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RESEARCH PAPER

Analytical Solutions for Heat Transfer and Flow of Thin Film on an Inclined Wall Using the Optimal Homotopy Asymptotic Method

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Abstract

This study explores the analytical solutions for non-isothermal couple stress fluid flow in thin films over an inclined plane. The strongly nonlinear ordinary differential equations governing momentum and energy transport are derived and solved analytically using the Optimal Homotopy Asymptotic Method (OHAM) under appropriate boundary conditions. The study provides explicit expressions for temperature distribution, vorticity, shear stress, volume flow rate, and velocity profile. A comprehensive comparison of numerical and graphical results demonstrates good agreement, validating the accuracy of the proposed method. The findings contribute to a deeper understanding of heat transfer and fluid dynamics in industrial, biomedical, and engineering applications. Additionally, the influence of key parameters such as couple stress effects, heat transfer rates, and variations in thin-film thickness are analyzed in detail. The study's results can be applied to lubrication systems, microfluidics, and coating technologies, where non-Newtonian fluid behavior plays a crucial role. The effectiveness of OHAM in addressing nonlinear problems is highlighted, showcasing its advantages over conventional numerical techniques. The study also emphasizes the significance of thermophysical properties in determining flow characteristics, offering valuable insights for researchers and engineers in applied fluid mechanics.

Keywords: Couple Stress Fluid, Thin Film Flow, OHAM, Temperature Distribution

1. Introduction

In recent years, people worldwide have shown a growing interest in studying non-Newtonian fluids from both theoretical and applied perspectives [1-3]. The study of non-Newtonian fluids is crucial because they are widely used in various industrial and technological applications. These fluids include, but are not limited to, paint, shampoo, mud, ketchup, polymer melts, blood, clay coatings, certain oils and greases, and other mixtures. A careful flow analysis is essential for these fluids, both in theory and in practice. In theory, the study of these flow behaviors is

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fundamental to the field of fluid mechanics [4-7]. These flows are useful in many industrial manufacturing processes, in our opinion. Researchers conduct extensive investigations on non-Newtonian fluids, primarily by examining the differential equations that arise from these fluids. In practical domains such as atmospheric rheology and physics, fluid mechanics is studied through the measurement of material coefficients through an experimental setup. Non-Newtonian fluids exhibit a vast variety of physical structures, making it challenging to offer a single constitutive equation that captures all of their properties. As a result, several fluid models that forecast the non-Newtonian behavior of various materials have been made available. [8] The one that has usual the extreme attention is the generalized second-grade fluid model.

Recently, a number of specialists have been interested in thin film flows. Its widespread popularity in industrial manufacturing processes is the reason for this. The literature on thin-film flows for Newtonian fluids is abundant, but non-Newtonian fluids haven't gotten as much attention in this area [9]. Siddiqui et al. [10-13] and Hayat et al. [14, 15] have hardly attempted to handle non-Newtonian fluid thin film flows.

A large number of researchers must be interested in thin-film flow and heat transfer [16-18]. This is because of their many technical and industrial applications, which include the processing of food products, the creation of coatings for wire and fiber, the fluidization of reactors, the cooling of transpiration, the processing of polymers, heat pipes, gaseous diffusion, and the fluidic cells that are a part of many biochemical and organic finding classifications. Lavrik et al. [19] looked at the issue of places for organic and biotic micro cantilevers, like as fluidic cells. These were natural and live discovery systems. In most flow and heat transfer study problems, the non-Newtonian fluid is represented by the power-law fluid model. While several examples, like polymer processing, have shown the importance of this phenomena, studies that combine the effects of viscous dissipation have gotten very little attention.

This paper is organized into multiple sections. Section 2 contains nomenclature. Section 3 depicted the basic equation. Section 4 contains the formulation of the problem. The solution to the problem and the basic phenomena of the technique (OHAM) are covered in Section 5. Section 6 provides the problem's shear stress, volumetric flow rate, average velocity and vorticity. While Sections 7 deal with the result and discussion. Section 8 provides the conclusion.

2. Nomenclature

$\boldsymbol{u}$	Fluid velocity
$\boldsymbol{L}$	Gradient of V
η	Couple Stress Parameter
Ψ	Temperature
B_r	Brinkman number
ρ	Density
$\boldsymbol{f}$	Body force
P	Pressure
C_p	Specific heat
κ	Thermal conductivity
$\frac{D}{Dt}$	Material time derivative
Ψ^*	Dimensionless temperature
u^*	Dimensionless velocity
$\boldsymbol{A}_1$	First Rivilin–Erickson tensor

3. Basic Equations

The main idea equations guiding a pair stress liquids movement, taking into consideration thermal effects, are

$$\nabla.\boldsymbol{u} = 0, \tag{1}$$

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \boldsymbol{f} - \nabla P - \nabla.\boldsymbol{T} - \eta \nabla^4 \boldsymbol{u}, \tag{2}$$

$$\rho C_p \frac{D\Psi}{Dt} = \kappa \nabla^2 \Psi - \boldsymbol{T}.\boldsymbol{L} \tag{3}$$

The fluid velocity is symbolizes by $"\boldsymbol{u}"$ the couple stress parameter is denoted $"\eta"$, the body force is signified by $"\boldsymbol{f}"$, the fluid density is represents by $"\rho"$, $"\Psi"$ stand for temperature, the specific heat is signified by $"C_p"$, $"P"$ is the pressure, the material time derivative denoted $"\frac{D}{Dt}"$ and $"\kappa"$ stands for conductivity. These values are clear as

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \boldsymbol{u}.\nabla \right),$$

where
$$\boldsymbol{T} = \mu(\boldsymbol{A}_1),$$

$$\boldsymbol{A}_1 = \boldsymbol{L} + \boldsymbol{L}^T.$$

4. Problem Formulation

Let us assume a couple stress liquid flowing on an inclined surface in the form of thin film. The film is assumed to be of uniform thickness δ, and the only force moving it is gravity. The geometry of the problem can be seen in figure 1 below. The assumptions on flow are

$$\boldsymbol{V} = \left[u(y), 0, 0 \right] \qquad \Psi = \Psi(y), \quad \mathrm{T} = \mathrm{T}(y). \tag{4}$$

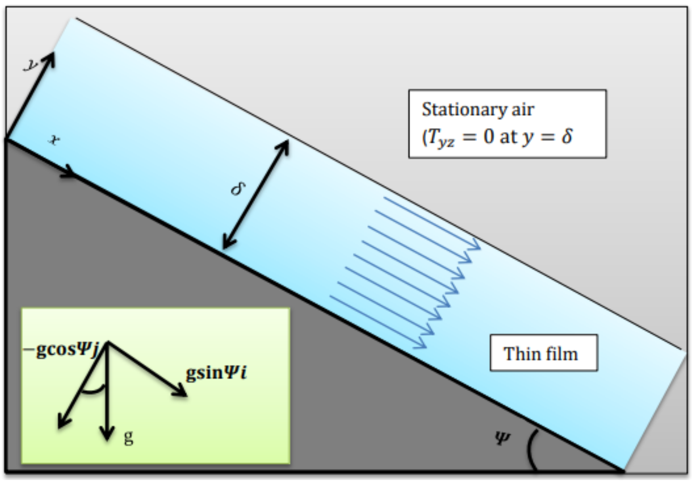

Figure 1: Couple Stress fluid flowing in a thin layer down an inclined plane.

Equation (1) is now satisfied exactly when using equations (4). Additionally, Eq. (2) is reduced to

$$\eta \frac{d^4 u}{dy^4} - \mu \frac{d^2 u}{dy^2} - \rho g \sin\theta = 0. \tag{5}$$

The equation (3) becomes

$$\kappa \frac{d^2\Psi}{dy^2} + \mu\left(\frac{du}{dy}\right)^2 = 0. \tag{6}$$

The linked boundary conditions are

$$\text{At} \quad y = 0\ ,\ \ u = U\ ,\ \ \frac{d^2u}{dx^2} = 0\ ,\ \ \ \Psi = 0. \tag{7}$$

$$\text{At} \quad y = 1,\ \ \frac{du}{dy} = 0\ ,\ \ \frac{d^3u}{dy^3} = 0,\ \ \ \frac{d\Psi}{dy} = 0. \tag{8}$$

For the non-dimensionalization of the equations, the following non-dimensional parameters are presented.

$$y^* = \frac{x}{\delta}\ ,\ u^* = \frac{u}{U},\ \Psi^* = \frac{\Psi - \Psi_0}{\Psi_1 - \Psi_0}, \alpha^2 = \frac{\mu\delta^2}{\eta},\ k = \frac{\rho g \delta^4}{\eta U}\sin\theta,\ B_r = \frac{\mu U^2}{k\left(\Psi_1 - \Psi_0\right)}.$$

"B_r" Represent the Brinkman number. Presenting these values in equation (5 - 8), and removing the "*" one gets

$$\frac{d^4u}{dy^4} - \alpha^2\frac{d^2u}{dy^2} = k, \tag{9}$$

$$\frac{d^2\Psi}{dy^2} = -B_r\left(\frac{du}{dy}\right)^2, \tag{10}$$

$$\text{At} \quad y = 0\ ,\ \ u = 1\ ,\ \ \frac{d^2u}{dx^2} = 0\ ,\ \ \ \Psi = 0. \tag{11}$$

$$\text{At} \quad y = 1,\ \ \frac{du}{dy} = 0\ ,\ \ \frac{d^3u}{dy^3} = 0,\ \ \ \frac{d\Psi}{dy} = 0. \tag{12}$$

5. Solutions of the Problem

5.1. Exact Solution

The exact result of Eq. (9) is

$$u = \frac{1}{2\left(1 + e^{2\alpha}\right)\alpha^4}\left(e^{-y\alpha}\left(\begin{array}{l} 2e^{2\alpha}k - 2e^{y\alpha}k + 2e^{2y\alpha}k - 2e^{2\alpha + y\alpha}k + 2e^{y\alpha}ky\alpha^2 \\ +2e^{2\alpha + y\alpha}ky\alpha^2 - e^{y\alpha}ky^2\alpha^2 - e^{2\alpha + y\alpha}ky^2\alpha^2 + 2e^{y\alpha}\alpha^4 + 2e^{2\alpha + y\alpha}\alpha^4 \end{array}\right)\right). \tag{13}$$

Now, solve (10) with the appropriate boundary conditions to get the following answer.

$$\begin{aligned}\Psi = \frac{1}{12\left(1+\mathrm{e}^{2\alpha}\right)^{2}\alpha^{8}}\mathrm{e}^{-2y\alpha}k^{2}(&-3\mathrm{e}^{4\alpha}+51\mathrm{e}^{2y\alpha}-48\mathrm{e}^{3y\alpha}-3\mathrm{e}^{4y\alpha}-48\mathrm{e}^{(4+y)\alpha}-48\mathrm{e}^{(2+y)\alpha}\\
&+96\mathrm{e}^{2\alpha+2y\alpha}-48\mathrm{e}^{(2+3y)\alpha}+51\mathrm{e}^{4\alpha+2y\alpha}+24\mathrm{e}^{2y\alpha}\alpha-24\mathrm{e}^{3y\alpha}\alpha+24\mathrm{e}^{(2+y)\alpha}\alpha+24\mathrm{e}^{(4+y)\alpha}\alpha\\
&-24\mathrm{e}^{(2+3y)\alpha}\alpha+24\mathrm{e}^{3y\alpha}y\alpha-24\mathrm{e}^{4\alpha+2y\alpha}\alpha-24\mathrm{e}^{(2+y)\alpha}y\alpha-24\mathrm{e}^{(4+y)\alpha}y\alpha+24\mathrm{e}^{(2+3y)\alpha}y\alpha\\
&-24\mathrm{e}^{2\alpha+2y\alpha}y\alpha^{2}+12\mathrm{e}^{2(1+y)\alpha}y^{2}\alpha^{2}+4\mathrm{e}^{2y\alpha}y\alpha^{4}+8\mathrm{e}^{2\alpha+2y\alpha}y\alpha^{4}+4\mathrm{e}^{4\alpha+2y\alpha}y\alpha^{4}\\
&-6\mathrm{e}^{2y\alpha}y^{2}\alpha^{4}-12\mathrm{e}^{2(1+y)\alpha}y^{2}\alpha^{4}-6\mathrm{e}^{2(2+y)\alpha}y^{2}\alpha^{4}+4\mathrm{e}^{2y\alpha}y^{3}\alpha^{4}+8\mathrm{e}^{2(1+y)\alpha}y^{3}\alpha^{4}\\
&+4\mathrm{e}^{2(2+y)\alpha}y^{3}\alpha^{4}-\mathrm{e}^{2y\alpha}y^{4}\alpha^{4}-2\mathrm{e}^{2(1+y)\alpha}y^{4}\alpha^{4}-\mathrm{e}^{2(2+y)\alpha}y^{4}\alpha^{4})B_{r}.\end{aligned} \tag{14}$$

Now, the Optimal Homotopy Asymptotic method (OHAM) is used to determine the estimated answers of equations (9) and (10) subject to the boundary conditions. However, first we give the basics of the OHAM.

5.2. Basics Phenomena of Optimal Homotopy Asymptotic Method (OHAM)

Supposing that the differential equation is non-linear.

$$L\left(\Psi\left(\tau\right)\right)+F\left(\tau\right)+N\left(\Psi\left(\tau\right)\right)=0,\qquad B\left(\Psi,\frac{d\Psi}{d\tau}\right)=0. \tag{15}$$

The non-linear operator is represented by $N\left(\Psi\left(\tau\right)\right)$, the linear operator is denoted by L, the provided function is $F\left(\tau\right)$, the unknown function is still $\Psi\left(\tau\right)$, and the boundary operator is B.

The Optimal Homotopy Asymptotic Method (OHAM) was utilized to produce the following results.

$$\left(1-r\right)\left[L\left(\Psi\left(\tau,r\right)\right)+F\left(\tau\right)\right]=H\left(r\right)\left[L\left(\Psi\left(\tau,r\right)\right)+F\left(\tau\right)+N\left(\Psi\left(\tau,r\right)\right)\right],\ B\left(\Psi\left(\tau,r\right),\frac{d\Psi\left(\tau,r\right)}{d\tau}\right)=0. \tag{16}$$

When $r=0$ (i.e. $H\left(0\right)=0$), the secondary function $H\left(r\right)$, is definitely given. In the same way, for $r\neq 0$, it is non-zero. We can express the embedding variable $r\in\left[0,1\right]$ as follows:

$$\Psi\left(\tau,0\right)=\Psi_{0}\left(\tau\right),\ \ \Psi\left(\tau,1\right)=\Psi\left(\tau\right). \tag{17}$$

The range of the solution $\Psi\left(\tau,r\right)$ is $\Psi_{0}\left(\tau\right)$ and $\Psi\left(\tau\right)$, where r diverges from 0 to 1. By altering $r=0$ in Eq. (15), we can derive $\Psi_{0}\left(\tau\right)$.

$$L\left(\Psi_{0}\left(\tau\right)\right)+F\left(\tau\right)=0\ ,\ B\left(\Psi,\frac{d\Psi_{0}}{d\tau}\right)=0 \tag{18}$$

The auxiliary function $H\left(r\right)$ has the following expression:

$$H\left(r\right)=rc_{1}+r^{2}c_{2}+r^{3}c_{3}+\ldots \tag{19}$$

Since it is necessary to determine the constants in this case, $c_1,c_2,c_3,\ldots$ Eq. (16) can be expressed as

$$\Psi\left(\tau,r,c_{i}\right)=\Psi_{0}\left(\tau\right)+\sum_{j\geq 1}\Psi_{j}\left(\tau,c_{i}\right)r^{j},\ \ i=1,2,\ldots \tag{20}$$

We get the following system by analyzing comparable powers of r and replacing Eq. (20) to Eq. (16).

$$L\left(\Psi_1(\tau)\right) = c_1 N_0\left(\Psi_0(\tau)\right), \quad B\left(\Psi_1, \frac{d\Psi_1}{d\tau}\right) = 0 \tag{21}$$

$$L(\Psi_j(\tau) - \Psi_{j-1}(\tau)) = c_j N_0\left(\Psi_0(\tau)\right) + \sum_{k=1}^{j-1} c_k [L\left(\Psi_{j-k}(\tau)\right) N_{j-k}\left(\Psi_0(\tau), \Psi_1(\tau)\ldots, \Psi_{j-1}(\tau)\right)],$$
$$B\left(\Psi_1, \frac{d\Psi}{d\tau}\right) = 0, \; j = 1, 2, \ldots \tag{22}$$

In Eq. (22) the term $N_m\left(\Psi_0(\tau), \Psi_1(\tau), \ldots, \Psi_n(\tau)\right)$ is the coefficient of r^m in the expansion of

$$N\left(\Psi(\tau, r, c_k)\right) = N_0\left(\Psi_0(\tau)\right) + \sum_{j\geq 1} N_j(\Psi_0(\tau), \Psi_1(\tau), \ldots, \Psi_j(\tau)) r^j, \qquad k = 1, 2, \ldots \tag{23}$$

For $\Psi_j(\tau), \; j \geq 0,$ the result of Eq. (20) may be simply determined as follows: the convergence is solely dependent on the constants $c_1, c_2, c_3, \ldots$; if it is convergent at $r = 1,$ then from Eq. (20) we get

$$\Psi(\tau, c_k) = \Psi_0(\tau) + \sum_{j\geq 1} \Psi_j(\tau, c_i). \tag{24}$$

In common, the result of equation (15) is estimated by

$$\Psi^n(\tau, c_k) = \Psi_0(\tau) + \sum_{j=1}^{n} \Psi_j(\tau, c_k), \qquad k = 1, 2, \ldots, n. \tag{25}$$

Eq. (18) is replaced into Eq. (25) and the residual is attained.

$$R(\tau, c_k) = L\left(\Psi^n(\tau, c_k)\right) + F(\tau) + N\left(\Psi^n(\tau, c_k)\right), \; k = 1, 2, \ldots, n. \tag{26}$$

We obtain the exact solution $\Psi^n(\tau, c_k)$, when $R(\tau, c_k) = 0,$ that is, the residual is zero; but, if $R(\tau, c_k) \neq 0$, that is, We can reduce in the following way if the residual is not zero.

$$J(c_l) = \int_a^b R^2(\omega, c_l)\, dx. \tag{27}$$

The unknown constants $c_1, c_2, c_3, \ldots,$ and so on, where a and b are constants based on the problem under discussion, can be calculated using the criteria..

$$\frac{\partial J}{\partial c_l} = 0, \quad l = 1, 2, \ldots, n. \tag{28}$$

Equation (18) can be used to get the approximated answer after we have the values of these constants. Furthermore, we use the collocation approach to determine the value of the unknown constant $c_1, c_2, c_3, \ldots.$

5.3. Solution of the Problem by OHAM

Using OHAM, the resulting component equations up to third order and their results are found.

5.3.1. Zero Component Problems

Zero component equations of velocity and temperature, along with the related boundary conditions, are

$$\frac{d^4 u_0}{dy^4} - k = 0, \tag{29}$$

$$\frac{d^2\Psi_0}{dy^2}=0, \tag{30}$$

At $\quad y=0, \qquad u_0=1\,, \qquad \dfrac{d^2u_0}{dy^2}=0, \qquad \Psi_0=0,$ (31)

At $\quad y=1\,, \qquad \dfrac{du_0}{dy}=0\,, \qquad \dfrac{d^3u_0}{dy^3}=0\,, \qquad \dfrac{d\Psi_0}{dy}=0.$ (32)

5.3.2. First Component Problems

First component equations of velocity and temperature, along with the related boundary conditions, are

$$\frac{d^4u_1}{dy^4}-C_1\left(\frac{d^4u_0}{dy^4}\right)-\frac{d^4u_0}{dy^4}+\alpha^2C_1\left(\frac{d^2u_0}{dy^2}\right)+k+kC_1=0,, \tag{33}$$

$$\frac{d^2\Psi_1}{dy^2}-B_rC_3\left(\frac{du_0}{dy}\right)^2-\frac{d^2\Psi_0}{dy^2}-C_3\left(\frac{d\Psi_0}{dy}\right)=0,, \tag{34}$$

At $\quad y=0, \qquad u_1=0\,, \qquad \dfrac{d^2u_1}{dy^2}=0, \qquad \Psi_1=0,$ (35)

At $\quad y=1\,, \qquad \dfrac{du_1}{dy}=0\,, \qquad \dfrac{d^3u_1}{dy^3}=0\,, \qquad \dfrac{d\Psi_1}{dy}=0.$ (36)

5.3.3. Second Component Problems

Second component equations of velocity and temperature, along with the related boundary conditions, are

$$\frac{d^4u_2}{dy^4}-C_1\left(\frac{d^4u_1}{dy^4}\right)-\frac{d^4u_1}{dy^4}-C_2\left(\frac{d^4u_0}{dy^4}\right)+\alpha^2C_1\left(\frac{d^2u_1}{dy^2}\right)+\alpha^2C_2\left(\frac{d^2u_0}{dy^2}\right)+kC_2=0, \tag{37}$$

$$\frac{d^2\Psi_2}{dy^2}+2B_rC_3\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)-B_rC_4\left(\frac{du_0}{dy}\right)^2-C_4\left(\frac{d^2\Psi_0}{dy^2}\right)-\frac{d^2\Psi_1}{dy^2}-C_3\left(\frac{d^2\Psi_1}{dy^2}\right)=0, \tag{38}$$

At $\quad y=0, \qquad u_2=0\,, \qquad \dfrac{d^2u_2}{dy^2}=0, \qquad \Psi_2=0,$ (38)

At $\quad y=1\,, \qquad \dfrac{du_2}{dy}=0\,, \qquad \dfrac{d^3u_2}{dy^3}=0\,, \qquad \dfrac{d\Psi_2}{dy}=0.$ (40)

5.3.4. Third Component Problems

Third component equations of velocity and temperature, along with the related boundary conditions, are

$$\frac{d^4u_3}{dy^4}-C_1\left(\frac{d^4u_2}{dy^4}\right)-\frac{d^4u_2}{dy^4}-C_2\left(\frac{d^4u_1}{dy^4}\right)+\alpha^2C_1\left(\frac{d^2u_2}{dy^2}\right)+\alpha^2C_2\left(\frac{d^2u_1}{dy^2}\right)=0, \tag{41}$$

$$\frac{d^2\Psi_3}{dy^2} - 2B_rC_3\left(\frac{du_0}{dy}\right)\left(\frac{du_2}{dy}\right) - B_rC_3\left(\frac{du_1}{dy}\right)^2 - C_4\left(\frac{d^2\Psi_1}{dy^2}\right) - \frac{d^2\Psi_2}{dy^2} - C_3\left(\frac{d^2\Psi_2}{dy^2}\right) - 2B_rC_4\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right) = 0, \quad (42)$$

At $y = 0,\quad u_3 = 0,\quad \frac{d^2u_3}{dy^2} = 0,\quad \Psi_3 = 0,$ (43)

At $y = 1,\quad \frac{du_3}{dy} = 0,\quad \frac{d^3u_3}{dy^3} = 0,\quad \frac{d\Psi_3}{dy} = 0.$ (44)

The solutions obtained for these equations by OHAM are cumbersome and we have tabulated these solutions in table 1 and 2.

6. Shear Stress, Vorticity, Average Velocity, Volumetric Flow Rate

6.1. Shear Stress on Inclined Surface

On inclined surface the shear stress is obtained by

$$T_{xy}\,|_{y=0} = \mu\left(\frac{du}{dy}\right)|_{y=0} \quad (45)$$

Table 1: Residual of velocity $\alpha = 0.03$, $k = 0.0005$, using OHAM technique.

y	u_{OHAM}	Residual u_{OHAM}
0.	1.	0.
0.1	1.0000165794438058	$3.618087999306943\times10^{-21}$
0.2	1.0000326882031383	$1.753622898612418\times10^{-22}$
0.4	1.0000479014226578	$2.580803511165446\times10^{-22}$
0.4	1.000061844239002	$8.341289296984732\times10^{-21}$
0.5	1.0000741917774127	$2.441837168256537\times10^{-20}$
0.6	1.0000846691488114	$4.575301404153307\times10^{-20}$
0.7	1.0000930514473263	$6.872216529090554\times10^{-20}$
0.8	1.0000991637482664	$8.863405699628203\times10^{-20}$
0.9	1.0001028811065487	$1.022494498789048\times10^{-19}$
1.	1.0001041285555734	$1.069379095908554\times10^{-19}$

Table 2: Residual of temperature for $\alpha = 1.5$, $k = 0.0007$, $B_r = 0.01$, using OHAM technique.

y	Ψ_{OHAM}	Residual Ψ_{OHAM}
0.	0.	$-2.182174213833026\times10^{-18}$
0.1	$2.00840010999138\times10^{-11}$	$-8.42690249038397\times10^{-23}$
0.2	$3.570421770329235\times10^{-11}$	$9.60632426661381\times10^{-19}$
0.4	$4.72288942660287\times10^{-11}$	$4.717322111474067\times10^{-19}$
0.4	$5.520679842799398\times10^{-11}$	$-5.407693254565418\times10^{-23}$
0.5	$6.030112255356151\times10^{-11}$	$-9.921457591023609\times10^{-20}$
0.6	$6.322016366344039\times10^{-11}$	$-4.050920594582972\times10^{-20}$
0.7	$6.465007880451779\times10^{-11}$	$-1.631096770378001\times10^{-23}$
0.8	$6.519409020719452\times10^{-11}$	$3.080343052965048\times10^{-21}$
0.9	$6.532164647906939\times10^{-11}$	$-1.925779863600017\times10^{-24}$
1.	$6.533022200484851\times10^{-11}$	0.

By replacing the value we have

$$T_{xy} = \mu\left(0.+\frac{k}{3}+\frac{1}{3628800}k\alpha^2\left(\begin{array}{l}-0.055797971403916465+269.804588954112\\ \left(5376+2176\alpha^2\right)-0.9989145795294502\\ \left(483840+391680\alpha^2+79360\alpha^4\right)-179.93485119479615\\ \left(8064-5.766159412607109\times10^{-8}\left(5376+2176\alpha^2\right)\right)\end{array}\right)\right) \tag{46}$$

Table 3: Absolute difference for $\alpha = 0.03$,$k = 0.0005$, velocity profile and exact solution.

y	u_{OHAM}	Exact_u	**Absolute difference**
0.	1.	1.	$6.949996134\times10^{-14}$
0.1	1.00001657	1.000016579	$6.9603648338\times10^{-14}$
0.2	1.00003268	1.000032688	$7.5428216190\times10^{-14}$
0.4	1.000047901	1.000047901	$1.2981876316\times10^{-13}$
0.4	1.000061844	1.000061844	$1.1038620104\times10^{-13}$
0.5	1.000074191	1.000074191	$8.075479304\times10^{-14}$
0.6	1.000084669	1.000084669	$7.5187184494\times10^{-14}$
0.7	1.000093051	1.000093051	$2.9990048530\times10^{-14}$
0.8	1.000099163	1.000099163	$1.910770803\times10^{-14}$
0.9	1.000102881	1.000102881	$4.2930502902\times10^{-14}$
1.	1.000104128	1.000104128	$1.5434349761\times10^{-13}$

Table 4: Absolute difference for $\alpha = 1.5$, $k = 0.0007$, $B_r = 0.01$, velocity profile and exact solution.

y	Ψ_{OHAM}	Exact$_\Psi$	Absolute difference
0.	0.	1.30369379×10^{-25}	0.
0.1	2.008400×10^{-11}	$6.479863761\times10^{-12}$	$1.36041373387\times10^{-11}$
0.2	3.570421×10^{-11}	1.14934354×10^{-11}	$2.42107822144\times10^{-11}$
0.3	4.7228894×10^{-11}	$1.517090474\times10^{-11}$	$3.20579895163\times10^{-11}$
0.4	5.5206798×10^{-11}	$1.770141199\times10^{-11}$	$3.75053864344\times10^{-11}$
0.5	6.030112×10^{-11}	1.93080387×10^{-11}	$4.09930838062\times10^{-11}$
0.6	6.322016×10^{-11}	2.02238833×10^{-11}	$4.29962802776\times10^{-11}$
0.7	6.465007×10^{-11}	2.0670554×10^{-11}	$4.39795239607\times10^{-11}$
0.8	6.519409×10^{-11}	2.0839909×10^{-11}	$4.43541802851\times10^{-11}$
0.9	6.5321646×10^{-11}	2.0879525×10^{-11}	$4.4442120716\times10^{-11}$
1.	6.533022×10^{-11}	2.08821857×10^{-11}	$4.44480362761\times10^{-11}$

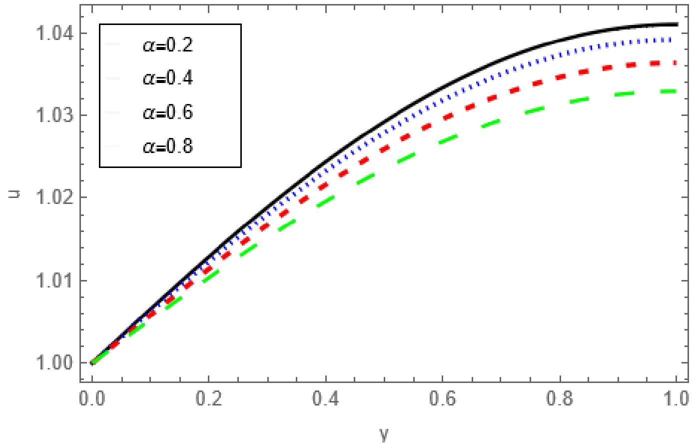

Figure 2: Velocity variation at different α keeping $k = 0.2$.

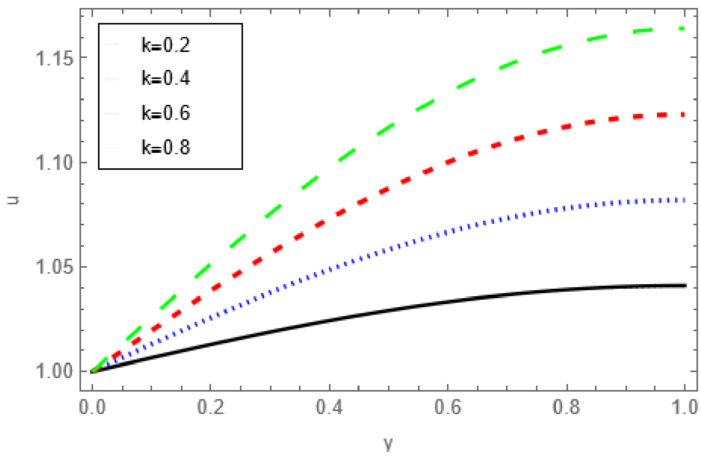

Figure 3: Velocity variation at different k keeping $\alpha = 0.2$.

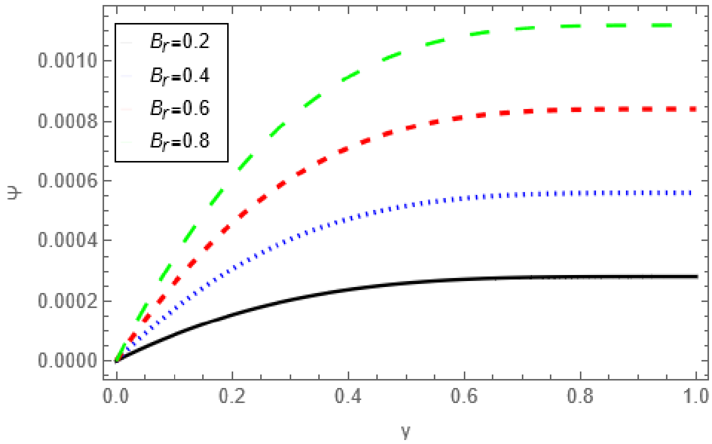

Figure 4: Temperature variation at different B_r keeping $B_r = 0.2, k = 0.3, \alpha = 0.4.$

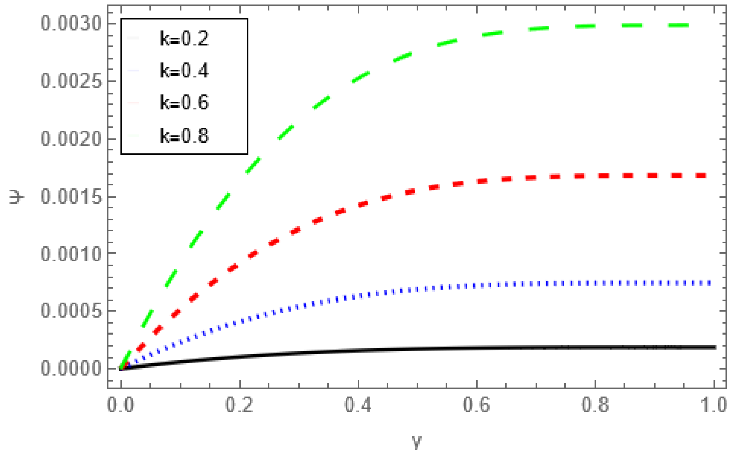

Figure 5: Temperature variation at different k keeping $B_r = 0.3, \alpha = 0.4.$

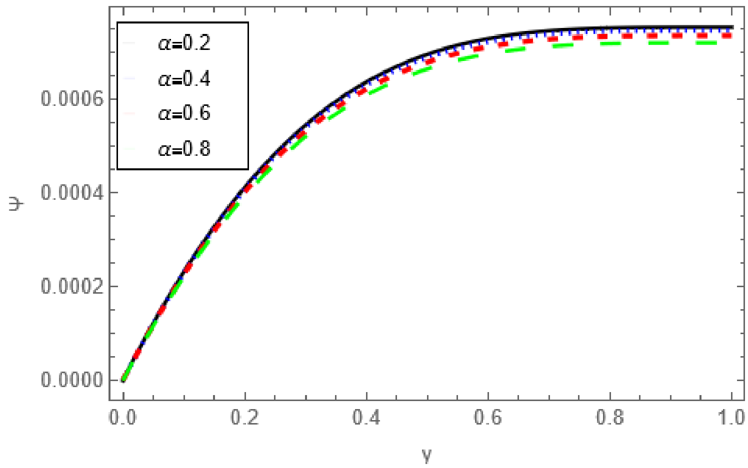

Figure 6: Variation at different on temperature α keeping $B_r = 0.3, k = 0.4.$

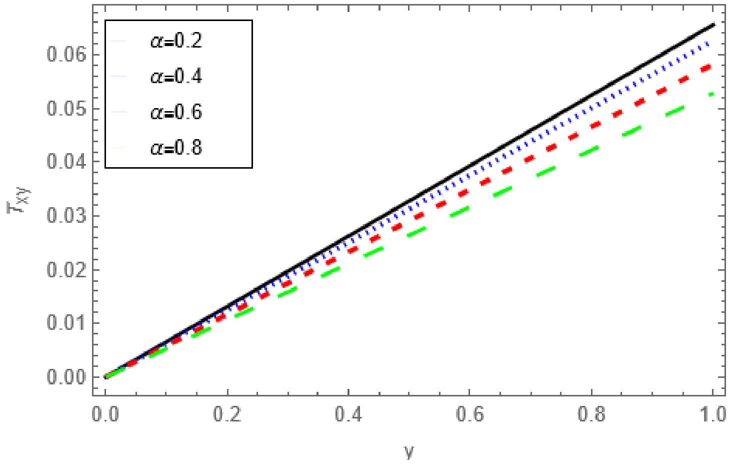

Figure 7: Variation of shear stress at different α keeping $k = 0.2$.

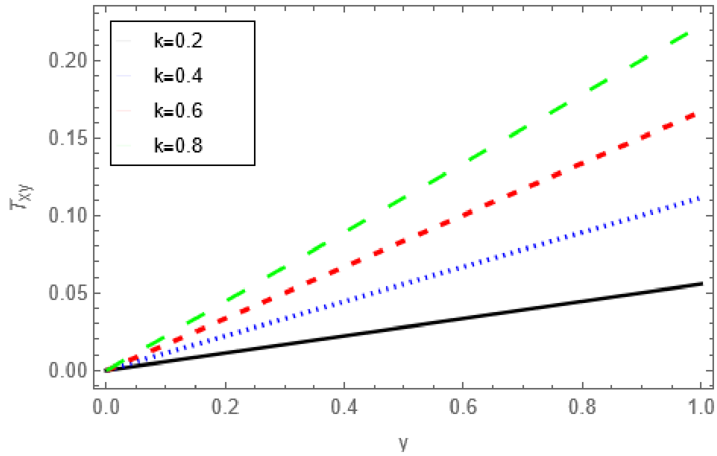

Figure 8: Variation of shear stress at different k keeping $\alpha = 0.7$.

6.2. Vorticity

The vorticity of the inclined is

$$\vec{\boldsymbol{\omega}} = \nabla \times \boldsymbol{V} = -\left(\frac{du}{dy}\right)\hat{\boldsymbol{k}}. \tag{47}$$

By putting the value we have

$$\vec{\boldsymbol{\omega}} = -\hat{\boldsymbol{k}}(\frac{1}{24}ky\left(2\left(-4+y\right)y+y^2\right)+\frac{1}{24}k\left(8+\left(-4+y\right)y^2\right)+\frac{1}{3628800}$$
$$ky\alpha^2(0.0005812288687907965\left(80y-24y^3+5y^4\right)+269.804588954112$$
$$(-56\left(80y-24y^3+5y^4\right)+\left(-1792y+448y^3-48y^5+7y^6\right)\alpha^2)-0.9989145795294502$$
$$\left(\begin{array}{l}-5040\left(80y-24y^3+5y^4\right)+180\left(-1792y+448y^3-48y^5+7y^6\right)\alpha^2\\-\left(65280y-16128y^3+1440y^5-80y^7+9y^8\right)\alpha^4\end{array}\right)$$

$$-179.93485119479615\begin{pmatrix}-84\left(80y-24y^3+5y^4\right)-5.766159412607109\times10^{-8}\\ \left(-56\left(80y-24y^3+5y^4\right)+\left(-1792y+448y^3-48y^5+7y^6\right)\alpha^2\right)\end{pmatrix})$$

$$+\frac{1}{3628800}k\alpha^2(0.0005812288687907965\left(-96+40y^2-6y^4+y^5\right)$$

$$+269.804588954112\left(-56\left(-96+40y^2-6y^4+y^5\right)+\begin{pmatrix}2176-896y^2\\+112y^4-8y^6+y^7\end{pmatrix}\alpha^2\right) \quad (48)$$

$$-0.9989145795294502\begin{pmatrix}-5040\left(-96+40y^2-6y^4+y^5\right)+180\begin{pmatrix}2176-896y^2\\+112y^4-8y^6+y^7\end{pmatrix}\alpha^2\\ -\left(-79360+32640y^2-4032y^4+240y^6-10y^8+y^9\right)\alpha^4\end{pmatrix}$$

$$-179.93485119479615(-84\left(-96+40y^2-6y^4+y^5\right)-5.766159412607109\times10^{-8}$$

$$\left(-56\left(-96+40y^2-6y^4+y^5\right)+\left(2176-896y^2+112y^4-8y^6+y^7\right)\alpha^2)))\right).$$

6.3. Volumetric Flow Rate

Inclined volumetric flow rate is determined by

$$Q=\int_0^1 u\,dy. \quad (49)$$

By putting the value we have

$$Q_{(OHAM)}=1+\frac{2k}{15}-0.05396825396794774\,4k\alpha^2+0.021869482464733 53k\alpha^4$$
$$-0.00885361519262273 8k\alpha^6. \quad (50)$$

6.4. Average Velocity

For an inclined problem, the average velocity in dimensional form is provided by

$$\bar{u}=\frac{Q}{\delta}. \quad (51)$$

The average velocity and flow rate coincide in non-dimensional form, thus

$$\bar{u}_{(OHAM)}=1+\frac{2k}{15}-0.053968253967947744k\alpha^2+0.02186948246473353k\alpha^4$$
$$-0.008853615192622738k\alpha^6. \quad (52)$$

7. Results and Discussion

In the current work, we approximated the thin-film flow results using non-isothermal couple stress fluid thin-film flow on an inclined plane. This problem develops differential equations and associated boundary conditions. One method for solving non-linear differential equations is the Optimal Homotopy Asymptotic Method (OHAM). The variation in temperature distribution and velocity profiles due to changed parameters like α, k and B_r. Tables 1 and 2 present the velocity profile and temperature distribution OHAM results along with the corresponding problem residuals. The absolute difference between the OHAM solution and the exact solutions for velocity and temperature are displayed in Tables 3 and 4. Figures 2 and 3 show how the values of α and k affect the velocity profile. Figure 2 shows that the velocity profile and α have an inverse relationship, but Figure 3 shows that k and velocity have a direct relationship. Figures 4, 5, and 6 show how the parameters B_r, k and α affect the temperature profiles. There is a direct relationship between $\bar{u} > 0$ and the temperature profile in Figures 4 and 5, and an opposite relationship between α and the temperature profile in Figure 6. Figures 7 and 8 show how variables α and k affect shear stress. Figure 7 shows that shear stress and α have an inverse relationship, but Figure 8 shows that k and shear stress have a direct relationship.

8. Conclusion

This work has used OHAM to study the non-isothermal couple stress fluid thin-film flow on an inclined plane. An incredible arrangement is produced by calculating the OHAM findings both mathematically and graphically. The main findings are enumerated below.

- It's crucial to remember that regular stresses don't support a stable couple stress fluid flow.
- In the event that the average velocity, $\bar{u} > 0$, there will be a net upward flow. Equations (52) provide it.
- A comparable decrease in velocity is observed as parameter α increases.
- The increases in parameter k cause spikes in velocity.
- With a surge in the Brinkman number B_r and k the temperature rises

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