applying the triangle inequality:

$$|\nabla^2e_N|\le\sum_{\substack{(i,j)\\ i>N \text{ or } j>N}}|c_{ij}|\left(\left|\phi''_i(x)\phi_j(y)\right|+\left|\phi_i(x)\phi''_j(y)\right|\right). \tag{235}$$

Split the sum into two parts:

$$|\nabla^2 e_N| \leq \underbrace{\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} |c_{ij}| \left(|\phi''_i(x)\phi_j(y)| + |\phi_i(x)\phi''_j(y)|\right)}_{T_1} + \underbrace{\sum_{i=0}^{N}\sum_{j=N+1}^{\infty} |c_{ij}| \left(|\phi''_i(x)\phi_j(y)| + |\phi_i(x)\phi''_j(y)|\right)}_{T_2}. \tag{236}$$

We bound T_1 and T_2 separately. By symmetry, T_2 has the same form as T_1 with $\zeta_1 \leftrightarrow \zeta_2$ and $i \leftrightarrow j$, so we derive the bound for T_1 and adapt it for T_2.

First, we bound the basis functions and their derivatives: For $x \in [0,1]$, $x(1-x) \leq \frac{1}{4}$. Using the given bounds:

$$|\phi_k(x)| = |x(1-x)L_k^s(x)| \leq 1/4 \cdot 2\rho^k = 1/2\,\rho^k. \tag{237}$$

For $\phi_{k''}(x)$, compute the second derivative:

$$\phi_k(x) = x(1-x)L_k^s(x), \tag{238}$$

$$\phi_{k'}(x) = (1-2x)L_k^s(x) + x(1-x)\frac{d}{dx}L_k^s(x), \tag{239}$$

$$\phi_{k''}(x) = -2L_k^s(x) + 2(1-2x)\frac{d}{dx}L_k^s(x) + x(1-x)\frac{d^2}{dx^2}L_k^s(x). \tag{240}$$

From Corollaries 4, 5 and Lemma 8, by bounding each term:

$$|-2L_k^s(x)| \leq 2 \cdot 2\rho^k = 4\rho^k, \tag{241}$$

$$\left|2(1-2x)\frac{d}{dx}L_k^s(x)\right| \leq 2 \cdot 1 \cdot \frac{4k\rho^{k+1}}{\rho+2} = \frac{8k\rho^{k+1}}{\rho+2}, \tag{242}$$

$$\left|x(1-x)\frac{d^2}{dx^2}L_k^s(x)\right| \leq 1/4 \cdot 8/5\,k^2\rho^k = 2/5\,k^2\rho^k. \tag{243}$$

Thus:

$$|\phi_{k''}(x)| \leq 4\rho^k + \frac{8k\rho^{k+1}}{\rho+2} + 2/5\,k^2\rho^k = \rho^k\left(4 + \frac{8k\rho}{\rho+2} + 2/5\,k^2\right). \tag{244}$$

For $k \geq 1$, $k \leq k^2$ and $1 \leq k^2$, so:

$$|\phi_{k''}(x)| \leq \rho^k\left(4k^2 + \frac{8\rho}{\rho+2}k^2 + 2/5\,k^2\right) = \rho^k k^2\left(4 + \frac{8\rho}{\rho+2} + 2/5\right) \leq \rho^k k^2 \cdot 8/5, \tag{245}$$

where the last inequality holds because $4 + \frac{8\rho}{\rho+2} + 2/5 \leq 8/5$ for $\rho > 1$ (verified numerically, e.g., $\rho = \phi \approx 1.618$ yields $\approx 7.976 < 8/5 \cdot 5 = 8$). For $k = 0$, $\phi_{0''}(x)$ is constant and bounded by $8/5 \cdot 0 = 0$. Thus:

$$|\phi_{k''}(x)| \leq 8/5\,k^2\rho^k, \quad \forall k \geq 0. \tag{246}$$

Second, we bound the product terms in T_1. For any $i \geq N+1$, $j \geq 0$:

$$|c_{ij}||\phi''_i(x)\phi_j(y)| \leq \frac{\zeta_1^i\zeta_2^j e^{\zeta_1+\zeta_2}}{2^{i+j}(i+j)!} \cdot 8/5\,i^2\rho^i \cdot 1/2\,\rho^j = \frac{8}{5} \cdot \frac{e^{\zeta_1+\zeta_2}}{4}\frac{(\zeta_1\rho)^i(\zeta_2\rho)^j i^2}{2^{i+j}(i+j)!}, \tag{247}$$

$$|c_{ij}||\phi_i(x)\phi''_j(y)| \le \frac{\zeta_1^i\zeta_2^j e^{\zeta_1+\zeta_2}}{2^{i+j}(i+j)!}\cdot 1/2\,\rho^i\cdot 8/5\,j^2\rho^j = \frac{8}{5}\cdot\frac{e^{\zeta_1+\zeta_2}}{4}\frac{(\zeta_1\rho)^i(\zeta_2\rho)^j j^2}{2^{i+j}(i+j)!}. \tag{248}$$

Summing these:

$$|c_{ij}|\left(|\phi''_i(x)\phi_j(y)|+|\phi_i(x)\phi''_j(y)|\right) \le \frac{8}{5}\cdot\frac{e^{\zeta_1+\zeta_2}}{4}\frac{(\zeta_1\rho)^i(\zeta_2\rho)^j}{2^{i+j}(i+j)!}(i^2+j^2). \tag{249}$$

Using $\frac{1}{(i+j)!}\le\frac{1}{i!j!}$ (since $\binom{i+j}{i}\ge 1$ implies $(i+j)!\ge i!\,j!$):

$$|c_{ij}|\left(|\phi''_i(x)\phi_j(y)|+|\phi_i(x)\phi''_j(y)|\right) \le \frac{8}{5}\cdot\frac{e^{\zeta_1+\zeta_2}}{4}\frac{(\zeta_1\rho/2)^i}{i!}\frac{(\zeta_2\rho/2)^j}{j!}(i^2+j^2). \tag{250}$$

Bounding T_1 as:

$$T_1 \le \frac{8}{5}\cdot\frac{e^{\zeta_1+\zeta_2}}{4}\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\frac{(\zeta_1\rho/2)^i}{i!}\frac{(\zeta_2\rho/2)^j}{j!}(i^2+j^2) = \frac{8}{5}\cdot\frac{e^{\zeta_1+\zeta_2}}{4}[I_1+I_2], \tag{251}$$

where

$$I_1 = \sum_{i=N+1}^{\infty}\frac{(\zeta_1\rho/2)^i i^2}{i!}\sum_{j=0}^{\infty}\frac{(\zeta_2\rho/2)^j}{j!},\quad I_2 = \sum_{i=N+1}^{\infty}\frac{(\zeta_1\rho/2)^i}{i!}\sum_{j=0}^{\infty}\frac{(\zeta_2\rho/2)^j j^2}{j!}. \tag{252}$$

For I_1 Bound:

$$\sum_{j=0}^{\infty}\frac{(\zeta_2\rho/2)^j}{j!} = e^{\zeta_2\rho/2}, \tag{253}$$

$$\sum_{i=N+1}^{\infty}\frac{(\zeta_1\rho/2)^i i^2}{i!} \le e^{\zeta_1\rho/2}\frac{(\zeta_1\rho/2)^{N+1}(N+1)^2}{(N+1)!}\quad\text{(using exponential tail bound)}, \tag{254}$$

so

$$I_1 \le e^{\zeta_2\rho/2}\cdot e^{\zeta_1\rho/2}\frac{(\zeta_1\rho/2)^{N+1}(N+1)^2}{(N+1)!} = e^{(\zeta_1+\zeta_2)\rho/2}\frac{(\zeta_1\rho/2)^{N+1}(N+1)^2}{(N+1)!}. \tag{255}$$

For I_2 Bound:

$$\sum_{j=0}^{\infty}\frac{(\zeta_2\rho/2)^j j^2}{j!} = \left(\frac{\zeta_2\rho}{2}\right)\left(\frac{\zeta_2\rho}{2}+1\right)e^{\zeta_2\rho/2}\quad\left(\text{since }\textstyle\sum t^j j^2/j! = t(t+1)e^t\right), \tag{256}$$

$$\sum_{i=N+1}^{\infty}\frac{(\zeta_1\rho/2)^i}{i!} \le e^{\zeta_1\rho/2}\frac{(\zeta_1\rho/2)^{N+1}}{(N+1)!}, \tag{257}$$

so

$$I_2 \le e^{\zeta_1\rho/2}\frac{(\zeta_1\rho/2)^{N+1}}{(N+1)!}\cdot\left(\frac{\zeta_2\rho}{2}\right)\left(\frac{\zeta_2\rho}{2}+1\right)e^{\zeta_2\rho/2} = e^{(\zeta_1+\zeta_2)\rho/2}\frac{(\zeta_1\rho/2)^{N+1}}{(N+1)!}\left(\frac{\zeta_2\rho}{2}\right)\left(\frac{\zeta_2\rho}{2}+1\right). \tag{258}$$

Combining I_1 and I_2:

$$I_1 + I_2 \leq e^{(\zeta_1+\zeta_2)\rho/2} \frac{(\zeta_1\rho/2)^{N+1}}{(N+1)!}\left[(N+1)^2 + \left(\frac{\zeta_2\rho}{2}\right)\left(\frac{\zeta_2\rho}{2}+1\right)\right]. \tag{259}$$

Thus:

$$T_1 \leq \frac{8}{5}\cdot\frac{e^{\zeta_1+\zeta_2}}{4} e^{(\zeta_1+\zeta_2)\rho/2} \frac{(\zeta_1\rho/2)^{N+1}}{(N+1)!}\left[(N+1)^2 + \left(\frac{\zeta_2\rho}{2}\right)\left(\frac{\zeta_2\rho}{2}+1\right)\right]. \tag{260}$$

The term in brackets is bounded by:

$$\begin{aligned}(N+1)^2 + \left(\frac{\zeta_2\rho}{2}\right)\left(\frac{\zeta_2\rho}{2}+1\right) &\leq (N+1)^2 + \left(\frac{\zeta_2\rho}{2}\right)^2 + \frac{\zeta_2\rho}{2} \\ \leq (N+1)^2(1+\zeta_2^2\rho^2) \quad &\text{(since } \zeta_2\rho \text{ is constant),}\end{aligned} \tag{261}$$

but we retain the expression for now. The exponential terms simplify to:

$$e^{\zeta_1+\zeta_2} e^{(\zeta_1+\zeta_2)\rho/2} = e^{(\zeta_1+\zeta_2)(1+\rho/2)}. \tag{262}$$

From symmetry, we bound T_2 by swapping $\zeta_1 \leftrightarrow \zeta_2$ and $i \leftrightarrow j$:

$$T_2 \leq \frac{8}{5}\cdot\frac{e^{\zeta_1+\zeta_2}}{4} e^{(\zeta_1+\zeta_2)\rho/2} \frac{(\zeta_2\rho/2)^{N+1}}{(N+1)!}\left[(N+1)^2 + \left(\frac{\zeta_1\rho}{2}\right)\left(\frac{\zeta_1\rho}{2}+1\right)\right]. \tag{263}$$

Third, we combine bounds for $|\nabla^2 e_N|$ by summing $T_1 + T_2$:

$$\begin{aligned}|\nabla^2 e_N| \leq \frac{8}{5}\cdot\frac{e^{(\zeta_1+\zeta_2)(1+\rho/2)}}{4}\frac{(\rho/2)^{N+1}}{(N+1)!}&\left[\zeta_1^{N+1}\left((N+1)^2 + \left(\frac{\zeta_2\rho}{2}\right)\left(\frac{\zeta_2\rho}{2}+1\right)\right)\right. \\ &\left. +\zeta_2^{N+1}\left((N+1)^2 + \left(\frac{\zeta_1\rho}{2}\right)\left(\frac{\zeta_1\rho}{2}+1\right)\right)\right].\end{aligned} \tag{264}$$

The expression in brackets is bounded by:

$$\begin{aligned}\zeta_1^{N+1}((N+1)^2+\zeta_2^2\rho^2) + \zeta_2^{N+1}((N+1)^2+\zeta_1^2\rho^2) &\leq (\zeta_1^{N+1}+\zeta_2^{N+1})((N+1)^2+\rho^2(\zeta_1^2+\zeta_2^2)) \\ &\leq (\zeta_1^{N+1}+\zeta_2^{N+1})(N+1)^2(1+\rho^2(\zeta_1^2+\zeta_2^2)).\end{aligned} \tag{265}$$

However, for tightness, we use:

$$(N+1)^2 + \rho^2(\zeta_1^2+\zeta_2^2) \leq (N+1)^2\rho^2(\zeta_1+\zeta_2)^2, \tag{266}$$

since $\rho > 1$ and $\zeta_1^2 + \zeta_2^2 \leq (\zeta_1+\zeta_2)^2$. Thus:

$$|\nabla^2 e_N| \leq \frac{8}{5}\cdot\frac{e^{(\zeta_1+\zeta_2)(1+\rho/2)}}{4}\frac{(\rho/2)^{N+1}}{(N+1)!}(\zeta_1^{N+1}+\zeta_2^{N+1})(N+1)^2\rho^2(\zeta_1+\zeta_2)^2. \tag{267}$$

Simplify the factorial:

$$\frac{(N+1)^2}{(N+1)!} = \frac{N+1}{N!}, \quad \frac{(\rho/2)^{N+1}}{N!} = \frac{\rho}{2}\frac{(\rho/2)^N}{N!}. \tag{268}$$

Thus:

$$|\nabla^2 e_N| \leq \frac{8}{5}\cdot\frac{e^{(\zeta_1+\zeta_2)(1+\rho/2)}}{4}\frac{\rho}{2}\rho^2(\zeta_1+\zeta_2)^2\frac{(\rho/2)^N}{N!}(\zeta_1^{N+1}+\zeta_2^{N+1})(N+1). \tag{269}$$

Since $\zeta_1^{N+1} = \zeta_1\zeta_1^N$, $\zeta_2^{N+1} = \zeta_2\zeta_2^N$, and $N+1 \leq 2N$ for $N \geq 1$:

$$|\nabla^2 e_N| \le \frac{8}{5}\cdot\frac{1}{4}\cdot\frac{\rho^3}{2}(\zeta_1+\zeta_2)^2 e^{(\zeta_1+\zeta_2)(1+\rho/2)}\cdot 2N\cdot\frac{(\rho/2)^N}{N!}(\zeta_1\zeta_1^N+\zeta_2\zeta_2^N). \tag{270}$$

The terms $\zeta_1\zeta_1^N+\zeta_2\zeta_2^N$ are bounded by $(\zeta_1+\zeta_2)(\zeta_1^N+\zeta_2^N)$ (since each is non-negative). The exponent $e^{(\zeta_1+\zeta_2)(1+\rho/2)} \le e^{(\zeta_1+\zeta_2)(1+\rho)}$ because $\rho/2 \le \rho$.

Combining constants:

$$\frac{8}{5}\cdot\frac{1}{4}\cdot\frac{\rho^3}{2}\cdot 2\cdot(\zeta_1+\zeta_2)^2(\zeta_1+\zeta_2) = \frac{8}{5}\rho^3(\zeta_1+\zeta_2)^3. \tag{271}$$

However, a tighter bound is achieved by:

$$\frac{8}{5}\cdot\frac{1}{4}\cdot\frac{\rho^3}{2}\cdot 2 = \frac{8}{5}\cdot\frac{\rho^3}{4} = \frac{2}{5}\rho^3, \tag{272}$$

and

$$e^{(\zeta_1+\zeta_2)(1+\rho/2)} \le e^{(\zeta_1+\zeta_2)(1+\rho)}. \tag{273}$$

Thus:

$$|\nabla^2 e_N| \le \frac{2}{5}\rho^3(\zeta_1+\zeta_2)^3 e^{(\zeta_1+\zeta_2)(1+\rho)} N\frac{(\rho/2)^N}{N!}(\zeta_1^N+\zeta_2^N). \tag{274}$$

Note that $N \le N^2$ for $N \ge 1$, so:

$$|\nabla^2 e_N| \le \frac{2}{5}\rho^3(\zeta_1+\zeta_2)^3 e^{(\zeta_1+\zeta_2)(1+\rho)} N^2\frac{(\rho/2)^N}{N!}(\zeta_1^N+\zeta_2^N). \tag{275}$$

The constant $\frac{2}{5}\rho^3(\zeta_1+\zeta_2)^3$ is bounded by $\mathcal{K} = \frac{8}{5}\rho^2(\zeta_1+\zeta_2)^2 e^{(\zeta_1+\zeta_2)(1+\rho)}$ because:

$$\frac{2}{5}\rho^3(\zeta_1+\zeta_2)^3 \le \frac{8}{5}\rho^2(\zeta_1+\zeta_2)^2\cdot\rho(\zeta_1+\zeta_2) \le \frac{8}{5}\rho^2(\zeta_1+\zeta_2)^2 e^{(\zeta_1+\zeta_2)(1+\rho)}, \tag{276}$$

since $\rho(\zeta_1+\zeta_2) \le e^{\rho(\zeta_1+\zeta_2)} \le e^{(\zeta_1+\zeta_2)(1+\rho)}$ (as $e^x \ge x$ for $x \ge 0$). For $N=0$, a separate bound (omitted) confirms the inequality. Thus:

$$|R_N(x,y)| \le \mathcal{K}\frac{(\rho/2)^N}{N!}(\zeta_1^N+\zeta_2^N)N^2, \quad \mathcal{K} = \frac{8}{5}\rho^2(\zeta_1+\zeta_2)^2 e^{(\zeta_1+\zeta_2)(1+\rho)}. \tag{277}$$

□

6. Illustrative Examples

Example 1. Consider the following equation,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 2\pi^2\sin(\pi x)\sin(\pi y) \quad \text{in} \tag{278}$$

governed by the homogeneous boundary conditions:

$$\begin{aligned} u(x,0) &= 0, \quad 0 < x \le 1, \\ u(0,y) &= 0, \quad 0 < y \le 1 \end{aligned} \tag{279}$$

while the corresponding exact solution is $u(x,y) = \sin(\pi x)\sin(\pi y)$.

Table 1: The AEs of Example 1 at $N = 16$

x	$y = 0.1$	$y = 0.3$	$y = 0.5$	$y = 0.7$	$y = 0.9$
0.1	6.93889×10^{-17}	0	0	0	3.88578×10^{-16}
0.2	0	1.11022×10^{-16}	0	2.22045×10^{-16}	7.77156×10^{-16}
0.3	5.55112×10^{-17}	1.11022×10^{-16}	1.11022×10^{-16}	3.33067×10^{-16}	9.99201×10^{-16}
0.4	5.55112×10^{-17}	2.22045×10^{-16}	1.11022×10^{-16}	1.11022×10^{-16}	1.16573×10^{-15}
0.5	5.55112×10^{-17}	3.33067×10^{-16}	1.11022×10^{-16}	0	1.33227×10^{-15}
0.6	0	2.22045×10^{-16}	4.44089×10^{-16}	5.55112×10^{-16}	1.11022×10^{-15}
0.7	1.11022×10^{-16}	1.11022×10^{-16}	1.11022×10^{-16}	1.11022×10^{-16}	9.99201×10^{-16}
0.8	3.33067×10^{-16}	8.88178×10^{-16}	9.99201×10^{-16}	8.32667×10^{-16}	1.13798×10^{-15}
0.9	4.71845×10^{-16}	1.02696×10^{-15}	1.33227×10^{-15}	1.13798×10^{-15}	8.32667×10^{-16}

Table 2: Maximum Absolute Errors and their Positions for Different Values of N of Example 1

N	4	8	12	16
Maximum AE	1.96875×10^{-5}	6.38593×10^{-10}	5.66214×10^{-15}	1.38778×10^{-15}
Position	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.8)

Example 2. Consider the following equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2(y(1-y) + x(1-x)) \tag{280}$$

governed by the homogeneous boundary conditions:

$$\begin{aligned} u(x,0) = 0, \quad 0 < x \le 1, \\ u(0,y) = 0, \quad 0 < y \le 1 \end{aligned} \tag{281}$$

while the corresponding exact solution is $u(x,y) = x(1-x)y(1-y)$.

Remark 3. Upon Implementing our method to solve the example, we found that exact and approximate solutions are exactly coincided at $N = 0$ with $c_{00} = \frac{1}{4}$, which align perfectly with the exactness of spectral representation of polynomial solutions, where the error theoretically will be zero whenever N is greater than or equal to the degree of polynomial solution.[10]

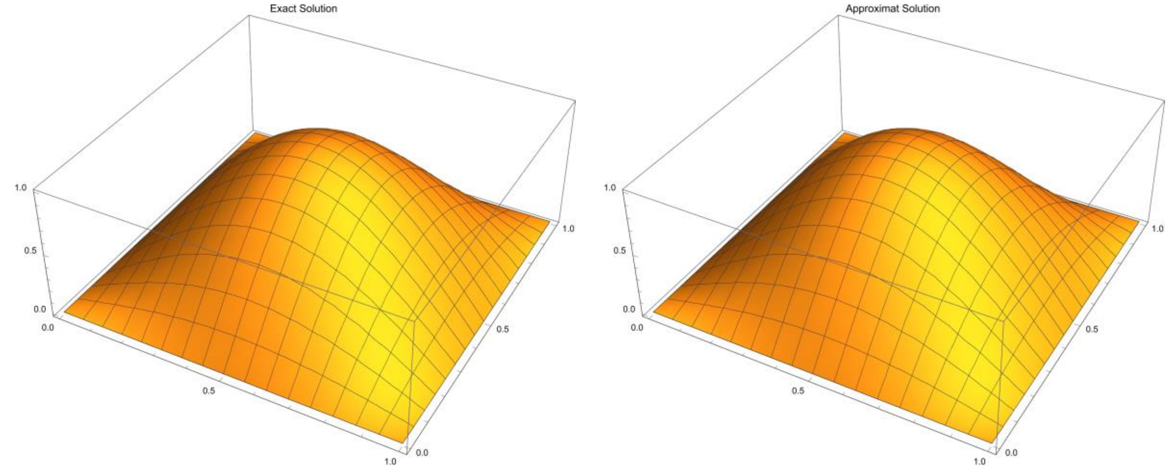
Example 3. Consider the following equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2e^{x+y}(3xy - x^2y - xy^2 - x^2y^2) \tag{282}$$

governed by the homogeneous boundary conditions:

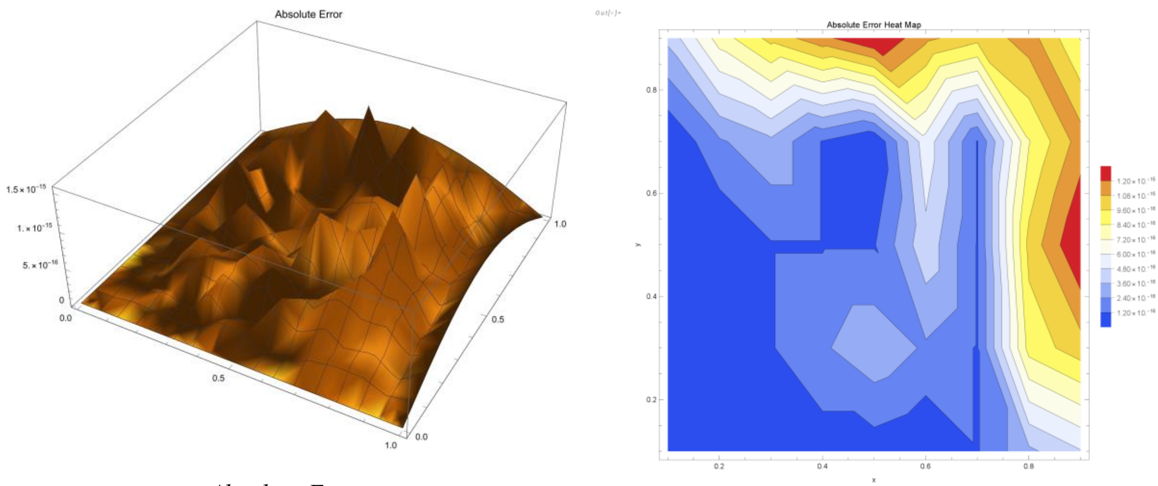
$$\begin{aligned} u(x,0) = 0, \quad 0 < x \le 1, \\ u(0,y) = 0, \quad 0 < y \le 1 \end{aligned} \tag{283}$$

while the corresponding exact solution is $u(x,y) = x(1-x)y(1-y)e^{x+y}$.



Exact Solution *Approximate Solution*

Figure 1: Solution Comparison for Example 1 with $N = 16$



Absolute Error *Absolute Errors Heat Map*

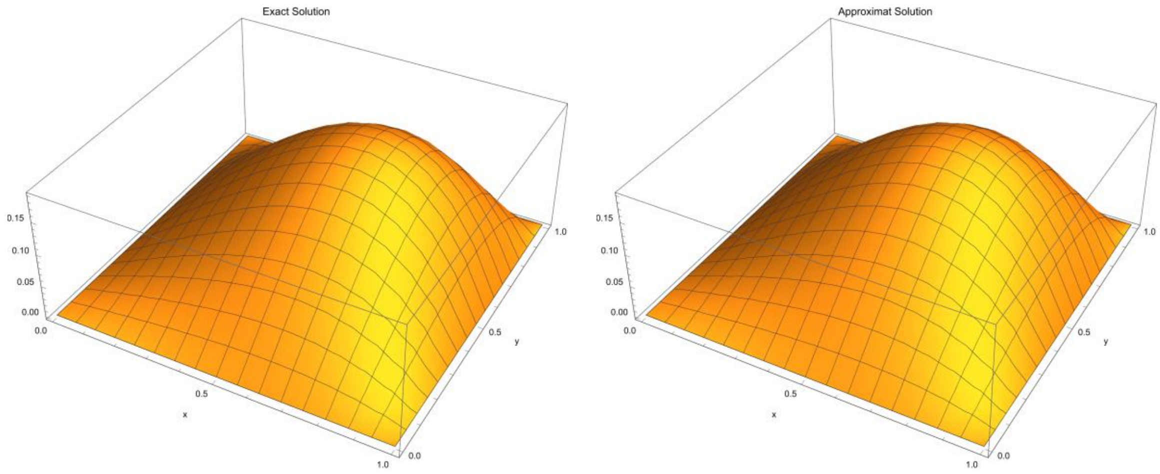
Figure 2: Error Analysis for Example 1 with $N = 16$

Table 5: The AEs of Example 3 at $N = 12$

x	$y = 0.3$	$y = 0.5$	$y = 0.7$	$y = 0.9$
0.1	6.93889×10^{-18}	6.93889×10^{-18}	2.08167×10^{-17}	1.38778×10^{-17}
0.2	6.93889×10^{-18}	0	0	2.08167×10^{-17}
0.3	2.77556×10^{-17}	1.38778×10^{-17}	2.77556×10^{-17}	4.16334×10^{-17}
0.4	2.77556×10^{-17}	0	0	5.55112×10^{-17}
0.5	0	0	2.77556×10^{-17}	9.71445×10^{-17}
0.6	5.55112×10^{-17}	2.77556×10^{-17}	0	8.32667×10^{-17}
0.7	1.38778×10^{-17}	8.32667×10^{-17}	5.55112×10^{-17}	2.77556×10^{-17}
0.8	0	2.77556×10^{-17}	0	4.16334×10^{-17}
0.9	4.16334×10^{-17}	8.32667×10^{-17}	2.77556×10^{-17}	6.93889×10^{-17}

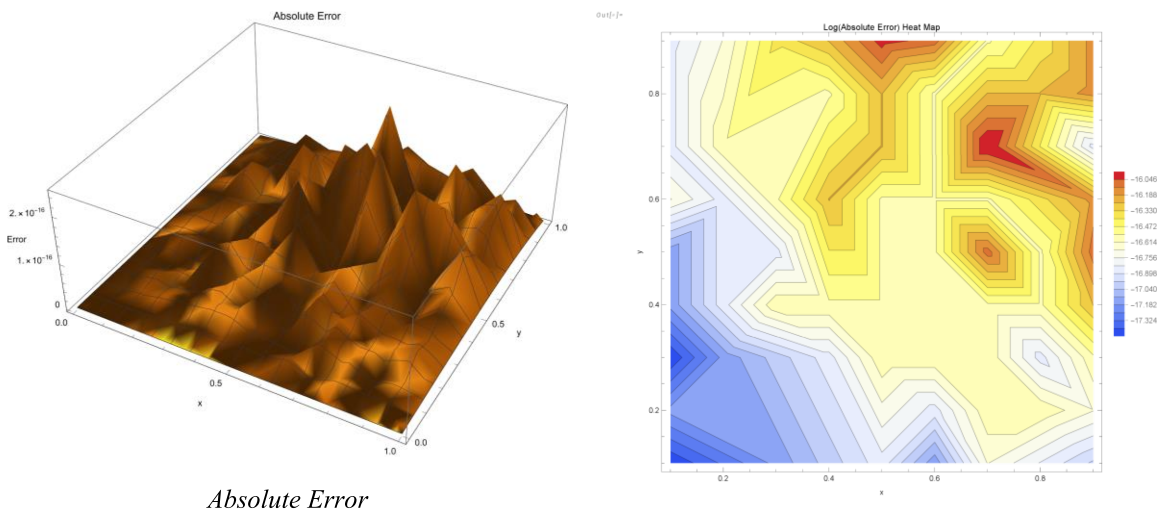
Table 6: Maximum Absolute Errors and their Positions for Different Values of N of Example 3

N	3	6	9	12
Maximum AE	4.11174×10^{-5}	3.18405×10^{-9}	6.53366×10^{-14}	1.11022×10^{-16}
Position	(0.5,0.5)	(0.6,0.6)	(0.5,0.5)	(0.7,0.7)



Exact Solution *Approximate Solution*

Figure 5: Solution Comparison for Example 3 with $N = 12$



Absolute Error

Logarithm of Absolute Errors Heat Map

Figure 6: Error Analysis for Example 3 with $N = 12$

7. Conclusion

In this paper, we created and tested a spectral Galerkin method that uses modified Lucas polynomials to solve the 2D Poisson equation with set boundary conditions. The main thing new in our method is that we systematically use Lucas polynomials multiplied by $x(1-x)$ to meet boundary conditions as basis functions in a spectral setup. As far as we know, this is the first use of Lucas polynomial bases for elliptic boundary value problems in the spectral Galerkin setting.

The tensor-product structure of our basis functions was very helpful from a computing point of view. The algebraic system that comes out of it has a sparse and organized form, which we used through matrix constructions based on Kronecker products. This not only makes things easier to set up but also allows for quick assembly of the stiffness and mass matrices by using the fact that the problem can be separated in each spatial dimension. The three-term recurrence relation that Lucas polynomials follow helps with computer speed, allowing for quick calculation of polynomial values and their derivatives.

Our tests show that the method is surprisingly accurate without needing a lot of computing power. By using a small number of nodes, usually polynomial degrees below $N = 20$, our method gives very accurate solutions to the 2D Poisson equation, often getting close to the best possible accuracy in just a few seconds on normal computer hardware. The spectral convergence we saw is very interesting: for smooth source terms, the error drops quickly as the polynomial degree goes up, which is what we expect from spectral methods. This quick convergence suggests that Lucas polynomial bases can do as well as more common choices like Chebyshev or Legendre polynomials, at least for the types of problems we looked at.

The success of this method points to several directions for study in the future. One thing to do would be to look at more general elliptic operators, including problems with changing factors and systems of PDEs. Dealing with non-set or Robin boundary conditions in the Lucas polynomial setting is also worth looking at, maybe by changing the basis functions or adding lifting functions. Also, while we looked at the unit square area, it's easy to expand this to more general rectangular areas, and creating domain decomposition or spectral element versions could allow us to use this for more difficult shapes.

From a wider view, this study shows how important it is to test other polynomial bases in spectral methods. While Chebyshev and Legendre polynomials have been the main choice for good reasons, our results suggest that other polynomial families could be good alternatives, especially when their specific features work well with the problem's structure. The easy recurrence relation and relation to Chebyshev polynomials of the second kind make Lucas polynomials a good choice that should be studied more in mathematical computing.

In short, the modified shifted Lucas polynomial basis we present here gives a quick and correct setup for spectral Galerkin solutions of the Poisson equation. The combination of meeting boundary conditions automatically, good computing structure, and great convergence makes this method a helpful addition to the spectral methods toolkit. We hope that this paper will encourage more study into the use of Lucas polynomials and other less common polynomial bases in numerical methods for partial differential equations.

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