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RESEARCH PAPER

A spatial decay estimates for a thermoelastic Cosserat body without energy dissipation

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Abstract

A cylinder with a prismatic structure is considered and this is "filled" with a thermoelastic material with the Cosserat structure. It is supposed that on the lateral surface of the cylinder there are no body forces, or body couple nor heat supply. But, on the base of cylinder, a microrotation is given, which is time-dependent, a displacement, which is also time-dependent, and a thermal displacement is also prescribed. All these loads are assumed to be harmonic functions in time and from their corroborated action, the movement of the body under consideration is induced. We will define a measure associated with the vibration that corresponds to the steady state. Assuming that there is a certain critical frequency and we can suppose that any excitation frequency is lower than the critical one, we will be able to obtain an estimation regarding the spatial decay.

Keywords: Thermoelastic Cosserat Media; Micropolar Vibration Analysis; Spatial Attenuation in Continuum Media; Non-classical Elasticity Theories; Critical Frequency Effects; Thermomechanical Wave Propagation; Thermoelasticity; Cosserat Theory; Spatial Decay; Microrotation; Vibration Analysis

1. Main text

In the classical theory of thermoelasticity, heat flow is based on Fourier's law, which implies energy dissipation and allows heat to be transmitted in the form of thermal waves at infinite speed. In contrast to the classical theory thermoelasticity, the theory of thermoelastic bodies without energy dissipation permits the transmission of heat as thermal waves at finite speed.

The theory of this type of body that we are addressing in our study was introduced by the French Cosserat brothers since 1909. It is known, the deformation in this theory is evaluated by a vector of displacement and a vector of independent rotation. In this theory the heat flow is described with finite propagation speed. There are other hyperbolic theories that describe this type of propagation which some researchers call theories of second sound. Some of these theories are described in the study [1] by Chandrasekharaiah. It is considered that the first approaches in the theory of thermoelastic bodies without energy dissipation were made by Green and Naghdi [2]. They introduced a so-called thermal displacement, with regards to the usual temperature. In Green and Naghdi [3] the same authors postulated a general entropy balance. Nappa in [4] approaces the linear theory of homogeneous and

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isotropic materials without energy dissipation and obtain some spatial energy bounds. In the same context of thermoelasticity without energy dissipation Chandrasekharaiah [5] obtained the uniqueness of solution for the mixed problem, Iesan [6] deduced some continuous dependence estimates, while Quintanilla [7] addresses the existence of the solution.

In this present study the spatial behavior of some vibrations, which are harmonic in time, within the linear theory of thermoelasticity, for Cosserat bodies without dissipation energy.

Several "a priori" estimations of the amplitude of vibrations, which are harmonic in time, are demonstrated with the help of some auxiliary identities. The evolution of the vibration amplitude is related to the base, which is exposed to the loads mentioned above, respecting the condition that the vibration frequency is higher than the declared critical frequency.

Chirita in [8] studied spatial evolution vibrations in the linear classical thermoelasticity. He uses the same technique as that exhibited by Flavin and Knops in [9], in the case of low frequency. The exponential estimates are obtained provided that the constitutive coefficients are positive definite tensors.

Ciarletta proposed in [10] a theory of thermoelastic micropolar bodies that, because it is without energy dissipation, can assure the propagation of the waves, of thermal type, at a finite speed.

Some concrete and practical issues related to bodies with generalized structures can be found in the papers [11-20].

Regarding the plane of our study, as usual, first are written down the main differential equations, initial data and boundary conditions for the mixed problem, considered within context of thermoelastic Cosserat bodies without energy dissipation. After that, there are proven certain differential identities for certain integrals in cross-sectional domains. With the help of these identities, some estimates are obtained that describe the evolution of the amplitude evolves, regarding the distance until the base that is exciting. The condition is respected that the frequency of vibrations must be at least as frequent as is considered critical. be greater than a certain critical value.

2. Main equations and conditions

We will work in a domain D of three-dimensional Euclidian space, which is occupied, in its initial configuration by a Cosserat homogeneous material. The boundary of D is denoted by ∂D and $\overline{D}$ is the notation for the closure of $D, \overline{D} = D \cup \partial D$.

A system of fixed rectangular Cartesian axes is used, and vectors and tensors are denoted using boldface notation. All points in D are depending on spatial variable with the components x_m and the temporal variable $t \in [0, \infty)$. In the case there is no likehood of confusion the time argument and the spatial argument of a function can be omitted. The differentiation with respect to time t and the diferentiation with respect to the variable x_m is denoted by the subscript m preceded by a comma, $f, m = \frac{\partial f}{\partial x_m}$.

As in [5], the basic equations governing the theory of thermoelasticity of Cosserat bodies without energy dissipation are:

- the motion equations:

$$\begin{aligned} &\tau_{mn,n} + \rho f_m = \rho \ddot{v}_m, \\ &\sigma_{mn,n} + \epsilon_{mnk}\tau_{nk} + \rho g_m = I_{mn}\ddot{\phi}_n; \end{aligned} \tag{1}$$

- the energy equation:

$$\rho \dot{S} = \frac{\rho}{T_0} r - q_{m,m}. \tag{2}$$

All these equations are satisfied for any $(t, x) \in (0, \infty) \times D$.

The constitutive equations, in the case that the solid, in its initial reference has a center of symmetry at each point, are defined for any $(t, x) \in (0, \infty) \times D$ by:

$$\begin{aligned} \tau_{mn} &= C_{mnkl}e_{kl} + B_{mnkl}\varepsilon_{kl} - D_{mn}T, \\ \sigma_{mn} &= B_{mnkl}e_{kl} + A_{mnkl}\varepsilon_{kl} - E_{mn}T, \\ \rho S &= D_{mn}e_{mm} + E_{mn}\varepsilon_{mn} + \frac{a}{T_0}T, \\ q_m &= -\frac{1}{T_0}\kappa_{mn}\beta_m. \end{aligned} \tag{3}$$

The tensors of deformation, e_{mm} and ε_{mn}, which appear in equations (3) are defined by means of the following kinematic equations:

$$e_{mm} = v_{l,k} + \epsilon_{nmk}\phi_k, \ \ \varepsilon_{mn} = \phi_{l,k}, \tag{4}$$

for all $(t,x) \in [0,\infty) \times D$.

In order to complete the above system of equations, we add the following heat flow equation:

$$\dot{\beta}_m = T_{,m}, \tag{5}$$

which is satisfied by any $(t,x) \in (0,\infty) \times D$.

The notations used in the above equations are: (v_m) - the displacement vector, (ϕ_i) - the microrotatia vector, (τ_{mn}) - the tensor of stress, (σ_{mn}) - the couple tensor of stress, (q_m) - the vector of the heat conduction, S - the entropy per unit mass, ρ - the density mass, constant in the reference state, T - the temperature, which in the reference state has constant value T_0, I_{mn} - the inertia tensor, β_m - the vector of the thermal displacement gradient, f_m - the vector of the external body force, g_m - the vector of the external body couple, r - external rate of the heat supply and ϵ_{mnk} represents the alternating symbol.

The coefficients from equations (3), i.e., C_{mnkl}, B_{mnkl}, A_{mnkl}, D_{mn}, E_{mn}, c and κ_{mn} are constant material characteristics which satisfy the next symmetry relations:

$$C_{mnkl} = C_{klmn}, \ \ A_{mnkl} = A_{klmn}, \ \ I_{mn} = I_{nm} \ ; \kappa_{mn} = \kappa_{nm}. \tag{6}$$

In order to obtain the above constitutive equations, it is used the free energy Ψ defined by:

$$\rho\Psi = \frac{1}{2}C_{mnkl}e_{mm}e_{kl} + B_{mnkl}e_{mm}\varepsilon_{kl} + \frac{1}{2}A_{mnkl}\varepsilon_{mn}\varepsilon_{kl} - \\ -D_{mn}e_{mm}T - E_{mn}\varepsilon_{mn}T - \frac{a}{2T_0}T^2 + \frac{a}{2T_0}\kappa_{mn}\tau_{,m}\tau_{,n}. \tag{7}$$

Above we denoted by a the specific heat and τ is the notation for the thermal displacement with regard to the variation of the temperature, namely:

$$\dot{\tau} = T. \tag{8}$$

Considering the constitutive equations (3) and the kinematic equations (4), from motion equations (1) and the energy equation (2), it is obtained the following system of differential equations relative to the displacements v_m, the microrotations ϕ_m and the thermal displacements τ:

$$\begin{aligned} &\left[C_{mnkl}\left(v_{l,k} + \epsilon_{klj}\phi_j\right) + B_{mnkl}\phi_{l,k} - D_{mn}\dot{\tau}\right]_{,n} + \rho f_m = \rho\ddot{v}_m, \\ &\left[B_{mnkl}\left(v_{l,k} + \epsilon_{klj}\phi_j\right) + A_{mnkl}\phi_{l,k} - E_{mn}\dot{\tau}\right]_{,n} + \\ &+\epsilon_{mnj}\left[C_{njkl}\left(v_{l,k} + \epsilon_{kli}\phi_i\right) + B_{njkl}\phi_{l,k} - D_{nj}\dot{\tau}\right] + \rho g_m = I_{mn}\ddot{\phi}_n, \\ &\frac{1}{T_0}\left(\kappa_{mn}\tau_{,n}\right)_{,m} - D_{mn}\left(\dot{v}_{l,k} + \epsilon_{nmk}\dot{\phi}_k\right) - E_{mn}\dot{\phi}_{l,k} + \frac{\rho}{T_0}r = \frac{a}{T_0}\ddot{\tau}, \end{aligned} \tag{9}$$

that are satisfied for all $(t,x) \in (0,\infty) \times D$.

3. Preliminary results

A cross-section Ω of a prismatic cylinder is considered, having a boundary of section denoted by ∂D, which is supposed be piecewise continuously differentiable. The Cartesian rectangular system of axes is chosen so that its origin is in the center of the cylinder base. Also, to simplify writing, the positive x_3 $-$axis will be denoted by z and it is directed along the cylinder. The length of the cylinder is considered to be L, so that the lateral border of cylinder is then $\Sigma = \partial\Omega \times [0,L]$. The material from the prismatic cylinder is a homogeneous and anisotropic Cosserat body.

It is supposed that that on the lateral boundary surface of cylinder there is null body force, null couple force, null heat supply and, zero displacements, zero microrotations and zero thermal displacements, that is the cylinder is free of load on the lateral boundary surface. The displacements, microrotations and thermal displacements appear over the base of cylinder and are assumed harmonic in time.

Therefore, we need to add the following boundary relations on the lateral surface

$$v_m(t,x) = 0, \ \ \phi_m(t,x) = 0, \ \ \tau(t,x) = 0, \ \ (t,x) \in (0,\infty) \times \Sigma, \tag{10}$$

and the next boundary relations on the base:

$$v_m(x_1,x_2,0,t) = \tilde{v}_m(x_1,x_2)e^{\iota\omega t},\ \phi_m(x_1,x_2,0,t) = \tilde{\phi}_m(x_1,x_2)e^{\iota\omega t},$$
$$\tau(x_1,x_2,0,t) = \tilde{\tau}(x_1,x_2)e^{\iota\omega t},\ (x_1,x_2) \in D(0),\ t > 0, \tag{11}$$

in which $\tilde{u}_m(x_1,x_2)$, $\tilde{\phi}_m(x_1,x_2)$ and $\tilde{\tau}(x_1,x_2)$ are given smooth functions, $\omega > 0$ is a given constant and ι is the notation for the complex unit.

Due to the loads considered in (11), inside the cylinder appear some vibrations which are harmonic in time and have the following form:

$$v_m(x_1,x_2,z,t) = V_m(x_1,x_2,z)e^{\iota\omega t},\ \phi_m(x_1,x_2,z,t) = \Phi_m(x_1,x_2,z)e^{\iota\omega t},$$
$$\tau(x_1,x_2,z,t) = \Theta(x_1,x_2,z)e^{\iota\omega t},\ (x_1,x_2,z,t) \in D \times (0,\infty). \tag{12}$$

The components of the vibrations amplitude, that is, (V_m, Φ_m, Θ), satisfy the next system of equations:

$$\left[C_{mnkl}\left(V_{l,k} + \epsilon_{klj}\Phi_j\right) + B_{mnkl}\Phi_{l,k} - \iota\omega D_{mn}\Theta\right]_{,n} + \rho\omega^2 V_m = 0,$$
$$\left[B_{mnkl}\left(V_{l,k} + \epsilon_{klj}\Phi_j\right) + A_{mnkl}\Phi_{l,k} - \iota\omega E_{mn}\Theta\right]_{,n} +$$
$$+ \epsilon_{mnj}\left[C_{njkl}\left(V_{l,k} + \epsilon_{kli}\Phi_i\right) + B_{njkl}\Phi_{l,k} - \iota\omega D_{nj}\Theta\right] + I_{mn}\omega^2\Phi_n = 0, \tag{13}$$
$$\left(\frac{1}{T_0}\kappa_{mn}\Theta_{,n}\right)_{,m} - \iota\omega D_{mn}\left(V_{n,m} + \epsilon_{nmk}\Phi_k\right) - \iota\omega E_{mn}\Phi_{n,m} + \frac{a}{T_0}\omega^2\Theta = 0.$$

For the lateral boundary we have the conditions:

$$V_m(x) = 0,\ \Phi_m(x) = 0,\ \Theta(x) = 0,\ x \in \Sigma, \tag{14}$$

while the base boundary relations get the form:

$$V_m(x_1,x_2,0) = \tilde{V}_m(x_1,x_2),\ \Phi_m(x_1,x_2,0) = \tilde{\Phi}_m(x_1,x_2),$$
$$\Theta(x_1,x_2,0) = \tilde{\Theta}(x_1,x_2),\ (x_1,x_2) \in D(0). \tag{15}$$

If the cylinder is finite, a condition on its upper base is required, i.e., on $D(L)$. Chirita in [4] and Ciarletta in [5] studied the spatial behavior of the amplitude for a forced oscillation in the case of rhombic thermoelastic materials, provided that the excitation frequency is lower than a certain critical frequency.

The main goal of this study is to estimate how the amplitude of oscillation evolves with axial distance from the point of excitation.

In what follows we consider the mixed problem denoted by P and consists of the system of differential equations (13), the lateral boundary relations (14) and the conditions on the base boundary (15). The following notation is used: $\mathcal{V}_{l,k} = V_{l,k} + \epsilon_{klm}\Phi_m$.

The four auxiliary identities which we demonstrate in the following will prove to be useful in obtaining the main result of our study.

Theorem 1. *If the ordered array* (V_m, Φ_m, Θ) *is a solution of the boundary value problem* P*, then there are satisfied the following four equalities:*

$$2\int_{D(z)}\left\{C_{mnkl}\mathcal{V}_{l,k}\bar{\mathcal{V}}_{l,k} + A_{mnkl}\Phi_{l,k}\bar{\Phi}_{l,k} +\right.$$
$$\left.+ B_{mnkl}\left[\mathcal{V}_{l,k}\bar{\Phi}_{l,k} + \bar{\mathcal{V}}_{l,k}\Phi_{l,k}\right] - \rho\omega^2 V_m\bar{V}_m - I_{mn}\omega^2\Phi_m\bar{\Phi}_n\right\}dA +$$
$$+ \int_{D(z)}\left\{\iota\omega D_{mn}\left(\bar{\Theta}\mathcal{V}_{l,k} - \Theta\bar{\mathcal{V}}_{l,k}\right) + \iota\omega E_{mn}\left(\bar{\Theta}\Phi_{m,n} - \Theta\bar{\Phi}_{m,n}\right)\right\}dA =$$
$$= \frac{d}{dz}\int_{D(z)}\left\{\left[C_{3mkl}\mathcal{V}_{l,k} + B_{3mkl}\Phi_{l,k} - \iota\omega D_{3m}\Theta\right]\bar{V}_m\right\}dA + \tag{16}$$

$$+\frac{d}{dz}\int_{D(z)}\left\{\left[C_{3mkl}\bar{\mathcal{V}}_{l,k}+B_{3mkl}\bar{\Phi}_{l,k}+\iota\omega D_{3m}\bar{\Theta}\right]V_m\right\}dA+$$

$$+\frac{d}{dz}\int_{D(z)}\left\{\left[B_{3mkl}\mathcal{V}_{l,k}+A_{3mkl}\Phi_{l,k}-\iota\omega E_{3m}\Theta\right]\bar{\Phi}_m\right\}dA+$$
$$+\frac{d}{dz}\int_{D(z)}\left\{\left[B_{3mkl}\bar{\mathcal{V}}_{l,k}+A_{3mkl}\bar{\Phi}_{l,k}+\iota\omega E_{3m}\bar{\Theta}\right]\Phi_m\right\}dA;$$

$$\begin{aligned}&\int_{D(z)}\left[\iota\omega D_{mn}\left(\bar{\Theta}\mathcal{V}_{l,k}+\Theta\bar{\mathcal{V}}_{l,k}\right)+\iota\omega E_{mn}\left(\bar{\Theta}\Phi_{m,n}+\Theta\bar{\Phi}_{m,n}\right)\right]dA=\\&=\frac{d}{dz}\int_{D(z)}\left\{\left[C_{3mkl}\bar{\mathcal{V}}_{l,k}+B_{3jmn}\bar{\Phi}_{l,k}+\iota\omega D_{3m}\bar{\Theta}\right]V_m\right\}dA-\\&-\frac{d}{dz}\int_{D(z)}\left\{\left[C_{3mkl}\mathcal{V}_{l,k}+B_{3mkl}\Phi_{l,k}-\iota\omega D_{3m}\Theta\right]\bar{V}_m\right\}dA+\\&+\frac{d}{dz}\int_{D(z)}\left\{\left[B_{3mkl}\bar{\mathcal{V}}_{l,k}+A_{3mkl}\bar{\Phi}_{l,k}+\iota\omega E_{3m}\bar{\Theta}\right]\Phi_m\right\}dA-\\&-\frac{d}{dz}\int_{D(z)}\left\{\left[B_{3mkl}\mathcal{V}_{k,l}+A_{3mkl}\Phi_{l,k}-\iota\omega E_{3m}\Theta\right]\bar{\Phi}_m\right\}dA;\end{aligned}\tag{17}$$

$$\begin{aligned}&2\int_{D(z)}\frac{1}{T_0}\left(\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n}-c\omega^2\Theta\bar{\Theta}\right)dA+\int_{D(z)}\iota\omega E_{mn}\left(\Phi_{l,k}\bar{\Theta}-\bar{\Phi}_{l,k}\Theta\right)dA+\\&\quad+\int_{D(z)}\iota\omega D_{mn}\left(\mathcal{V}_{l,k}\bar{\Theta}-\bar{\mathcal{V}}_{l,k}\Theta\right)dA=\\&\qquad=\frac{d}{dz}\int_{D(z)}\frac{1}{T_0}\kappa_{33}\left(\bar{\Theta}T_{,3}+\Theta\bar{\Theta}_{,3}\right)dA;\end{aligned}\tag{18}$$

$$\begin{aligned}&\int_{D(z)}\iota\omega D_{mn}\left(\mathcal{V}_{l,k}\bar{\Theta}+\mathcal{V}_{l,k}\Theta\right)dA+\\&\quad+\int_{D(z)}\iota\omega E_{mn}\left(\bar{\Phi}_{l,k}\Theta+\Phi_{l,k}\bar{\Theta}\right)dA=\frac{d}{dz}\int_{D(z)}\frac{1}{T_0}\kappa_{3m}\left(\bar{\Theta}\Theta_{,m}-\Theta\bar{\Theta}_{,m}\right)dA.\end{aligned}\tag{19}$$

Above, for a field f, its complex conjugate was denoted by $\bar{f}$.

Proof. With the help of equations (13) 1 and (13) 2 we can obtain the next equality:

$$\begin{aligned}&\left\{\left[C_{mnkl}\mathcal{V}_{l,k}+B_{mnkl}\Phi_{l,k}-\iota\omega D_{mn}\Theta\right]_{,n}+\rho\omega^2V_m\right\}\bar{V}_m+\\&+\left\{\left[C_{mnkl}\bar{\mathcal{V}}_{l,k}+B_{mnkl}\bar{\Phi}_{l,k}+\iota\omega D_{mn}\bar{\Theta}\right]_{,n}+\rho\omega^2\bar{V}_m\right\}V_m+\\&+\left[B_{mnkl}\mathcal{V}_{l,k}+A_{mnkl}\Phi_{l,k}-\iota\omega E_{mn}\Theta\right]_{,n}\bar{\Phi}_m+\\&+\epsilon_{mnj}\left[C_{njkl}\mathcal{V}_{l,k}+B_{njkl}\Phi_{l,k}-\iota\omega D_{nj}T\right]\bar{\Phi}_m+I_{mn}\omega^2\Phi_m\bar{\Phi}_n+\\&+\left[B_{mnij}\bar{\mathcal{V}}_{l,k}+A_{mnkl}\bar{\Phi}_{l,k}+\iota\omega E_{mn}\bar{\Theta}\right]_{,n}\Phi_m+\\&+\epsilon_{mnj}\left[C_{njkl}\bar{\mathcal{V}}_{l,k}+B_{njkl}\bar{\Phi}_{l,k}+\iota\omega D_{nj}\bar{\Theta}\right]\Phi_m+I_{mn}\omega^2\Phi_m\bar{\Phi}_n=0.\end{aligned}\tag{20}$$

After simple calculations, this relation receives the form:

$$\begin{aligned}&2\left\{C_{mnkl}\mathcal{V}_{l,k}\bar{\mathcal{V}}_{l,k}+B_{mnkl}\left[\mathcal{V}_{l,k}\bar{\Phi}_{l,k}+\bar{\mathcal{V}}_{l,k}\Phi_{l,k}\right]+\right.\\&\left.+A_{mnkl}\Phi_{l,k}\bar{\Phi}_{l,k}-\rho\omega^2V_m\bar{V}_m-I_{mn}\omega^2\Phi_m\bar{\Phi}_n\right\}+\\&+\iota\omega D_{mn}\left(\bar{\Theta}\mathcal{V}_{l,k}-T\bar{\mathcal{V}}_{l,k}\right)+\iota\omega E_{mn}\left(\bar{\Theta}\Phi_{m,n}-T\bar{\Phi}_{m,n}\right)=\\&=\left\{\left[C_{mnkl}\mathcal{V}_{l,k}+B_{mnkl}\Phi_{l,k}-\iota\omega D_{mn}\Theta\right]\bar{V}_m\right\}_{,n}+\end{aligned}\tag{21}$$

$$+\{[C_{mnkl}\bar{\mathcal{V}}_{l,k} + B_{mnkl}\bar{\Phi}_{l,k} + \iota\omega D_{mn}\bar{\Theta}]V_m\}_{,n} +$$

$$+\{[B_{mnkl}\mathcal{V}_{l,k} + A_{mnkl}\Phi_{l,k} - \iota\omega E_{mn}\Theta]\bar{\Phi}_m\}_{,n} +$$

$$+\{[B_{mnkl}\bar{\mathcal{V}}_{l,k} + A_{mnkl}\bar{\Phi}_{l,k} + \iota\omega E_{mn}\bar{\Theta}]\Phi_m\}_{,n}.$$

Now, we integrate the equality (21) over the domain $D(z)$. Applying the theorem of divergence and using the lateral relations from (14), we obtain the equality (16).

Taking into account once again the equations (13) 1 and (13) 2, we get the next equality:

$$\begin{aligned}&\left\{[C_{mnkl}\mathcal{V}_{l,k} + B_{mnkl}\Phi_{l,k} - \iota\omega D_{mn}\Theta]_{,n} + \rho\omega^2 V_m\right\}\bar{V}_m -\\&-\left\{[C_{mnkl}\bar{\mathcal{V}}_{l,k} + B_{mnkl}\bar{\Phi}_{l,k} + \iota\omega D_{mn}\bar{\Theta}]_{,n} + \rho\omega^2 \bar{V}_m\right\}V_m +\\&+[B_{mnkl}\mathcal{V}_{l,k} + A_{mnkl}\Phi_{l,k} - \iota\omega E_{mn}\Theta]_{,n}\bar{\Phi}_m +\\&+\epsilon_{mnj}[C_{njkl}\mathcal{V}_{l,k} + B_{njkl}\Phi_{l,k} - \iota\omega D_{nj}T]\bar{\Phi}_m + I_{mn}\omega^2\Phi_m\bar{\Phi}_n -\\&-[B_{mnkl}\bar{\mathcal{V}}_{l,k} + A_{mnkl}\bar{\Phi}_{l,k} + \iota\omega E_{mn}\bar{\Theta}]_{,n}\Phi_m -\\&-\epsilon_{mnj}[C_{njkl}\bar{\mathcal{V}}_{l,k} + B_{njkl}\bar{\Phi}_{l,k} + \iota\omega D_{nj}\bar{\Theta}]\Phi_m - I_{mn}\omega^2\Phi_m\bar{\Phi}_n = 0.\end{aligned} \tag{22}$$

After simple calculations, this relation receives the form:

$$\begin{aligned}&\iota\omega D_{mn}(\bar{\Theta}\mathcal{V}_{l,k} + \Theta\bar{\mathcal{V}}_{l,k}) + \iota\omega E_{mn}(\bar{\Theta}\Phi_{m,n} + T\bar{\Phi}_{m,n}) =\\&= +\{[C_{mnkl}\bar{\mathcal{V}}_{l,k} + B_{mnkl}\bar{\Phi}_{l,k} + \iota\omega D_{mn}\bar{\Theta}]V_m\}_{,n} -\\&-\{[C_{mnkl}\mathcal{V}_{l,k} + B_{mnkl}\Phi_{l,k} - \iota\omega D_{mn}\Theta]\bar{V}_m\}_{,n} +\\&+\{[B_{mnkl}\bar{\mathcal{V}}_{l,k} + A_{mnkl}\bar{\Phi}_{l,k} + \iota\omega E_{mn}\bar{\Theta}]\Phi_m\}_{,n} -\\&-\{[B_{mnkl}\mathcal{V} + A_{mnkl}\Phi_{l,k} - \iota\omega E_{mn}\Theta]\bar{\Phi}_m\}_{,n}.\end{aligned} \tag{23}$$

Now, we integrate equality (23) over the domain $D(z)$. Applying the theorem of divergence and using the lateral relations from (14), we arrive at the equality (17).

This time we consider the equation (13) 3, so that after similar calculations it is obtained the equality:

$$\begin{aligned}&\bar{\Theta}\left[\left(\frac{1}{T_0}\kappa_{mn}\Theta_{,n}\right)_{,m} - \iota\omega D_{mn}\mathcal{V}_{l,k} - \iota\omega E_{mn}\Phi_{l,k} + \frac{a}{T_0}\omega^2\Theta\right] +\\&+\Theta\left[\left(\frac{1}{T_0}\kappa_{mn}\bar{\Theta}_{,n}\right)_{,m} + \iota\omega D_{mn}\mathcal{V}_{l,k} + \iota\omega E_{mn}\bar{\Phi}_{l,k} + \frac{a}{T_0}\omega^2\bar{\Theta}\right] = 0.\end{aligned} \tag{24}$$

After simple calculations, this relation receives the form:

$$\begin{aligned}&\frac{2}{T_0}\left(\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n} - c\omega^2\Theta\bar{\Theta}\right) + \iota\omega D_{mn}(\mathcal{V}_{l,k}\bar{\Theta} - V_{l,k}\Theta) +\\&+\iota\omega E_{mn}(\Phi_{l,k}\bar{\Theta} - \bar{\Phi}_{l,k}\Theta) = \left[\frac{1}{T_0}\kappa_{mn}(\bar{\Theta}\Theta_{,n} + \Theta\bar{\Theta}_{,n})\right]_{,m}.\end{aligned} \tag{25}$$

Now, we integrate equality (25) over the domain $D(z)$. Applying the theorem of divergence and using the lateral relations from (14), we arrive at the equality (18).

At the end, we will use once again the equation (13) 3 to obtain:

$$\bar{\Theta}\left[\left(\frac{1}{T_0}\kappa_{mn}\Theta_{,n}\right)_{,m} - \iota\omega D_{mn}\mathcal{V}_{l,k} - \iota\omega E_{mn}\Phi_{l,k} + \frac{a}{T_0}\omega^2\Theta\right] -$$

$$-\Theta\left[\left(\frac{1}{T_0}\kappa_{mn}\bar{\Theta}_{,n}\right)_{,m}+\iota\omega D_{mn}\mathcal{V}_{l,k}+\iota\omega E_{mn}\bar{\Phi}_{l,k}+\frac{a}{T_0}\omega^2\bar{\Theta}\right]=0. \tag{26}$$

After simple calculations, this relation receives the form:

$$\iota\omega D_{mn}\left(\mathcal{V}_{l,k}\bar{\Theta}+\bar{\mathcal{V}}_{l,k}\Theta\right)+$$
$$+\iota\omega E_{mn}\left(\bar{\Phi}_{l,k}\Theta+\Phi_{l,k}\bar{\Theta}\right)=\left[\frac{1}{T_0}\kappa_{mn}\left(\bar{\Theta}\Theta_{,n}-\Theta\bar{\Theta}_{,n}\right)\right]_{,m}. \tag{27}$$

Finally, we integrate equality (27) over the domain $D(z)$. Applying the theorem of divergence and using the lateral relations from (14), we arrive at equality (19).

In this way, the proof of Theorem 1 is finished.

Two more auxiliary identities will be obtained in the next theorem. Our main result will also be based on these two auxiliary identities.

Theorem 2. *If the ordered array* (V_m,Φ_m,Θ) *is a solution of the boundary value problem* P*, then there are satisfied the following two equalities:*

$$\int_{D(z)}\left[C_{mnkl}\mathcal{V}_{l,k}\bar{\mathcal{V}}_{l,k}+A_{mnkl}\Phi_{n,m}\bar{\Phi}_{l,k}\right]dA+$$
$$+\int_{D(z)}\left\{B_{mnkl}\left[\mathcal{V}_{l,k}\bar{\Phi}_{m,n}+\bar{\mathcal{V}}_{l,k}\Phi_{m,n}\right]-3\omega^2\left(\rho V_m\bar{V}_m+I_{mn}\Phi_m\bar{\Phi}_n\right)\right\}dA$$
$$-2\iota\omega\int_{D(z)}\left\{D_{mn}\left(\Theta\bar{\mathcal{V}}_{l,k}-\bar{\Theta}\mathcal{V}_{l,k}\right)+E_{mn}\left(\Theta\bar{\Phi}_{n,m}-\bar{\Theta}\Phi_{n,m}\right)\right\}dA-$$
$$-\iota\omega\int_{D(z)}\left\{D_{mn}x_p\left(\Theta_{,p}\bar{\mathcal{V}}_{l,k}-\bar{\Theta}_{,p}\mathcal{V}_{l,k}\right)+E_{mn}x_p\left(\Theta_{,p}\bar{\Phi}_{n,m}-\bar{\Theta}_{,p}\Phi_{n,m}\right)\right\}dA$$
$$=-\frac{d}{dz}\int_{D(z)}\left\{\left[C_{3mkl}\mathcal{V}_{l,k}+B_{3mkl}\Phi_{l,k}-\iota\omega D_{3j}\Theta\right]x_p\bar{V}_{m,p}\right\}dA-$$
$$-\frac{d}{dz}\int_{D(z)}\left\{\left[C_{3mkl}\bar{\mathcal{V}}_{l,k}+B_{3mkl}\bar{\Phi}_{l,k}+\iota\omega D_{3m}\bar{\Theta}\right]x_pV_{m,p}\right\}dA-$$
$$-\frac{d}{dz}\int_{D(z)}\left\{\left[B_{3mkl}\mathcal{V}_{l,k}+A_{3mkl}\Phi_{l,k}-\iota\omega E_{3m}\Theta\right]x_p\bar{\Phi}_{m,p}\right\}dA-$$
$$-\frac{d}{dz}\int_{D(z)}\left\{\left[B_{3mkl}\bar{\mathcal{V}}_{l,k}+A_{3mkl}\bar{\Phi}_{l,k}+\iota\omega E_{3m}\bar{\Theta}\right]x_p\Phi_{m,p}\right\}dA+q$$
$$+\frac{d}{dz}\int_{D(z)}z\left[C_{mnkl}\bar{\mathcal{V}}_{l,k}\mathcal{V}_{l,k}+A_{mnkl}\Phi_{n,m}\bar{\Phi}_{l,k}\right]dA+$$
$$+\frac{d}{dz}\int_{D(z)}z\left\{B_{mnkl}\left[\mathcal{V}_{l,k}\bar{\Phi}_{l,k}+\bar{\mathcal{V}}_{l,k}\Phi_{l,k}\right]-\rho\omega^2V_m\bar{V}_m\right\}dA$$
$$-\frac{d}{dz}\int_{D(z)}\left\{\iota\omega zD_{mn}\left(\Theta\bar{\mathcal{V}}_{l,k}-\bar{\Theta}\mathcal{V}_{l,k}\right)\right\}dA-$$
$$-\frac{d}{dz}\int_{D(z)}\left\{\iota\omega zE_{mn}\left(\Theta\bar{\Phi}_{n,m}-\bar{\Theta}\Phi_{n,m}\right)-zI_{mn}\omega^2\Phi_m\bar{\Phi}_n\right\}dA \tag{28}$$

$$+\int_{\partial D(z)}x_pn_p\left(A_{l\alpha k\beta}n_\alpha n_\beta\frac{\partial V_l}{\partial n}\frac{\partial\bar{V}_k}{\partial n}+B_{l\alpha k\beta}n_\alpha n_\beta\frac{\partial V_l}{\partial n}\frac{\partial\bar{\Phi}_k}{\partial n}+C_{l\alpha k\beta}n_\alpha n_\beta\frac{\partial\Phi_l}{\partial n}\frac{\partial\bar{\Phi}_k}{\partial n}\right)ds,$$

$$\int_{D(z)}\frac{1}{T_0}\left(\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n}-3c\omega^2\Theta\bar{\Theta}\right)dA+$$
$$+\int_{D(z)}\iota\omega E_{mn}\left(\bar{\Phi}_{l,k}\Theta_{,p}-\Phi_{l,k}\bar{\Theta}_{,p}\right)dA+$$
$$+\int_{D(z)}\iota\omega D_{mn}x_p\left(\mathcal{V}_{l,k}\Theta_{,p}-\mathcal{V}_{l,k}\bar{\Theta}_{,p}\right)dA$$
$$+\int_{\partial D(z)}\frac{1}{T_0}x_pn_p\kappa_{\alpha\beta}n_\alpha n_\beta\frac{\partial\Theta}{\partial n}\frac{\partial\bar{\Theta}}{\partial n}ds= \tag{29}$$

$$= -\frac{d}{dz}\int_{D(z)} \frac{1}{T_0}\left[x_\alpha \kappa_{3\beta}\left(\bar{\Theta}_{,\alpha}\Theta_{,\beta} + \Theta_{,\alpha}\bar{\Theta}_{,\beta}\right) + x_\alpha \kappa_{33}\left(\Theta_{,3}\bar{\Theta}_{,\alpha} + \bar{\Theta}_3\Theta_{,\alpha}\right)\right]dA -$$
$$-\frac{d}{dz}\int_{D(z)} \frac{z}{T_0}\left(\kappa_{33}\Theta_{,3}\bar{\Theta}_3 - \kappa_{\alpha\beta}\Theta_{,\alpha}\bar{\Theta}_{,\beta} + c\omega^2\Theta\bar{\Theta}\right)dA.$$

Proof. Taking into account equations (13) 1 and (13) 2, the following equality is obtained:

$$\begin{aligned}
&\left\{\left[C_{mnkl}\mathcal{V}_{l,k} + B_{mnkl}\Phi_{l,k} - \iota\omega D_{mn}\Theta\right]_{,n} + \rho\omega^2 V_m\right\} x_p \bar{V}_{i,p} + \\
&+\left[B_{mnkl}\mathcal{V}_{l,k} + A_{mnkl}\Phi_{l,k} - \iota\omega E_{mn}\Theta\right]_{,n} x_p \bar{\Phi}_{i,p} + \\
&+\epsilon_{mnj}\left[C_{njkl}\mathcal{V}_{l,k} + B_{njkl}\Phi_{l,k} - \iota\omega D_{nj}\Theta\right] x_p \bar{\Phi}_{m,p} + I_{mn}\omega^2 x_p \bar{\Phi}_{m,p}\Phi_n + \\
&+\left\{\left[C_{mnkl}\bar{\mathcal{V}}_{l,k} + B_{mnkl}\bar{\Phi}_{l,k} + \iota\omega D_{mn}\bar{\Theta}\right]_{,n} + \rho\omega^2 \bar{V}_m\right\} x_p U_{m,p} + \\
&+\left[B_{mnkl}\bar{\mathcal{V}}_{l,k} + A_{mnkl}\bar{\Phi}_{l,k} + \iota\omega E_{mn}\bar{\Theta}\right]_{,n} x_p \Phi_{m,p} + \\
&+\epsilon_{mnj}\left[C_{njkl}\bar{\mathcal{V}}_{l,k} + B_{njkl}\bar{\Phi}_{l,k} + \iota\omega D_{nj}\bar{\Theta}\right] x_p \Phi_{m,p} + I_{mn}\omega^2 x_p \Phi_{m,p}\bar{\Phi}_n = 0.
\end{aligned} \tag{30}$$

After simple calculations, this relation receives the form:

$$\begin{aligned}
&\left\{\left[C_{mnkl}\mathcal{V}_{l,k} + B_{mnkl}\Phi_{l,k} - \iota\omega D_{mn}\Theta\right] x_p \bar{V}_{m,p}\right\}_{,n} - \\
&-\left[C_{mnkl}\mathcal{V}_{l,k} + B_{mnkl}\Phi_{l,k} - \iota\omega D_{mn}\Theta\right] x_p \bar{V}_{i,pj} + \rho\omega^2 x_p V_m \bar{V}_{m,p} + \\
&+\left\{\left[B_{mnkl}\mathcal{V}_{l,k} + A_{mnkl}\Phi_{l,k} - \iota\omega E_{mn}\Theta\right] x_p \bar{\Phi}_{m,p}\right\}_{,n} - \\
&-\left[B_{mnkl}\mathcal{V}_{l,k} + A_{mnkl}\Phi_{l,k} - \iota\omega E_{mn}\Theta\right] x_p \bar{\Phi}_{m,pj} + \\
&+\epsilon_{mnj}\left[C_{njkl}\mathcal{V}_{l,k} + B_{njkl}\Phi_{l,k} - \iota\omega D_{nj}\Theta\right] x_p \bar{\Phi}_{m,p} + I_{mn}\omega^2 x_p \bar{\Phi}_{m,p}\Phi_n + \\
&+\left\{\left[C_{mnkl}\bar{\mathcal{V}}_{l,k} + B_{mnkl}\bar{\Phi}_{l,k} + \iota\omega D_{mn}\bar{\Theta}\right] x_p V_{m,p}\right\}_{,n} - \\
&-\left[C_{mnkl}\bar{\mathcal{V}}_{l,k} + B_{mnkl}\bar{\Phi}_{l,k} + \iota\omega D_{mn}\bar{\Theta}\right] x_p V_{m,pn} + \rho\omega^2 x_p \bar{V}_m V_{m,p} + \\
&+\left\{\left[B_{mnkl}\bar{\mathcal{V}}_{l,k} + A_{mnkl}\bar{\Phi}_{l,k} + \iota\omega E_{mn}\bar{\Theta}\right] x_p \Phi_{m,p}\right\}_{,n} - \\
&-\left[B_{mnkl}\bar{\mathcal{V}}_{l,k} + A_{mnkl}\bar{\Phi}_{l,k} + \iota\omega E_{mn}\bar{\Theta}\right] x_p \Phi_{m,pn} + \\
&+\epsilon_{mnj}\left[A_{njkl}\bar{\mathcal{V}}_{l,k} + B_{njkl}\bar{\Phi}_{l,k} + \iota\omega D_{nj}\bar{\Theta}\right] x_p \Phi_{m,p} + I_{mn}\omega^2 x_p \Phi_{m,p}\bar{\Phi}_n = 0.
\end{aligned} \tag{31}$$

We can rewrite this equality as follows:

$$\begin{aligned}
&C_{mnkl}\mathcal{V}_{l,k}\bar{\mathcal{V}}_{l,k} + A_{mnkl}\Phi_{i,n}\bar{\Phi}_{l,k} + \\
&+B_{mnkl}\left(\mathcal{V}_{l,k}\bar{\Phi}_{i,n} + V_{l,k}\Phi_{m,n}\right) - 3\omega^2\left(\rho V_m \bar{U}_m + I_{mn}\Phi_m\bar{\Phi}_n\right) - \\
&-2\iota\omega D_{mn}\left(\Theta\bar{\mathcal{V}}_{l,k} - \bar{\Theta}\mathcal{V}_{l,k}\right) - 2\iota\omega E_{mn}\left(\Theta\bar{\Phi}_{n,m} - \bar{\Theta}\Phi_{n,m}\right) \\
&-\iota\omega D_{mn} x_p\left(\Theta_{,p}\bar{\mathcal{V}}_{l,k} - \bar{\Theta}_{,p}\mathcal{V}_{l,k}\right) - \iota\omega E_{mn} x_p\left(\Theta_{,p}\bar{\Phi}_{n,m} - \bar{\Theta}_{,p}\Phi_{n,m}\right) = \\
&= -\left\{\left[C_{mnkl}\mathcal{V}_{l,k} + B_{mnkl}\Phi_{l,k} - \iota\omega D_{mn}\Theta\right] x_p \bar{V}_{m,p}\right\}_{,n} - \\
&-\left\{\left[C_{mnkl}\bar{\mathcal{V}}_{l,k} + B_{mnkl}\bar{\Phi}_{l,k} + \iota\omega D_{mn}\bar{\Theta}\right] x_p V_{m,p}\right\}_{,n} - \\
&-\left\{\left[B_{mnkl}\mathcal{V}_{l,k} + A_{mnkl}\Phi_{l,k} - \iota\omega E_{mn}\Theta\right] x_p \bar{\Phi}_{m,p}\right\}_{,n} -
\end{aligned} \tag{32}$$

$$-\{[B_{mnkl}\bar{\mathcal{V}}_{l,k}+A_{mnkl}\bar{\Phi}_{l,k}+\iota\omega E_{mn}\bar{\Theta}]x_p\Phi_{m,p}\}_{,n}+$$
$$+[x_pC_{mnkl}\mathcal{V}_{l,k}\bar{\mathcal{V}}_{l,k}+x_pA_{mnkl}\Phi_{m,n}\bar{\Phi}_{l,k}]_{,p}+$$
$$+\{x_pB_{mnkl}[\mathcal{V}_{l,k}\bar{\Phi}_{m,n}+\bar{\mathcal{V}}_{l,k}\Phi_{m,n}]-x_p\rho\omega^2V_m\bar{V}_m\}_{,p}$$
$$-\{\iota\omega x_pD_{mn}(\Theta\bar{\mathcal{V}}_{n,m}-\bar{\Theta}\mathcal{V}_{n,m})\}_{,p}-$$
$$-\{\iota\omega x_pE_{mn}(\Theta\bar{\Phi}_{n,m}-\bar{\Theta}\Phi_{n,m})-x_pI_{mn}\omega^2\Phi_m\bar{\Phi}_n\}_{,p}.$$

Let us integrate equality (32) over the domain $D(z)$. Applying the theorem of divergence and using the lateral relations from (14), we arrive at equality:

$$\begin{aligned}
&\int_{D(z)}[C_{mnkl}\bar{\mathcal{V}}_{l,k}\bar{\mathcal{V}}_{l,k}+A_{mnkl}\Phi_{i,n}\bar{\Phi}_{l,k}]dA+\\
&+\int_{D(z)}\{B_{mnkl}[\mathcal{V}_{l,k}\bar{\Phi}_{i,n}+\bar{\mathcal{V}}_{l,k}\Phi_{m,n}]-3\omega^2(\rho V_m\bar{V}_m+I_{mn}\Phi_m\bar{\Phi}_n)\}dA\\
&-2\iota\omega\int_{D(z)}\{D_{mn}(\Theta\bar{\mathcal{V}}_{l,k}-\bar{\Theta}\mathcal{V}_{l,k})+E_{mn}(\Theta\bar{\Phi}_{n,m}-\bar{\Theta}\Phi_{n,m})\}dA-\\
&-\iota\omega\int_{D(z)}\{D_{mn}x_p(\Theta_{,p}\bar{\mathcal{V}}_{l,k}-\bar{\Theta}_{,p}\mathcal{V}_{l,k})+E_{mn}x_p(\Theta_{,p}\bar{\Phi}_{n,m}-\bar{\Theta}_{,p}\Phi_{n,m})\}dA\\
&=-\frac{d}{dz}\int_{D(z)}\{[C_{3mkl}\mathcal{V}_{l,k}+B_{3mkl}\Phi_{l,k}-\iota\omega D_{3m}\Theta]x_p\bar{V}_{m,p}\}dA-\\
&-\frac{d}{dz}\int_{D(z)}\{[C_{3mkl}\bar{\mathcal{V}}_{l,k}+B_{3mkl}\bar{\Phi}_{l,k}+\iota\omega D_{3m}\bar{\Theta}]x_pV_{m,p}\}dA-\\
&-\frac{d}{dz}\int_{D(z)}\{[B_{3mkl}\mathcal{V}_{l,k}+A_{3mkl}\Phi_{l,k}-\iota\omega E_{3m}\Theta]x_p\bar{\Phi}_{m,p}\}dA-\\
&-\frac{d}{dz}\int_{D(z)}\{[B_{3mkl}\bar{\mathcal{V}}_{l,k}+A_{3mkl}\bar{\Phi}_{l,k}+\iota\omega E_{3m}\bar{\Theta}]x_p\Phi_{m,p}\}dA+\\
&+\frac{d}{dz}\int_{D(z)}z[C_{mnkl}\bar{\mathcal{V}}_{l,k}\bar{\mathcal{V}}_{l,k}+A_{mnkl}\Phi_{m,n}\bar{\Phi}_{l,k}]dA+\\
&+\frac{d}{dz}\int_{D(z)}z\{B_{mnkl}[\mathcal{V}_{l,k}\bar{\Phi}_{i,n}+\bar{\mathcal{V}}_{l,k}\Phi_{m,n}]-\rho\omega^2V_m\bar{V}_m\}dA\\
&-\frac{d}{dz}\int_{D(z)}\{\iota\omega zD_{mn}(\Theta\bar{\mathcal{V}}_{l,k}-\bar{\Theta}\mathcal{V}_{l,k})\}dA-\\
&-\frac{d}{dz}\int_{D(z)}\{\iota\omega zE_{mn}(\Theta\bar{\Phi}_{n,m}-\bar{\Theta}\Phi_{n,m})-zI_{mn}\omega^2\Phi_m\bar{\Phi}_n\}dA\\
&-\int_{\partial D(z)}[x_p\bar{\mathcal{V}}_{s,p}A_{psmn}\bar{\mathcal{V}}_{l,k}+x_p\mathcal{V}_{s,p}A_{psmn}\bar{\mathcal{V}}_{s,p}]n_pds-\\
&-\int_{\partial D(z)}[x_p\bar{\Phi}_{s,p}B_{psmn}\mathcal{V}_{l,k}+x_p\Phi_{s,p}B_{psmn}\bar{\mathcal{V}}_{l,k}]n_pds\\
&-\int_{\partial D(z)}[x_p\bar{\Phi}_{s,p}C_{psmn}\Phi_{n,m}+x_p\Phi_{s,p}C_{psmn}\bar{\Phi}_{n,m}]n_pds+\\
&+\int_{\partial D(z)}x_pn_p\{C_{mnkl}\bar{\mathcal{V}}_{l,k}\mathcal{V}_{l,k}+A_{mnkl}\Phi_{l,k}\bar{\Phi}_{l,k}+\\
&+B_{mnkl}[\mathcal{V}_{l,k}\bar{\Phi}_{n,m}+\bar{\mathcal{V}}_{l,k}\Phi_{n,m}]\}ds.
\end{aligned} \tag{33}$$

With the help of the lateral boundary condition (14), we conclude that

$$V_{m,3}=0 \quad on \quad \partial D(z). \tag{34}$$

Denoting by n_α the components of the unit normal to ∂D and by τ_α the components of the unit vector tangent to ∂D, on the curve ∂D we have

$$V_{m,\alpha}=n_\alpha\frac{\partial V_m}{\partial n}+\tau_\alpha\frac{\partial V_m}{\partial\tau},$$

where $\partial/\partial\tau$ is the tangential derivative.

Considering the lateral boundary relation (14) we obtain that on the curve ∂D we have $\partial V_m/\partial\tau = 0$, so that the derivative becomes:

$$V_{m,\alpha} = n_\alpha \frac{\partial V_m}{\partial n} \quad on \quad \partial D. \tag{35}$$

If we take into account the relations (34) and (35), we can write the last integral from (33) in the form:

$$\int_{\partial D(z)} x_p n_p \left(C_{mnkl} \mathcal{V}_{n,m} \bar{\mathcal{V}}_{l,k} + B_{mnkl} \mathcal{V}_{n,m} \bar{\Phi}_{l,k} + A_{mnkl} \Phi_{n,m} \bar{\Phi}_{l,k}\right) ds =$$
$$= \int_{\partial D(z)} x_p n_p \left(C_{k\alpha l\beta} n_\alpha n_\beta \frac{\partial \mathcal{V}_k}{\partial n}\frac{\partial \bar{\mathcal{V}}_l}{\partial n} + B_{k\alpha l\beta} n_\alpha n_\beta \frac{\partial \mathcal{V}_k}{\partial n}\frac{\partial \bar{\Phi}_l}{\partial n} + \right.$$
$$\left. + A_{k\alpha l\beta} n_\alpha n_\beta \frac{\partial \Phi_k}{\partial n}\frac{\partial \bar{\Phi}_l}{\partial n}\right) ds. \tag{36}$$

The other integrals in (33) become:

$$\int_{\partial D(z)} \left[x_q \bar{\mathcal{V}}_{r,q} C_{prkl} \mathcal{V}_{l,k} + x_q \mathcal{V}_{r,q} C_{prkl} \bar{\mathcal{V}}_{l,k}\right] n_p \ ds =$$
$$= 2\int_{\partial D(z)} x_p n_p C_{k\alpha l\beta} n_\alpha n_\beta \frac{\partial \mathcal{V}_k}{\partial n}\frac{\partial \bar{\mathcal{V}}_l}{\partial n} \ ds,$$

$$\int_{\partial D(z)} \left[x_q \bar{\mathcal{V}}_{r,q} B_{prkl} \Phi_{l,k} + x_q \mathcal{V}_{r,q} B_{prkl} \bar{\Phi}_{l,k}\right] n_p \ ds =$$
$$= 2\int_{\partial D(z)} x_p n_p B_{k\alpha l\beta} n_\alpha n_\beta \frac{\partial \mathcal{V}_k}{\partial n}\frac{\partial \bar{\Phi}_l}{\partial n} \ ds, \tag{37}$$
$$\int_{\partial D(z)} \left[x_q \bar{\Phi}_{r,q} A_{prkl} \Phi_{l,k} + x_q \Phi_{r,q} A_{prkl} \bar{\Phi}_{l,k}\right] n_p \ ds =$$
$$= 2\int_{\partial D(z)} x_p n_p C_{k\alpha l\beta} n_\alpha n_\beta \frac{\partial \Phi_k}{\partial n}\frac{\partial \bar{\Phi}_l}{\partial n} \ ds.$$

The results from equalities (36) and (37) are replaced in identity (33), so that we obtain the relation (28), that is, first relation of Theorem 2.

In order to obtain relation (29), we begin with the following identity:

$$x_p \bar{\Theta}_p \left[\left(\frac{1}{T_0}\kappa_{mn}\Theta_{,n}\right)_{,m} - \iota\omega D_{mn}\mathcal{V}_{n,m} - \iota\omega E_{mn}\Phi_{n,m} + \frac{a}{T_0}\omega^2\Theta\right] +$$
$$+ x_p \Theta_p \left[\left(\frac{1}{T_0}\kappa_{mn}\bar{\Theta}_{,n}\right)_{,m} + \iota\omega D_{mn}\mathcal{V}_{n,m} - \iota\omega E_{mn}\bar{\Phi}_{n,m} + \frac{a}{T_0}\omega^2\bar{\Theta}\right] = 0. \tag{38}$$

After simple calculations, this relation receives the form:

$$\iota\omega D_{mn} x_p\left(\mathcal{V}_{n,m}\Theta_{,p} - \mathcal{V}_{n,m}\bar{\Theta}_{,p}\right) +$$
$$+ \iota\omega E_{mn} x_p\left(\bar{\Phi}_{n,m}\Theta_{,p} - \Phi_{n,m}\bar{\Theta}_{,p}\right) = -x_p\left(\frac{a}{T_0}\omega^2\Theta\bar{\Theta}\right)_{,p} + \frac{2}{T_0}\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n} -$$
$$- \left[\frac{1}{T_0} x_p \kappa_{mn}\left(\bar{\Theta}_{,p}\Theta_{,n} + \Theta_{,p}\bar{\Theta}_{,n}\right)\right]_{,m} + x_p\left(\frac{1}{T_0}\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n}\right)_{,p}. \tag{39}$$

We can rewrite this equality in the following form:

$$\frac{1}{T_0}\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n} - \frac{3c}{T_0}\omega^2\Theta\bar{\Theta} + \iota\omega E_{mn}\left(\bar{\Phi}_{n,m}\Theta_{,p} - \Phi_{n,m}\bar{\Theta}_{,p}\right) +$$
$$+ \iota\omega D_{mn} x_p\left(\mathcal{V}_{n,m}\Theta_{,p} - \mathcal{V}_{n,m}\bar{\Theta}_{,p}\right) =$$
$$= -\left(\frac{a}{T_0}\omega^2\Theta\bar{\Theta}\right)_{,p} - \left[\frac{1}{T_0} x_p \kappa_{mn}\left(\bar{\Theta}_{,p}\Theta_{,n} + \Theta_{,p}\bar{\Theta}_{,n}\right)\right]_{,m} + \left(\frac{x_p}{T_0}\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n}\right)_{,p}. \tag{40}$$

Let us integrate the relation (40) over the domain $D(z)$. Applying the theorem of divergence and using the lateral relations from (14), we arrive at equality:

$$\int_{D(z)} \frac{1}{T_0}\left(\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n} - 3c\omega^2\Theta\bar{\Theta}\right)dA +$$
$$+\int_{D(z)} \iota\omega E_{mn}x_p\left(\bar{\Phi}_{n,m}\Theta_{,p} - \Phi_{n,m}\bar{\Theta}_{,p}\right)dA$$
$$+\int_{D(z)} \iota\omega D_{mn}x_p\left(\mathcal{V}_{n,m}\Theta_{,p} - \mathcal{V}_{n,m}\bar{\Theta}_{,p}\right)dA =$$
$$= -\frac{d}{dz}\int_{D(z)} \left[\frac{1}{T_0}x_p\kappa_{3m}\left(\bar{\Theta}_{,p}\Theta_{,n} + \Theta_{,p}\bar{\Theta}_{,n}\right) - \frac{z}{T_0}\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n} + \frac{z}{T_0}c\omega^2\Theta\bar{\Theta}\right]dA \tag{41}$$
$$+\int_{\partial D(z)} \frac{1}{T_0}\left[x_pn_p\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n} - x_p\kappa_{mn}\left(\bar{\Theta}_{,m}\Theta_{,n} + \Theta_{,m}\bar{\Theta}_{,n}\right)n_p\right]ds.$$

As we showed above, on the curve $\partial D(z)$, the lateral boundary condition lead to:

$$\Theta_{,3} = 0, \quad \Theta_{,\alpha} = n_\alpha \frac{\partial\Theta}{\partial n},$$

so that, the relation (41) implies (29). The proof of Theorem 2 is finished. □

In the following theorem we will prove two conservation laws which will be used to derive *a priori* estimation on a solution of the mixed problem P.

Theorem 3. *If the ordered array* (V_m, Φ_m, Θ) *is a solution of the boundary value problem* P, *then take place the following two conservation laws:*

$$\frac{d}{dz}\int_{D(z)} \omega^2\left(\rho V_n\bar{V}_n + I_{mn}\Phi_m\bar{\Phi}_n + \frac{a}{T_0}\Theta\bar{\Theta}\right)dA +$$
$$+\frac{d}{dz}\int_{D(z)} \frac{a}{T_0}\left(\kappa_{33}\Theta_{,3}\bar{\Theta}_{,3} - \kappa_{\alpha\beta}\Theta_{,\alpha}\bar{\Theta}_{,\beta}\right)dA +$$
$$+\frac{d}{dz}\int_{D(z)} \left[C_{k3l3}\mathcal{V}_{k,3}\bar{\mathcal{V}}_{l,3} + B_{k3l3}\left(\mathcal{V}_{k,3}\bar{\Phi}_{l,3} + \bar{\mathcal{V}}_{k,3}\Phi_{l,3}\right) + A_{k3l3}\Phi_{k,3}\bar{\Phi}_{l,3}\right]dA \tag{42}$$
$$-\frac{d}{dz}\int_{D(z)} \left[C_{k\alpha l\beta}\mathcal{V}_{k,\alpha}\bar{\mathcal{V}}_{l,\beta} + B_{k\alpha l\beta}\left(\mathcal{V}_{k,\alpha}\bar{\Phi}_{l,\beta} + \bar{\mathcal{V}}_{k,\alpha}\Phi_{l,\beta}\right) + A_{k\alpha l\beta}\Phi_{k,\alpha}\bar{\Phi}_{l,\beta}\right]dA$$
$$+\frac{d}{dz}\int_{D(z)} \left[\iota\omega D_{m\alpha}\left(\Theta\bar{\mathcal{V}}_{m,\alpha} - \bar{\Theta}\mathcal{V}_{m,\alpha}\right) + \iota\omega E_{m\alpha}\left(\Theta\bar{\Phi}_{m,\alpha} - \bar{\Theta}\Phi_{m,\alpha}\right)\right]dA = 0,$$

$$\frac{d}{dz}\int_{D(z)} \left\{\left[C_{3mkl}\bar{\mathcal{V}}_{l,k} + B_{3mkl}\bar{\Phi}_{l,k} + \iota\omega D_{3m}\bar{\Theta}\right]V_m\right\}dA -$$
$$-\frac{d}{dz}\int_{D(z)} \left\{\left[C_{3mkl}\mathcal{V}_{l,k} + B_{3mkl}\Phi_{l,k} - \iota\omega D_{3m}\Theta\right]\bar{V}_m\right\}dA +$$
$$+\frac{d}{dz}\int_{D(z)} \left\{\left[B_{3mkl}\bar{\mathcal{V}}_{l,k} + A_{3mkl}\bar{\Phi}_{l,k} + \iota\omega E_{3m}\bar{\Theta}\right]\Phi_m\right\}dA - \tag{43}$$
$$-\frac{d}{dz}\int_{D(z)} \left\{\left[B_{3mkl}\mathcal{V}_{l,k} + A_{3mkl}\Phi_{l,k} - \iota\omega E_{3m}\Theta\right]\bar{\Phi}_m\right\}dA =$$
$$= \frac{d}{dz}\int_{D(z)} \left[\frac{1}{T_0}\kappa_{3m}\left(\bar{\Theta}\Theta_{,m} - \Theta\bar{\Theta}_{,m}\right)\right]dA.$$

Proof. In order to obtain the (42) we begin by considering again the equations (13) 1 and (13) 2 and deduce the next equality:

$$\left\{\left[C_{mnkl}\left(\mathcal{V}_{l,k} + \epsilon_{kli}\Phi_i\right) + B_{mnkl}\Phi_{l,k} - \iota\omega D_{mn}\Theta\right]_{,m} + \rho\omega^2U_n\right\}\bar{V}_{n,3} +$$
$$+\left\{\left[B_{mnkl}\mathcal{V}_{l,k} + A_{mnkl}\Phi_{l,k} - \iota\omega E_{mn}\Theta\right]_{,m} +\right.$$
$$\left.+\epsilon_{mnj}\left[C_{njkl}\left(\mathcal{V}_{l,k} + \epsilon_{kli}\Phi_i\right) + B_{njkl}\Phi_{l,k} - \iota\omega D_{nj}\Theta\right] + I_{mn}\omega^2\Phi_m\right\}\bar{\Phi}_{n,3} +$$
$$+\left\{\left[C_{mnkl}\left(\mathcal{V}_{l,k} + \epsilon_{kli}\Phi_i\right) + B_{mnkl}\bar{\Phi}_{l,k} + \iota\omega D_{mn}\bar{\Theta}\right]_{,m} + \rho\omega^2\bar{V}_n\right\}V_{n,3} + \tag{44}$$
$$+\left\{\left[B_{mnkl}\bar{\mathcal{V}}_{l,k} + A_{mnkl}\bar{\Phi}_{l,k} + \iota\omega E_{mn}\bar{\Theta}\right]_{,m} +\right.$$

$$+\epsilon_{mnj}\left[C_{njkl}\left(\mathcal{V}_{l,k}+\epsilon_{kli}\Phi_i\right)+B_{njkl}\bar{\Phi}_{l,k}+\iota\omega D_{nj}\bar{\Theta}\right]+I_{mn}\omega^2\bar{\Phi}_m\}\Phi_{n,3}=0.$$

After simple calculations, this relation receives the form:

$$\begin{aligned}&\frac{d}{dz}\left[\rho\omega^2 V_n\bar{V}_n+I_{mn}\omega^2\Phi_m\bar{\Phi}_n+C_{k3l3}\mathcal{V}_{l,3}\bar{\mathcal{V}}_{k,3}+B_{k3l3}\left(\mathcal{V}_{l,3}\bar{\Phi}_{k,3}+\bar{\mathcal{V}}_{l,3}\Phi_{k,3}\right)+\right.\\&+A_{k3l3}\Phi_{l,3}\bar{\Phi}_{k,3}-C_{k\alpha l\beta}\mathcal{V}_{l,\alpha}\bar{\mathcal{V}}_{k,\beta}-B_{k\alpha l\beta}\left(\mathcal{V}_{l,\alpha}\bar{\Phi}_{k,\beta}+\bar{\mathcal{V}}_{l,\alpha}\Phi_{k,\beta}\right)-\\&\left.-C_{k\alpha l\beta}\Phi_{l,\alpha}\bar{\Phi}_{k,\beta}+\iota\omega D_{m\alpha}\left(\Theta\bar{\mathcal{V}}_{m,\alpha}-\bar{\Theta}\mathcal{V}_{m,\alpha}\right)+\iota\omega E_{m\alpha}\left(\Theta\bar{\Phi}_{m,\alpha}-\bar{\Theta}\Phi_{m,\alpha}\right)\right]+\\&+\left[C_{k\alpha l3}\mathcal{V}_{l,3}\bar{\mathcal{V}}_{k,\alpha}+B_{k\alpha l3}\left(\mathcal{V}_{l,3}\bar{\Phi}_{k,\alpha}+\bar{\mathcal{V}}_{l,3}\Phi_{k,\alpha}\right)+A_{k\alpha l3}\Phi_{l,3}\bar{\Phi}_{k,\alpha}\right]_{,\alpha}+\\&+\left[\iota\omega D_{m\alpha}\left(\bar{\Theta}\mathcal{V}_{m,3}-\Theta\bar{\mathcal{V}}_{m,3}\right)\right]_{,\alpha}+\left[\iota\omega E_{m\alpha}\left(\bar{\Theta}\Phi_{m,3}-\Theta\bar{\Phi}_{m,3}\right)\right]_{,\alpha}+\\&+\iota\omega D_{mn}\left(\bar{\Theta}_{,3}\mathcal{V}_{m,n}-\Theta_{,3}\bar{\mathcal{V}}_{m,n}\right)+\iota\omega E_{mn}\left(\bar{\Theta}_{,3}\Phi_{m,n}-\Theta_{,3}\bar{\Phi}_{m,n}\right)=0.\end{aligned}\tag{45}$$

Let us integrate equality (45) over the domain $D(z)$. Applying the theorem of divergence and using the lateral relations from (14), we arrive at equality:

$$\begin{aligned}&\frac{d}{dz}\int_{D(z)}\left[\rho\omega^2 U_m\bar{\mathcal{V}}_m+I_{mn}\omega^2\Phi_m\bar{\Phi}_n+C_{k3l3}U_{k,3}\bar{\mathcal{V}}_{l,3}+B_{k3l3}\left(\mathcal{V}_{k,3}\bar{\Phi}_{l,3}+\bar{\mathcal{V}}_{k,3}\Phi_{l,3}\right)+\right.\\&+A_{k3l3}\Phi_{k,3}\bar{\Phi}_{l,3}-C_{k\alpha l\beta}\mathcal{V}_{k,\alpha}\bar{\mathcal{V}}_{l,\beta}-B_{k\alpha l\beta}\left(\mathcal{V}_{k,\alpha}\bar{\Phi}_{l,\beta}+\bar{\mathcal{V}}_{k,\alpha}\Phi_{l,\beta}\right)-\\&\left.-A_{k\alpha l\beta}\Phi_{k,\alpha}\bar{\Phi}_{l,\beta}+\iota\omega D_{m\alpha}\left(\Theta\bar{\mathcal{V}}_{m,\alpha}-\bar{\Theta}\mathcal{V}_{m,\alpha}\right)+\iota\omega E_{m\alpha}\left(\Theta\bar{\Phi}_{m,\alpha}-\bar{\Theta}\Phi_{m,\alpha}\right)\right]dA+\\&+\int_{D(z)}\left[\iota\omega D_{mn}\left(\bar{\Theta}_{,3}\mathcal{V}_{m,n}-\Theta_{,3}\bar{\mathcal{V}}_{m,n}\right)+\iota\omega E_{mn}\left(\bar{\Theta}_{,3}\Phi_{m,n}-\Theta_{,3}\bar{\Phi}_{m,n}\right)\right]dA=0.\end{aligned}\tag{46}$$

Based on equations (13) 3, we can write:

$$\begin{aligned}&\bar{\Theta}_{,3}\left[\frac{1}{T_0}\kappa_{mn}\Theta_{,mn}-\iota\omega\left(D_{mn}\mathcal{V}_{m,n}+E_{mn}\Phi_{m,n}\right)+\frac{a}{T_0}\omega^2\Theta\right]+\\&+T_{,3}\left[\frac{1}{T_0}\kappa_{mn}\bar{\Theta}_{,mn}+\iota\omega\left(D_{mn}\bar{\mathcal{V}}_{m,n}+E_{mn}\bar{\Phi}_{m,n}\right)+\frac{a}{T_0}\omega^2\bar{\Theta}\right]=0,\end{aligned}\tag{47}$$

and this equality can be written in the form:

$$\begin{aligned}&\frac{d}{dz}\left(\frac{a}{T_0}\omega^2\Theta\bar{\Theta}+\frac{1}{T_0}\kappa_{33}\Theta_{,3}\bar{\Theta}_{,3}-\frac{1}{T_0}\kappa_{\alpha\beta}\Theta_{,\alpha}\bar{\Theta}_{,\beta}\right)+\\&+\left(\frac{2}{T_0}\kappa_{\alpha 3}\Theta_{,3}\bar{\Theta}_{,3}\right)_{,\alpha}+\iota\omega D_{mn}\left(\Theta_{,3}\bar{\mathcal{V}}_{m,n}-\bar{\Theta}_{,3}\mathcal{V}_{m,n}\right)+\\&+\iota\omega E_{mn}\left(\Theta_{,3}\bar{\Phi}_{m,n}-\bar{\Theta}_{,3}\Phi_{m,n}\right)=0.\end{aligned}\tag{48}$$

Let us integrate equality (48) over the domain $D(z)$. Applying the theorem of divergence and using the lateral relations from (14), we arrive at equality:

$$\begin{aligned}&\frac{d}{dz}\int_{D(z)}\left(\frac{a}{T_0}\omega^2\Theta\bar{\Theta}+\frac{1}{T_0}\kappa_{33}\Theta_{,3}\bar{\Theta}_{,3}-\frac{1}{T_0}\kappa_{\alpha\beta}\Theta_{,\alpha}\bar{\Theta}_{,\beta}\right)dA+\\&+\int_{D(z)}\left[\iota\omega D_{mn}\left(\Theta_{,3}\bar{\mathcal{V}}_{m,n}-\bar{\Theta}_{,3}\mathcal{V}_{m,n}\right)+\iota\omega E_{mn}\left(\Theta_{,3}\bar{\Phi}_{m,n}-\bar{\Theta}_{,3}\Phi_{m,n}\right)\right]dA=0.\end{aligned}\tag{49}$$

By combining equalities (49) and (46) we are led to equality (42).

In order to obtain the conservation law (43) we just need to equalize the right-side members of relations (17) and (19). In this way, the proof of Theorem 3 is finished. □

We can obtain different measures for the amplitude (V_m, Φ_m, Θ) by combining the relations (16)-(19) from Theorem 1 with the identities (28)-(29) of Theorem 2 and the conservation laws (42)-(43) from Theorem 3. Based on these measures, it is possible to deduce some spatial estimations in order to characterize the spatial evolution of the respective amplitude.

First such kind of estimate is deduced in the next theorem.

Theorem 4. *If the ordered array* (V_m, Φ_m, Θ) *is a solution of the boundary value problem* P, *then takes place the following estimate:*

$$\begin{aligned}
&\int_{D(z)} \left[C_{mnkl}\mathcal{V}_{n,m}\bar{\mathcal{V}}_{l,k} + B_{mnkl}\left(\mathcal{V}_{n,m}\bar{\Phi}_{l,k} + \bar{\mathcal{V}}_{n,m}\Phi_{l,k}\right) + A_{mnkl}\Phi_{n,m}\bar{\Phi}_{l,k} -\right.\\
&\left.-\omega^2\left(\rho V_m\bar{V}_m + I_{mn}\Phi_m\bar{\Phi}_n + \frac{a}{T_0}\Theta\bar{\Theta}\right) + \frac{a}{T_0}\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n}\right]dA +\\
&+\int_{D(z)}\left[\iota\omega D_{mn}\left(\bar{\Theta}\mathcal{V}_{n,m} - \Theta\bar{\mathcal{V}}_{n,m}\right) + \iota\omega E_{mn}\left(\bar{\Theta}\Phi_{n,m} - \Theta\bar{\Phi}_{n,m}\right)\right]dA =\\
&= \frac{d}{dz}\int_{D(z)}\left\{\left[C_{3mkl}\mathcal{V}_{l,k} + B_{3mkl}\Phi_{l,k} - \iota\omega D_{3m}\Theta\right]\bar{V}_m\right\}dA +\\
&+\frac{d}{dz}\int_{D(z)} + \left\{\left[C_{3mkl}\bar{\mathcal{V}}_{l,k} + B_{3mkl}\bar{\Phi}_{l,k} + \iota\omega D_{3m}\bar{\Theta}\right]V_m\right\}dA +\\
&+\frac{d}{dz}\int_{D(z)}\left\{\left[B_{3mkl}\mathcal{V}_{l,k} + A_{3mkl}\Phi_{l,k} - \iota\omega E_{3m}\Theta\right]\bar{\Phi}_m\right\}dA +\\
&+\frac{d}{dz}\int_{D(z)}\left\{\left[B_{3mkl}\bar{\mathcal{V}}_{l,k} + A_{3mkl}\bar{\Phi}_{l,k} + \iota\omega E_{3m}\bar{\Theta}\right]\Phi_m\right\}dA +\\
&+\frac{d}{dz}\int_{D(z)}\frac{1}{T_0}\kappa_{33}\left(\Theta\bar{\Theta}_{,3} + \bar{\Theta}\Theta_{,3}\right)dA.
\end{aligned} \tag{50}$$

Proof. This estimation can be immediately obtained by combining relations (16) and (18). □

In the following theorem is formulated another *a priori* estimation on the amplitude.

Theorem 5. *If the ordered array* (V_m, Φ_m, Θ) *is a solution of the boundary value problem* P, *then takes place the following estimate:*

$$\begin{aligned}
&\int_{D(z)} \left[C_{mnkl}\mathcal{V}_{n,m}\bar{\mathcal{V}}_{l,k} + B_{mnkl}\left(\mathcal{V}_{n,m}\bar{\Phi}_{l,k} + \bar{\bar{\mathcal{V}}}_{n,m}\Phi_{l,k}\right) + A_{mnkl}\Phi_{n,m}\bar{\Phi}_{l,k} +\right.\\
&\left.+\frac{1}{T_0}\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n} + \omega^2\left(\rho V_m\bar{V}_m + I_{mn}\Phi_m\bar{\Phi}_n + \frac{1}{T_0}\Theta\bar{\Theta}\right)\right]dA -\\
&-\int_{\partial D(z)} x_p n_p\left(C_{k\alpha l\beta}n_\alpha n_\beta\frac{\partial\mathcal{V}_k}{\partial n}\frac{\partial\bar{\mathcal{V}}_l}{\partial n} + B_{k\alpha l\beta}n_\alpha n_\beta\frac{\partial\mathcal{V}_k}{\partial n}\frac{\partial\bar{\Phi}_l}{\partial n} +\right.\\
&\left.+A_{k\alpha l\beta}n_\alpha n_\beta\frac{\partial\Phi_k}{\partial n}\frac{\partial\bar{\Phi}_l}{\partial n}\right)ds - \int_{\partial D(z)}\frac{1}{T_0}x_p n_p\kappa_{\alpha\beta}n_\alpha n_\beta\frac{\partial\Theta}{\partial n}\ \frac{\partial\bar{\Theta}}{\partial n}ds =\\
&= \frac{d}{dz}\int_{D(z)}\left[\left(A_{3mkl}\mathcal{V}_{l,k} + B_{3mkl}\Phi_{l,k} - \iota\omega D_{3m}T\right)\left(\bar{\mathcal{V}}_n + x_p\bar{\mathcal{V}}_{m,p}\right) +\right.\\
&\left.+\left(A_{3mkl}\bar{\mathcal{V}}_{l,k} + B_{3mkl}\bar{\Phi}_{l,k} + \iota\omega D_{3m}\bar{\Theta}\right)\left(\mathcal{V}_n + x_p\mathcal{V}_{m,p}\right)\right]dA +\\
&+\frac{d}{dz}\int_{D(z)}\left[\left(B_{3mkl}\mathcal{V}_{l,k} + C_{3mkl}\Phi_{l,k} - \iota\omega E_{3m}T\right)\left(\bar{\Phi}_n + x_p\bar{\Phi}_{m,p}\right) +\right.\\
&\left.+\left(B_{3mkl}\bar{\mathcal{V}}_{l,k} + C_{3mkl}\bar{\Phi}_{l,k} + \iota\omega E_{3m}\bar{\Theta}\right)\left(\Phi_n + x_p\Phi_{m,p}\right)\right]dA +\\
&+\frac{d}{dz}\int_{D(z)}\frac{1}{T_0}\kappa_{33}\left(\Theta\bar{\Theta}_{,3} + \bar{\Theta}\Theta_{,3}\right)dA +\\
&+\frac{d}{dz}\int_{D(z)}\frac{x_\alpha}{T_0}\left[\kappa_{3\alpha}\left(\bar{\Theta}_{,\alpha}\Theta_{,\beta} + \Theta_{,\alpha}\bar{\Theta}_{,\beta}\right) + \kappa_{33}\left(\bar{\Theta}_{,\alpha}\Theta_{,3} + \Theta_{,\alpha}\bar{\Theta}_{,3}\right)\right]dA +\\
&+\frac{d}{dz}\int_{D(z)} z\left[A_{k3l3}\mathcal{V}_{m,3}\bar{\mathcal{V}}_{m,3} + B_{k3l3}\left(\mathcal{V}_{m,3}\bar{\Phi}_{m,3} + \bar{\mathcal{V}}_{i,3}\Phi_{m,3}\right) + C_{k3l3}\Phi_{m,3}\bar{\Phi}_{m,3}\right]\\
&+z\left[C_{k\alpha l\beta}\mathcal{V}_{m,\alpha}\bar{\mathcal{V}}_{m,\beta} + B_{k\alpha l\beta}\left(\mathcal{V}_{m,\alpha}\bar{\Phi}_{m,\beta} + \bar{\mathcal{V}}_{i,\alpha}\Phi_{m,\beta}\right) + C_{k\alpha l\beta}\Phi_{m,\alpha}\bar{\Phi}_{m,\beta}\right]\\
&+z\iota\omega\left[D_{i\alpha}\left(\Theta\bar{\mathcal{V}}_{i,\alpha} - \bar{\Theta}\mathcal{V}_{m,\alpha}\right) + E_{i\alpha}\left(\Theta\bar{\Phi}_{i,\alpha} - \bar{\Theta}\Phi_{m,\alpha}\right)\right] +
\end{aligned} \tag{51}$$

$$+\frac{z}{T_0}\left(\kappa_{33}\Theta_{,3}\bar{\Theta}_{,3}-\kappa_{\alpha\beta}\Theta_{,\alpha}\bar{\Theta}_{,\beta}\right)+z\omega^2\left(\rho V_m\bar{V}_m+I_{mn}\Phi_m\bar{\Phi}_n+\frac{a}{T_0}\Theta\bar{\Theta}\right)\Big]\,dA.$$

Proof. The estimate (51) is immediately obtained by combining the results from relations (28) and (29), from Theorem 2, with the estimate (50) of Theorem 4. □

Based on identity (51), we will obtain our main result regarding the spatial behavior of amplitude. First, we need to specify the assumptions we need to obtain rigorous results, which are not very restrictive, are frequently used in mechanics of solids. So, we suppose that the thermoelastic tensors of Cosserat bodies satisfy the condition of strong ellipticity, that is,

$$\begin{aligned}&C_{mnkl}\xi_m\xi_k\eta_n\eta_l>0,\\&B_{mnkl}\xi_m\xi_k\eta_n\eta_l>0,\ \mathit{for\ all\ non-zero\ vectors}\ (\xi_1,\xi_2,\xi_3),\ (\eta_1,\eta_2,\eta_3),\\&A_{mnkl}\xi_m\xi_k\eta_n\eta_l>0.\end{aligned}\tag{52}$$

Other conditions we impose to the specific heat a and to the tensor of conductivity κ_{mn}, namely:

$$c>0,\ \kappa_{mn}\xi_m\xi_n>0,\ \mathit{for\ any\ non-null\ vector}\ (\xi_1,\xi_2,\xi_3).\tag{53}$$

From (52) it can be deduced that

$$\begin{aligned}&C_{k3l3}\xi_k\xi_l>0,\\&B_{k3l3}\xi_k\xi_l>0,\quad \mathit{for\ any\ non-null\ vector}\quad (\xi_1,\xi_2,\xi_3),\\&A_{k3l3}\xi_k\xi_l>0.\end{aligned}\tag{54}$$

The regularity condition imposed on the curve ∂D ensures the existence of the constant $h_0>0$ so that $x_pn_p\geq h_0>0$. As such, the following inequalities can be obtained:

$$\begin{aligned}0&\leq\int_{\partial D(z)}x_pn_p\left(C_{k\alpha l\beta}n_\alpha n_\beta\frac{\partial V_k}{\partial n}\frac{\partial\bar{V}_l}{\partial n}+2B_{k\alpha l\beta}n_\alpha n_\beta\frac{\partial V_k}{\partial n}\frac{\partial\bar{\Phi}_l}{\partial n}+C_{k\alpha l\beta}n_\alpha n_\beta\frac{\partial\Phi_k}{\partial n}\frac{\partial\bar{\Phi}_l}{\partial n}\right)ds\\&\leq MC\int_{\partial D(z)}\left(\frac{\partial V_m}{\partial n}\frac{\partial\bar{V}_m}{\partial n}+\frac{\partial\Phi_m}{\partial n}\frac{\partial\bar{\Phi}_m}{\partial n}\right)ds.\end{aligned}\tag{55}$$

Above we used the notations:

$$C=\sup_{(\xi_1,\xi_2)\in\partial D}\sqrt{(\xi_1^2+\xi_2^2)},\tag{56}$$

$$M=\sqrt{C_{k\alpha l\beta}C_{k\alpha l\beta}+2B_{k\alpha l\beta}B_{k\alpha l\beta}+A_{k\alpha l\beta}A_{k\alpha l\beta}}.\tag{57}$$

Also, the tensor of conductivity κ_{mn} must satisfy the conditions:

$$0\leq\int_{\partial D(z)}\frac{1}{T_0}x_pn_p\kappa_{\alpha\beta}n_\alpha n_\beta\frac{\partial\Theta}{\partial n}\ \frac{\partial\bar{\Theta}}{\partial n}ds\leq\frac{CK}{T_0}\int_{\partial D(z)}\frac{\partial\Theta}{\partial n}\ \frac{\partial\bar{\Theta}}{\partial n}ds,\tag{58}$$

in which the constant M was introduced in (57) and the constant K is:

$$K=\sqrt{\kappa_{\alpha\beta}\kappa_{\alpha\beta}}.\tag{59}$$

Now we need to introduce the constants ω_0^*, m_0, ω_1^*, and m_1, through the following relations:

$$\omega_0^* = \frac{1}{\rho} CMm_0,\ m_0 = \max_{z\in[0,L]} \frac{\int_{\partial D(z)} \left(\frac{\partial U_m}{\partial n}\frac{\partial \bar{V}_m}{\partial n} + \frac{\partial \Phi_m}{\partial n}\frac{\partial \bar{\Phi}_m}{\partial n}\right) ds}{\int_{D(z)} (U_m \bar{V}_m + \Phi_m \bar{\Phi}_m) ds}, \tag{60}$$

$$\omega_1^* = \frac{1}{a} CKm_1,\ m_1 = \max_{z\in[0,L]} \frac{\int_{\partial D(z)} \frac{\partial \Theta \partial \bar{\Theta}}{\partial n \partial n} ds}{\int_{D(z)} \Theta \bar{\Theta} ds}. \tag{61}$$

We need to suppose that

$$m_0 \le m_0^*,\ m_1 \le m_1^*, \tag{62}$$

$$\omega > \omega^* = \max\{\omega_0^*,\ \omega_1^*\}, \tag{63}$$

in which the constants m_1^* and m_0^* are defined by:

$$m_1^* = \max_{\Theta \in H_0^1(D)} \frac{\int_{\partial D(z)} \frac{\partial \Theta \partial \bar{\Theta}}{\partial n \partial n} ds}{\int_{D(z)} T \bar{\Theta} ds}, \tag{64}$$

$$m_0^* = \max \frac{\int_{\partial D(z)} \left(\frac{\partial V_m}{\partial n}\frac{\partial \bar{V}_m}{\partial n} + \frac{\partial \Phi_m}{\partial n}\frac{\partial \bar{\Phi}_m}{\partial n}\right) ds}{\int_{D(z)} (V_m \bar{V}_m + \Phi_m \bar{\Phi}_m) ds}, \tag{65}$$

where the last maximum is computed for $U_m \in H_0^1(D),\ \Phi_m \in H_0^1(D)$, in which $H_0^1(D)$ is the known Sobolev space.

So, we get, for the frequency of vibration, an explicit critical value, which is:

$$\omega^* = \max\left\{\frac{1}{\rho} CMm_0^*,\ \frac{1}{a} CKm_1^*\right\}.$$

Now, we can obtain the main result of our study, namely, an estimation of the spatial behavior of the amplitude (U_m, Φ_m, Θ), which is deduced by combining the results from above relations (51), (55), (58) and (62):

$$\begin{aligned}
&\frac{d}{dz}\int_{D(z)} \left[\left(C_{3mkl}\mathcal{V}_{l,k} + B_{3mkl}\Phi_{l,k} - \iota\omega D_{3m}\Theta\right)\left(\bar{\mathcal{V}}_m + x_p \bar{\mathcal{V}}_{m,p}\right) + \right.\\
&\left.+\left(C_{3mkl}\bar{\mathcal{V}}_{l,k} + B_{3mkl}\bar{\Phi}_{l,k} + \iota\omega D_{3m}\bar{\Theta}\right)\left(\mathcal{V}_m + x_p \mathcal{V}_{m,p}\right)\right] dA +\\
&+\frac{d}{dz}\int_{D(z)} \left[\left(B_{3mkl}\mathcal{V}_{l,k} + A_{3mkl}\Phi_{l,k} - \iota\omega E_{3m}\Theta\right)\left(\bar{\Phi}_m + x_p \bar{\Phi}_{m,p}\right) + \right.\\
&\left.+\left(B_{3mkl}\bar{\mathcal{V}}_{l,k} + A_{3mkl}\bar{\Phi}_{l,k} + \iota\omega E_{3m}\bar{\Theta}\right)\left(\Phi_m + x_p \Phi_{m,p}\right)\right] dA +\\
&+\frac{d}{dz}\int_{D(z)} \frac{x_\alpha}{T_0}\left[\kappa_{3\beta}\left(\bar{\Theta}_{,\alpha}\Theta_{,\beta} + \Theta_{,\alpha}\bar{\Theta}_{,\beta}\right) + \kappa_{33}\left(\bar{\Theta}_{,\alpha}\Theta_{,3} + \Theta_{,\alpha}\bar{\Theta}_{,3}\right)\right] dA +\\
&+\frac{d}{dz}\int_{D(z)} \frac{1}{T_0}\left[\kappa_{33}\left(\Theta\bar{\Theta}_{,3} + \bar{\Theta}\Theta_{,3}\right) + z\omega^2\left(\rho V_m \bar{V}_m + I_{mn}\Phi_m\bar{\Phi}_n + \frac{1}{T_0}\Theta\bar{\Theta}\right)\right] dA\\
&+\frac{d}{dz}\int_{D(z)} z\left[C_{k3l3}\mathcal{V}_{k,3}\bar{\mathcal{V}}_{l,3} + B_{k3l3}\left(\mathcal{V}_{k,3}\bar{\Phi}_{l,3} + \bar{\mathcal{V}}_{k,3}\Phi_{l,3}\right) + A_{k3l3}\Phi_{k,3}\bar{\Phi}_{l,3}\right]\\
&-z\left[C_{k\alpha l\beta}\mathcal{V}_{k,\alpha}\bar{\mathcal{V}}_{l,\beta} + B_{k\alpha l\beta}\left(\mathcal{V}_{k,\alpha}\bar{\Phi}_{l,\beta} + \bar{\mathcal{V}}_{k,\alpha}\Phi_{l,\beta}\right) + A_{k\alpha l\beta}\Phi_{k,\alpha}\bar{\Phi}_{l,\beta}\right] +\\
&-z\iota\omega\left[D_{m\alpha}\left(\Theta\bar{\mathcal{V}}_{m,\alpha} - \bar{\Theta}\mathcal{V}_{m,\alpha}\right) + E_{m\alpha}\left(\Theta\bar{\Phi}_{m,\alpha} - \bar{\Theta}\Phi_{m,\alpha}\right)\right] +\\
&+\frac{z}{T_0}\left(\kappa_{33}\Theta_{,3}\bar{\Theta}_{,3} - \kappa_{\alpha\beta}\Theta_{,\alpha}\bar{\Theta}_{,\beta}\right) + z\omega^2\left(\rho V_m \bar{V}_m + I_{mn}\Phi_m\bar{\Phi}_n + \frac{a}{T_0}\Theta\bar{\Theta}\right)\Big] dA \ge\\
&\ge \int_{D(z)} \left[C_{mnkl}\mathcal{V}_{n,m}\bar{\mathcal{V}}_{l,k} + B_{mnkl}\left(\mathcal{V}_{n,m}\bar{\Phi}_{l,k} + \bar{\mathcal{V}}_{n,m}\Phi_{l,k}\right) + \right.\\
&\left.+A_{mnkl}\Phi_{n,m}\bar{\Phi}_{l,k} + \frac{1}{T_0}\kappa_{mn}\Theta_{,m}\bar{\Theta}_{,n}\right] dA.
\end{aligned} \tag{66}$$

In this way, the proof of Theorem 5 is finished. □

4. Conclusions

It is worth noting that the inequality (66), of differential type, differs essentially from the other inequalities that are used to obtain the estimates of Saint-Venant type. In order to obtain our spatial decay, we used some auxiliary identities which are based only on the strong ellipticity hypotheses imposed to the thermoelastic tensors. As such, our estimates can be applied to many different materials.

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