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RESEARCH PAPER

High-Accuracy Modified Spectral Techniques for Two-Dimensional Integral Equations

Ahmed M. Abbas [a,b], Y.H. Youssri [c,d,*], M. El-kady [a,b,e], M. Abdelhakem [a,b,e]

[a] *Mathematics Department, Faculty of Science, Helwan University, Cairo 11795, Egypt*
[b] *Basic Science Department, School of Engineering, Canadian International College, New Cairo, Egypt*
[c] *Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt*
[d] *Associate Fellow (AFHEA) of the Higher Education Academy (Advance HE), UK*
[e] *Helwan School of Numerical Analysis in Egypt (HSNAE)*

Abstract

This research introduces a numerical method for solving two-dimensional integral equations. The exact solution is assumed to be a limit point for the set of all polynomials and is approximated to be a finite series of constant multiples of basis functions for the polynomial functions space. Legendre's first derivative polynomials have been chosen in this work as the orthogonal basis functions. Some new relations are constructed, such as the linearization formula. Subsequently, applying the pseudo-Galerkin spectral method results in a system of algebraic equations in the constant coefficients of the approximated expansion. Lastly, we solve the algebraic system using the Gauss elimination method for linear systems or Newton's iteration method with zero initial guesses for nonlinear systems that are most likely to appear out of the presented procedure. This approach yields the desired semi-analytic approximate solution. Convergence and error analyses have been studied. To clarify the efficiency and accuracy of the presented method, we solved some numerical test problems.

Keywords: Legendre polynomials; spectral methods; pseudo-Galerkin spectral method

1. Introduction

Integral and differential equations have many applications in various fields [1-10]. Most of the time, the exact solution cannot be obtained analytically, and numerical methods have been constructed to introduce numerical and semi-analytic approximations as suitable alternatives, such as finite element [11-13], finite difference [14, 15], and spectral methods [16-20]. Finite element and finite difference methods give us a pure numerical approximation solution. However, spectral methods give us a semi-analytic approximation solution. The main idea in all spectral methods is expanding the dependent variable as a linear combination of a set of functions that form a basis for the space of polynomials. They should be orthogonal according to an inner product under a weight function $w(x)$. 2-D integral equations are used in many scientific and engineering domains, including mathematical physics, fluid mechanics, potential theory, and image processing. These equations, which are composed of unknown functions provided over 2-D domains, can be classified as either Volterra or Fredholm types according to the limitations of integration. Spectral methods convert the integral equation into an algebraic system of equations that may be solved rapidly by extending the unknown function in terms of orthogonal polynomial bases. This method has been effectively applied to issues like heat conduction in composite materials and electrostatic interactions that call for high precision and quick convergence. For 2-D functions we have: The function $u(x, y)$ can be represented as an infinite series expansion:

[*] Corresponding author. *E-mail address:* youssri@cu.edu.eg

$$u(x,y) = \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{i=0}^{N} \sum_{j=0}^{M} c_{i,j}\phi_i(x)\phi_j(y). \tag{1}$$

This expansion allows us to reformulate the integral or differential equation in the following form:

$$\lim_{\substack{N\to\infty \\ M\to\infty}} R_{N,M}(x,y) = 0, \tag{2}$$

where $R_{N,M}(x,y)$ represents the residual function.

Because of their high accuracy and exponential convergence for smooth problems, spectral methods are frequently used to solve differential and integral equations. These approaches use a finite set of orthogonal basis functions, selected according to the issue domain, to approximate the answer. Among the different types of spectral methods, the Galerkin, Petrov-Galerkin, and collocation approaches are commonly used. Many recent works have demonstrated the efficiency of spectral methods in various applications. To discuss some of the recent advances in spectral methods, we mention here, that Youssri and Atta developed a Chebyshev collocation algorithm to model human corneal shape via the Caputo fractional derivative [21]. Youssri et al. proposed efficient Petrov-Galerkin schemes using Chebyshev polynomials for the Euler-Bernoulli beam equation [22] and nonlinear time-fractional integro-differential equations [23]. Youssri et al. developed explicit collocation and spectral collocation methods using third-kind Chebyshev polynomials for fractional Duffing equations [24] and biological interactions [25]. They also proposed a radical Petrov-Galerkin approach for the time-fractional KdV-Burgers' equation [26] and analyzed small deflections in fourth-order beam equations [27]. Ahmed et al. derived formulas for expansion and connection coefficients in classical orthogonal polynomials [28], and Atta et al. developed a spectral collocation algorithm for the fractional Bratu equation using hexic shifted Chebyshev polynomials [29].

Spectral methods involve three types: the Galerkin, Tau, and collocation methods. A modification of a spectral method is called the pseudo-Galerkin spectral method is used through this work, in which a system of algebraic equations is constructed by setting $R_{N,M}(x_i,y_j) = 0$, for some suitable set of points. Legendre's first derivative polynomials are chosen to be the basis functions. New relations have been constructed such as the linearization formula. Finally, the presented method is applied to some applications.

2. Preliminaries

Through this work, some needed relations of the introduced basis functions, Legendre's first derivative polynomials, are presented, such as the moment relation and the integration formula. They are essential to produce the introduced algorithm.

The following definition describes the basis functions used through this work [30].

Definition 1. *Legendre's derivative polynomials of degree q, denoted by $DL_q(x)$, are defined to be the derivative of the Legendre polynomial that is one degree higher:*

$$DL_q(x) = \frac{d}{dx}\mathcal{L}_{q+1}(x), \tag{3}$$

where q is a non-negative integer.

These polynomials can be expressed in an explicit forms as follow:

$$DL_q(x) = \sum_{j=0}^{\lfloor\frac{q}{2}\rfloor} (-1)^j \frac{(2q-2j+2)!}{2^{q+1}(j)!(q-j+1)!(q-2j)!} x^{q-2j}. \tag{4}$$

Now, some important formulas of these polynomials are presented [31].

The next formula is the moment formula for the introduced basis polynomials.

$$x^m DL_q(x) = \sum_{k=0}^{\min\left(m,\lfloor\frac{q+m}{2}\rfloor\right)} F_{m,k+1,q}\, DL_{q+m-2k}(x), \tag{5}$$

where:

$$F_{m,k,q} = \begin{cases} \frac{q+k}{2q+3}, & m = 1, \\ F_{1,1,q}\, F_{m-1,1,q+1}, & m > 1, k = 1, \\ F_{1,1,0}\, F_{m-1,k,1}, & m > 1, 1 < k < m+1, q = 0, \\ F_{1,1,q}\, F_{m-1,k,q+1} + F_{1,2,q}\, F_{m-1,k-1,q-1}, & m > 1, 1 < k < m+1, q > 0, \\ F_{1,2,q}\, F_{m-1,m,q-1}, & m > 1, k = m+1. \end{cases} \tag{6}$$

Integrals of Legendre's derivatives can be presented as a linear combination of Legendre's derivatives themselves according to:

$$\int_{-1}^{x} DL_q(t)\mathrm{dt} = \sum_{k=0}^{2} \left[\frac{(-1)^k(1-\delta_{k,2})(1-\delta_{q,0}\,\delta_{k,1})}{2q+3} + (-1)^q \delta_{k,2}\right] DL_{(q+1-2k)(1-\delta_{k,2})(1-\delta_{q,0}\,\delta_{k,1})}(x). \tag{7}$$

The next formula presents the integral of the m^{th} moment on the interval $[-1,1]$.

For every $m, q \in \mathbb{N}$ we have:

$$\int_{-1}^{1} t^m DL_q(t)\mathrm{dt} = \begin{cases} 1 + (-1)^{q+m}, & q \geq m, \\ \sum_{k=0}^{\min\left(m,\lfloor\frac{q+m}{2}\rfloor\right)} F_{m,k+1,q}(1 + (-1)^{q+m-2k}), & 0 \leq q < m. \end{cases} \tag{8}$$

The following section introduces and proves some essential formulas that treat the nonlinear terms in an integral equation. These formulas are required to simplify the calculations of several complicated integrations, enhancing the presented method's efficiency.

3. Linearization Formulas of the Legendre Derivative Polynomials

This section starts with the linearization formula. Hence, other important relations concerning essential integrations will be developed.

Lemma 1. *The product of two Legendre's derivative polynomials of degrees r and q can be expressed as a linear combination of the Legendre's derivative polynomials themselves as follow:*

$$DL_r(x)DL_q(x) = \sum_{i=0}^{\min(r,q)} \sigma_{r,q,i}\, DL_{r+q-2i}(x), \tag{9}$$

where:

$$\sigma_{r,q,i} = \frac{4\left(r+q+\frac{3}{2}-2i\right)(r+q-2i)!\,\Gamma\left(\frac{3}{2}+i\right)B\left(r+\frac{3}{2}-i,q+\frac{3}{2}-i\right)\left(r+q+\frac{3}{2}-i\right)_{\frac{3}{2}}}{\pi\left(r+q+\frac{3}{2}-i\right)\, i!(r-i)!(q-i)!}, \tag{10}$$

such that $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$.

Proof. As known, Ultraspherical polynomials, $C_q^{(\nu)}(x)$, are related to Legendre polynomials as follows [32]:

$$C_q^{\left(\frac{1}{2}\right)}(x) = \mathcal{L}_q(x). \tag{11}$$

$$\frac{dC_q^{(\nu)}(x)}{dx} = 2\nu\, C_{q-1}^{(\nu+1)}(x). \tag{12}$$

From Eqs. (11) and (12), the introduced basis functions are related to the ultraspherical polynomials as follow:

$$C_q^{\left(\frac{3}{2}\right)}(x) = DL_q(x). \tag{13}$$

The next relation shows the expansion of the product of two ultraspherical polynomials:

$$C_r^{(\nu)}(x)C_q^{(\nu)}(x) = \sum_{i=0}^{\min(r,q)} \frac{(r+q+\nu-2i)(\nu)_i(\nu)_{r-i}(\nu)_{q-i}(2\nu)_{r+q-i}(r+q-2i)!}{(r+q+\nu-i)\, i!(r-i)!(q-i)!(\nu)_{r+q-i}(2\nu)_{r+q-2i}} C_{r+q-2i}^{(\nu)}(x). \tag{14}$$

Setting $\nu = \frac{3}{2}$ in relation (14) to get:

$$DL_r(x)\, DL_q(x) = \sum_{i=0}^{\min(r,q)} \frac{\left(r+q+\frac{3}{2}-2i\right)\left(\frac{3}{2}\right)_i\left(\frac{3}{2}\right)_{r-i}\left(\frac{3}{2}\right)_{q-i}(3)_{r+q-i}(r+q-2i)!}{\left(r+q+\frac{3}{2}-i\right) i!(r-i)!(q-i)!\left(\frac{3}{2}\right)_{r+q-i}(3)_{r+q-2i}} DL_{r+q-2i}(x). \tag{15}$$

Simplifying the pochhammer factors to get:

$$DL_r(x)DL_q(x) = \sum_{i=0}^{\min(r,q)} \frac{\left(r+q+\frac{3}{2}-2i\right)(r+q-2i)!\,\Gamma\left(\frac{3}{2}+i\right)\Gamma\left(r+\frac{3}{2}-i\right)\Gamma\left(q+\frac{3}{2}-i\right)\Gamma(r+q+3-i)}{\left(r+q+\frac{3}{2}-i\right) i!(r-i)!(q-i)!\left(\Gamma\left(\frac{3}{2}\right)\right)^2\Gamma\left(r+q+\frac{3}{2}-i\right)\Gamma(r+q+3-2i)} DL_{r+q-2i}(x). \tag{16}$$

Using the relation between the beta and gamma functions, $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, to get:

$$DL_r(x)DL_q(x) = \sum_{i=0}^{\min(r,q)} \frac{4\left(r+q+\frac{3}{2}-2i\right)(r+q-2i)!\,\Gamma\left(\frac{3}{2}+i\right)B\left(r+\frac{3}{2}-i,q+\frac{3}{2}-i\right)\left(r+q+\frac{3}{2}-i\right)_{\frac{3}{2}}}{\pi\left(r+q+\frac{3}{2}-i\right) i!(r-i)!(q-i)!} DL_{r+q-2i}(x). \tag{17}$$

This completes the proof.

To deal with the integral of the form $\int_{-1}^{1} t^m DL_r(t)DL_q(t)dt$, Lemma (1) is used as the first step, then relation (8) is applied. The next Theorem calculates such integrals directly.

Theorem 1. *The following integral can be calculated as:*

$$\int_{-1}^{1} t^m DL_r(t)DL_q(t)dt = \begin{cases} 0, & r+q+m \text{ is odd}, \\ \sum_{i=0}^{\min(r,q)} 2\sigma_{r,q,i}, & r+q+m \text{ is even}, 0 \le m \le |r-q|, \\ \sum_{i=0}^{\frac{r+q-m}{2}} 2\sigma_{r,q,i} + \sum_{i=\frac{r+q-m+2}{2}}^{\min(r,q)} \sum_{k=0}^{\frac{r+q+m-2i}{2}} 2\sigma_{r,q,i}\, F_{m,k+1,r+q-2i}, & r+q+m \text{ is even}, |r-q| < m \le r+q, \\ \sum_{i=0}^{\min(r,q)} \sum_{k=0}^{\frac{r+q+m-2i}{2}} 2\sigma_{r,q,i}\, F_{m,k+1,r+q-2i}, & r+q+m \text{ is even}, m > r+q, \end{cases} \tag{18}$$

where $F_{m,k,q}$ is defined in Eq. (6), and $\sigma_{r,q,i}$ is defined in Eq. (10).

Proof. In the beginning, Eq. (8) needs to be rewritten as follow:

$$\int_{-1}^{1} t^m DL_q(t)\mathrm{dt} = \begin{cases} 0, & q+m \text{ is odd}, \\ 2, & q+m \text{ is even}, q \ge m, \\ \sum_{k=0}^{\frac{q+m}{2}} 2\, F_{m,k+1,q}, & q+m \text{ is even}, 0 \le q < m. \end{cases} \tag{19}$$

Applying Lemma (1) to the integrand of the left-hand side of Eq. (18) to get:

$$\int_{-1}^{1} t^m DL_r(t)DL_q(t)dt = \sum_{i=0}^{\min(r,q)} \sigma_{r,q,i} \int_{-1}^{1} t^m DL_{r+q-2i}(t)dt. \tag{20}$$

From Eq. (20), it is clear that:

$$0 \leq i \leq \min(r, q), \tag{21}$$

multiplying by -2, adding $r + q$, and since $r + q = \max(r, q) + \min(r, q)$, and $|r - q| = \max(r, q) - \min(r, q)$ to get:

$$r + q \geq r + q - 2i \geq \max(r, q) - \min(r, q). \tag{22}$$

Thus:

$$|r - q| \leq r + q - 2i \leq r + q. \tag{23}$$

For the first case, $r + q + m - 2i$ is odd, Eq. (19) is applied to Eq. (20). Hence:

$$\int_{-1}^{1} t^m DL_r(t) DL_q(t) dt = 0. \tag{24}$$

While the even case of $r + q + m - 2i$ will be split into three sub-cases.

For $0 \leq m \leq |r - q|$, from Eq. (23):

$$r + q - 2i \geq m. \tag{25}$$

Applying Eq. (19) to Eq. (20) to get:

$$\int_{-1}^{1} t^m DL_r(t) DL_q(t) dt = \sum_{i=0}^{\min(r,q)} 2\sigma_{r,q,i}. \tag{26}$$

For $m > r + q$, from Eq. (23):

$$0 \leq r + q - 2i < m. \tag{27}$$

Eq. (20), after applying Eq. (19), becomes:

$$\int_{-1}^{1} t^m DL_r(t) DL_q(t) dt = \sum_{i=0}^{\min(r,q)} \sum_{k=0}^{\frac{r+q+m-2i}{2}} 2\sigma_{r,q,i} \, F_{m,k+1,r+q-2i}. \tag{28}$$

For $|r - q| < m \leq r + q$, multiplying the inequality by a negative sign, then adding $r + q$ and dividing by 2 to get:

$$0 \leq \frac{r+q-m}{2} < \min(r, q). \tag{29}$$

Setting $r + q - 2i \geq m$ leads to:

$$i > \frac{r+q-m}{2}. \tag{30}$$

Applying Eq. (19) to the integral in Eq. (20):

$$\int_{-1}^{1} t^m DL_{r+q-2i}(t) dt = \begin{cases} 2, & 0 \leq i \leq \frac{r+q-m}{2}, \\ \sum_{k=0}^{\frac{r+q+m-2i}{2}} 2\, F_{m,k+1,r+q-2i}, & \frac{r+q-m}{2} < i \leq \min(r, q). \end{cases} \tag{31}$$

Therefore:

$$\int_{-1}^{1} t^m DL_r(t) DL_q(t) dt = \sum_{i=0}^{\frac{r+q-m}{2}} 2\sigma_{r,q,i} + \sum_{i=\frac{r+q-m+2}{2}}^{\min(r,q)} \sum_{k=0}^{\frac{r+q+m-2i}{2}} 2\sigma_{r,q,i} \, F_{m,k+1,r+q-2i}. \tag{32}$$

This completes the proof.

Lemma 2. *The following integral can be calculated as:*

$$\int_{-1}^{x} t^m DL_q(t)dt = \begin{cases} \Upsilon_q(x), & m = 0, \\ \sum_{k=0}^{\min\left(m,\lfloor\frac{q+m}{2}\rfloor\right)} F_{m,k+1,q}\Upsilon_{q+m-2k}(x), & m > 0. \end{cases} \tag{33}$$

where:

$$\Upsilon_q(x) = \frac{1}{2q+3} DL_{q+1}(x) + \frac{\delta_{q,0}-1}{2q+3} DL_{(q-1)(1-\delta_{q,0})}(x) + (-1)^q. \tag{34}$$

Proof. For $m = 0$:

From Eq. (7):

$$\int_{-1}^{x} DL_q(t)\mathrm{dt} = \sum_{j=0}^{2} \alpha_{q,j} DL_{\lambda_{q,j}}(x) = \alpha_{q,0} DL_{\lambda_{q,0}}(x) + \alpha_{q,1} DL_{\lambda_{q,1}}(x) + \alpha_{q,2} DL_{\lambda_{q,2}}(x), \tag{35}$$

where:

$$\alpha_{q,j} = \frac{(-1)^j(1-\delta_{j,2})(1-\delta_{q,0}\,\delta_{j,1})}{2q+3} + (-1)^q\delta_{j,2}\,, \tag{36}$$

and

$$\lambda_{q,j} = (q+1-2j)\big(1-\delta_{j,2}\big)\big(1-\delta_{q,0}\,\delta_{j,1}\big). \tag{37}$$

Therefore, Eq. (35) takes the following form:

$$\int_{-1}^{x} DL_q(t)\mathrm{dt} = \Upsilon_q(x), \tag{38}$$

which completes the proof for the $m = 0$ case.

For $m > 0$:

Integrating both sides of Eq. (5):

$$\int_{-1}^{x} t^m DL_q(t)dt = \sum_{k=0}^{\min\left(m,\lfloor\frac{q+m}{2}\rfloor\right)} F_{m,k+1,q} \int_{-1}^{x} DL_{q+m-2k}(t)dt. \tag{39}$$

Using Eq. (38), to get:

$$\int_{-1}^{x} t^m DL_q(t)dt = \sum_{k=0}^{\min\left(m,\lfloor\frac{q+m}{2}\rfloor\right)} F_{m,k+1,q}\Upsilon_{q+m-2k}(x), \tag{40}$$

which completes the proof for $m > 0$.

The following Theorem can be considered as a generalization of Theorem (1). This generalization allows us to integrate over the interval $[-1, x]$ instead of the interval $[-1,1]$.

Theorem 2. *The integral of the product of two Legendre's derivative polynomials by a factor of* t^m *over the interval* $[-1, x]$ *is calculated as follows:*

$$\int_{-1}^{x} t^m DL_r(t)DL_q(t)dt = \begin{cases} \sum_{i=0}^{\min(r,q)} \sigma_{r,q,i}\Upsilon_{r+q+m-2i}(x), & m = 0, \\ \sum_{i=0}^{\min(r,q)} \sum_{k=0}^{\min\left(m,\lfloor\frac{r+q+m-2i}{2}\rfloor\right)} \sigma_{r,q,i}\, F_{m,k+1,r+q-2i}\, \Upsilon_{r+q+m-2i-2k}(x), & m > 0, \end{cases} \tag{41}$$

where $F_{m,k,q}$ is defined in Eq. (6), $\sigma_{r,q,i}$ is defined in Eq. (10), and $\Upsilon_q(x)$ is defined in Eq. (34).

Proof. From Lemma (1) we have the following:

$$\int_{-1}^{x} t^m DL_r(t)DL_q(t)dt = \sum_{i=0}^{\min(r,q)} \sigma_{r,q,i} \int_{-1}^{x} t^m DL_{r+q-2i}(t)dt. \tag{42}$$

According to Eq. (42), the integration (41) can be calculated using Lemma (2) which completes the proof.

4. Pseudo-Galerkin Approach for Solving Integral Equations

This section presents a method by which some types of integral equations can be solved. The presented method is based on the Legendre's first derivative polynomials as basis functions. The relations, Lemmas, and Theorems of Section (3) take place in creating the solving algorithm.

Consider the two-dimensional integral equation on the following form:

$$f_1\big(x, y, u(x, y)\big) + \int_a^g \int_c^h f_2\big(x, s, y, t, u(s, t)\big)dt\, ds = 0, \tag{43}$$

where $x \in [a, b], y \in [c, d], g \in \{x, b\}, h \in \{y, d\}, f_1$ is a polynomial with respect to u, and f_2 is a polynomial with respect to s, t, and u.

The integral equation should be shifted to the domain $x, y \in [-1,1]$. Then the unknown function is expanded, as illustrated below.

Expanding the unknown function of Eq. (43), $u(x, y)$, as a double summation truncated series as follows:

$$u(x, y) \approx u_{N,M}(x, y) = \sum_{i=0}^{N} \sum_{j=0}^{M} c_{i,j}\, DL_i(x)DL_j(y), \tag{44}$$

for some $N, M \in \mathbb{N}$.

Substituting from Eq. (44) into the integral equation (43) to get the residue function:

$$R_{N,M}(x, y) = f_1\left(x, y, \sum_{i=0}^{N} \sum_{j=0}^{M} c_{i,j}\, DL_i(x)DL_j(y)\right) + \int_{-1}^{G} \int_{-1}^{H} f_2\left(x, s, y, t, \sum_{i=0}^{N} \sum_{j=0}^{M} c_{i,j}\, DL_i(s)DL_j(t)\right) dtds. \tag{45}$$

where G and H are the shifted values of g and h.

Collocating the residue function, (45), by the set of $N + 1$ and $M + 1$ Legendre Gauss Lobatto (LGL) quadrature points for the independent variables x and y, respectively, to get a system of $(N + 1)(M + 1)$ algebraic equations. Solving this system to obtain the unknown coefficients, $c_{i,j}$. Hence, the semi-analytic approximate solution is ready.

In the following section, some test problems will be solved to ensure the accuracy and efficiency of the introduced method.

5. Examples

This section will apply the introduced method to approximate some types of two-dimensional integral equations. The approximate solutions are compared with the exact ones, and the error is calculated and presented in tables and graphs to show the accuracy and efficiency of the introduced method.

Example 1. *Consider the following nonlinear two-dimensional integral equation [33]:*

$$u(x,y) = x + y - \frac{1}{12}xy(x^3 + 4x^2y + 4xy^2 + y^3) + \int_0^y \int_0^x (x + y - t - s)\big(u(s,t)\big)^2 ds\, dt, \tag{46}$$

where $x, y \in [0,1]$, *with exact solution* $u(x,y) = x + y$. *Shifting the domain to the region* $\{(x,y)|\ x, y \in [-1,1]\}$ *to get:*

$$\begin{aligned} u(x,y) = \frac{1}{2}x + \frac{1}{2}y - \frac{1}{384}(x+1)(y+1)((x+1)^3 + 4(x+1)^2(y+1) + 4(x+1)(y+1)^2 + (y+1)^3) \\ + \frac{1}{8}\int_{-1}^y \int_{-1}^x (x + y - t - s)\big(u(s,t)\big)^2 ds\, dt, \end{aligned} \tag{47}$$

with exact solution $u(x,y) = \frac{1}{2}x + \frac{1}{2}y + 1$.

Using the expansion (44), the residue function takes the form:

$$\begin{aligned} R_{N.M}(x,y) = \frac{1}{2}x + \frac{1}{2}y - \frac{1}{384}(x+1)(y+1)((x+1)^3 + 4(x+1)^2(y+1) + 4(x+1)(y+1)^2 + (y+1)^3) \\ - \sum_{i=0}^N \sum_{j=0}^M c_{i,j}\, DL_i(x)DL_j(y) \\ + \frac{1}{8}\sum_{i=0}^N \sum_{j=0}^M \sum_{a=0}^N \sum_{b=0}^M c_{i,j}\, c_{a,b}\, ((x+y)\int_{-1}^x DL_i(s)DL_a(s)\, ds \int_{-1}^y DL_j(t)DL_b(t)\, dt \\ - \int_{-1}^x DL_i(s)DL_a(s)\, ds \int_{-1}^y t\, DL_j(t)DL_b(t)\, dt - \int_{-1}^x s\, DL_i(s)DL_a(s)\, ds \int_{-1}^y DL_j(t)DL_b(t)\, dt). \end{aligned} \tag{48}$$

Finally, the integrals involved in Eq. (48) can be calculated from Theorem (2).

Exact solution is obtained at $N = M = 1$ *using the presented method, while the authors of [33] achieved an error of* 10^{-5} *after 32 iterations.*

Example 2. *Consider the following nonlinear two-dimensional integral equation [34]:*

$$u(x,y) = x^2e^{2y} + \frac{1}{5}y^3 - \int_0^y \int_0^1 y^2e^{-4t}(u(s,t))^2 dsdt, \tag{49}$$

where $x, y \in [0,1]$, *with exact solution* $u(x,y) = x^2e^{2y}$.

$$u(x,y) = \frac{(x+1)^2}{4}e^{y+1} + \frac{1}{40}(y+1)^3 - \frac{e^{-2}}{16}(y+1)^2 \int_{-1}^1 \int_{-1}^y e^{-2t}(u(s,t))^2 dtds, \tag{50}$$

where $x, y \in [-1,1]$, *with exact solution* $u(x,y) = \frac{(x+1)^2}{4}e^{y+1}$.

Using the expansion (44), the residue function takes the form:

$$\begin{aligned} R_{N,M}(x,y) = \frac{(x+1)^2}{4}e^{y+1} + \frac{1}{40}(y+1)^3 - \sum_{i=0}^N \sum_{j=0}^M c_{i,j}\, DL_i(x)DL_j(y) \\ - \frac{e^{-2}}{16}(y+1)^2 \sum_{i=0}^N \sum_{\alpha=0}^N \sum_{j=0}^M \sum_{\beta=0}^M c_{i,j}c_{\alpha,\beta} \int_{-1}^1 DL_i(s)DL_\alpha(s)ds \int_{-1}^y e^{-2t}DL_j(t)DL_\beta(t)dt, \end{aligned} \tag{51}$$

By expanding the exponential function e^{-2t} *as follows:*

$$e^{-2t} \approx \sum_{k=0}^L \frac{(-2t)^k}{k!}. \tag{52}$$

The residue function becomes:

$$\begin{aligned} R_{N,M}(x,y) = \frac{(x+1)^2}{4}e^{y+1} + \frac{1}{40}(y+1)^3 - \sum_{i=0}^N \sum_{j=0}^M c_{i,j}\, DL_i(x)DL_j(y) \\ - \frac{e^{-2}}{16}(y+1)^2 \sum_{i=0}^N \sum_{\alpha=0}^N \sum_{j=0}^M \sum_{\beta=0}^M \sum_{k=0}^L \frac{(-2)^k}{k!} c_{i,j}c_{\alpha,\beta} \int_{-1}^1 DL_i(s)DL_\alpha(s)ds \int_{-1}^y t^k DL_j(t)DL_\beta(t)dt. \end{aligned} \tag{53}$$

From Theorem (1) for $m = 0$, *and Theorem (2), integrals in Eq. (53) can be calculated.*

Table 1 presents the point-wise absolute error ($|E|$) compared with other methods.

Table 1: $|E|$ for Example (2).

$x = y$	[35]	[36]	Presented Method
0.0	$0.00E-00$	$6.43E-03$	$0.00E-00$
0.1	$9.39E-04$	$6.94E-03$	$3.12E-06$
0.2	$6.35E-04$	$4.31E-02$	$2.30E-05$
0.3	$6.61E-04$	$9.82E-02$	$6.92E-05$
0.4	$1.02E-03$	$1.68E-01$	$1.47E-04$
0.5	$1.69E-04$	$2.63E-01$	$2.60E-04$
0.6	$1.24E-03$	$3.88E-01$	$4.00E-04$
0.7	$1.13E-03$	$5.33E-01$	$5.44E-04$
0.8	$1.42E-03$	$6.91E-01$	$6.72E-04$
0.9	$2.17E-03$	$8.85E-01$	$7.81E-04$

Figure 1 shows the log error, which confirms the stability of the presented method.

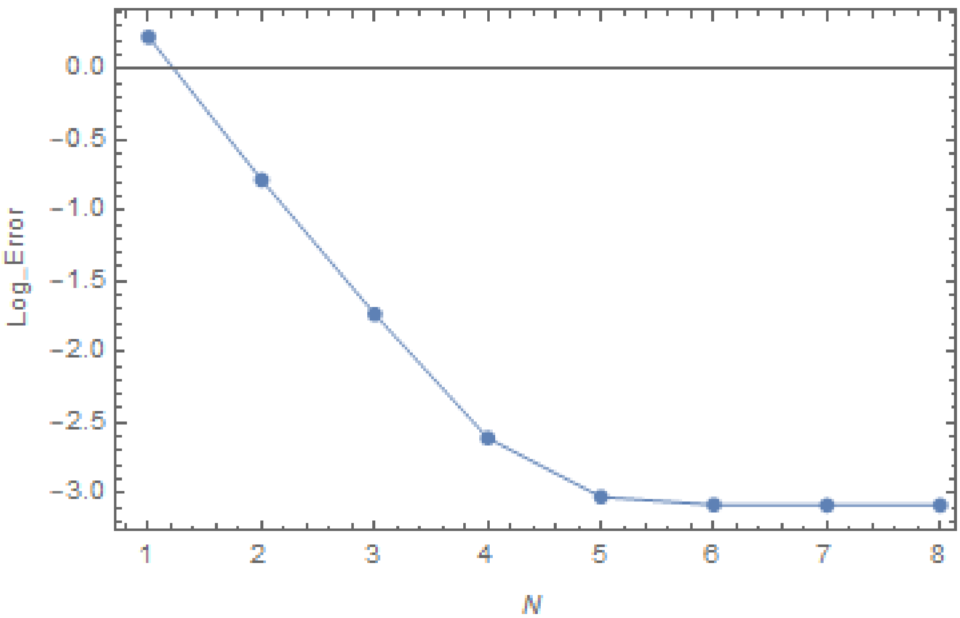

Figure 1: Log error graph for Example (2).

Example 3. *Consider the following nonlinear two-dimensional integral equation [37]:*

$$u(x,y) = x\cos y + \frac{1}{20}(\cos^4 1 - 1) - \frac{1}{12}\sin 1\ (\cos^2 1 + 2) + \int_0^1\int_0^1 (s\sin t + 1)(u(s,t))^3 ds\, dt \tag{54}$$

where $x, y \in [0,1]$, with exact solution $u(x,y) = x\cos y$.

$$\begin{aligned} u(x,y) = {} & \frac{1}{2}x\cos\left(\frac{y+1}{2}\right) + \frac{1}{2}\cos\left(\frac{y+1}{2}\right) + \frac{1}{20}(\cos^4 1 - 1) - \frac{1}{12}\sin 1\,(\cos^2 1 + 2) \\ & + \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\left(\frac{1}{2}s\sin\left(\frac{t+1}{2}\right) + \frac{1}{2}\sin\left(\frac{t+1}{2}\right) + 1\right)(u(s,t))^3 ds\, dt, \end{aligned} \tag{55}$$

where $x, y \in [-1,1]$, with exact solution $u(x,y) = \left(\frac{x+1}{2}\right)\cos\left(\frac{y+1}{2}\right)$.

Using the expansion (44), the residue function takes the form:

$$R_{N,M}(x,y) = \left(\frac{x+1}{2}\right) cos\left(\frac{y+1}{2}\right) + \frac{1}{20}(cos^4 1 - 1) - \frac{1}{12} sin\, 1\,(cos^2 1 + 2) - \sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\, DL_i(x)DL_j(y)$$
$$+ \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{a=0}^{N}\sum_{b=0}^{M}\sum_{\alpha=0}^{N}\sum_{\beta=0}^{M} \frac{c_{i,j}c_{a,b}c_{\alpha,\beta}}{4}\left(\int_{-1}^{1} DL_i(s)DL_a(s)DL_\alpha(s)\, ds \int_{-1}^{1} DL_j(t)DL_b(t)DL_\beta(t)\, dt \right. \quad (56)$$
$$\left. + \int_{-1}^{1}\left(\frac{s+1}{2}\right) DL_i(s)DL_a(s)DL_\alpha(s)\, ds \int_{-1}^{1} \sin\left(\frac{t+1}{2}\right) DL_j(t)DL_b(t)DL_\beta(t)\, dt\right).$$

Using the formula of the sine of the summation of two angles and using Taylor series to get:

$$\sin\left(\frac{t+1}{2}\right) = \sum_{k=0}^{L} \frac{(-1)^k}{2^{2k}(2k)!}\left(\left(\frac{\cos\frac{1}{2}}{2(2k+1)}\right) t^{2k+1} + \left(\sin\frac{1}{2}\right) t^{2k}\right) \quad (57)$$

Using Eq. (57), the residue function in Eq. (56) takes the form:

$$R_{N,M}(x,y) = \left(\frac{x+1}{2}\right) cos\left(\frac{y+1}{2}\right) + \frac{1}{20}(cos^4 1 - 1) - \frac{1}{12} sin\, 1\,(cos^2 1 + 2) - \sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\, DL_i(x)DL_j(y)$$
$$+ \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{a=0}^{N}\sum_{b=0}^{M}\sum_{\alpha=0}^{N}\sum_{\beta=0}^{M} \frac{c_{i,j}c_{a,b}c_{\alpha,\beta}}{4}\left(\int_{-1}^{1} DL_i(s)DL_a(s)DL_\alpha(s)\, ds \int_{-1}^{1} DL_j(t)DL_b(t)DL_\beta(t)\, dt \right.$$
$$+ \frac{1}{2}\sum_{k=0}^{L} \frac{(-1)^k}{2^{2k}(2k)!}\left(\int_{-1}^{1} s\, DL_i(s)DL_a(s)DL_\alpha(s)\, ds + \int_{-1}^{1} DL_i(s)DL_a(s)DL_\alpha(s)\, ds\right) \quad (58)$$
$$\left. \times\left(\frac{cos\frac{1}{2}}{2(2k+1)}\int_{-1}^{1} t^{2k+1} DL_j(t)DL_b(t)DL_\beta(t)\, dt + \left(sin\frac{1}{2}\right)\int_{-1}^{1} t^{2k} DL_j(t)DL_b(t)DL_\beta(t)\, dt\right)\right).$$

Using Lemma (1) together with Theorem (1) to calculate the integrals in Eq. (58).

Table 2 presents the maximum absolute error (MAE) for $N = 2$ and $L = 3$, compared with other methods for different values of M.

Table 2: MAE for Example (3).

M	[38]	[33]	[37]	Presented Method
2	−	$9.60E-02$	−	$2.42E-03$
4	$5.20E-02$	$4.30E-02$	$6.70E-05$	$9.12E-06$
8	$2.10E-02$	$1.70E-02$	$2.43E-06$	$9.28E-09$
16	$6.80E-03$	$5.40E-03$	$7.09E-08$	−

Figure 2 shows the log error, which confirms the accuracy of the presented method.

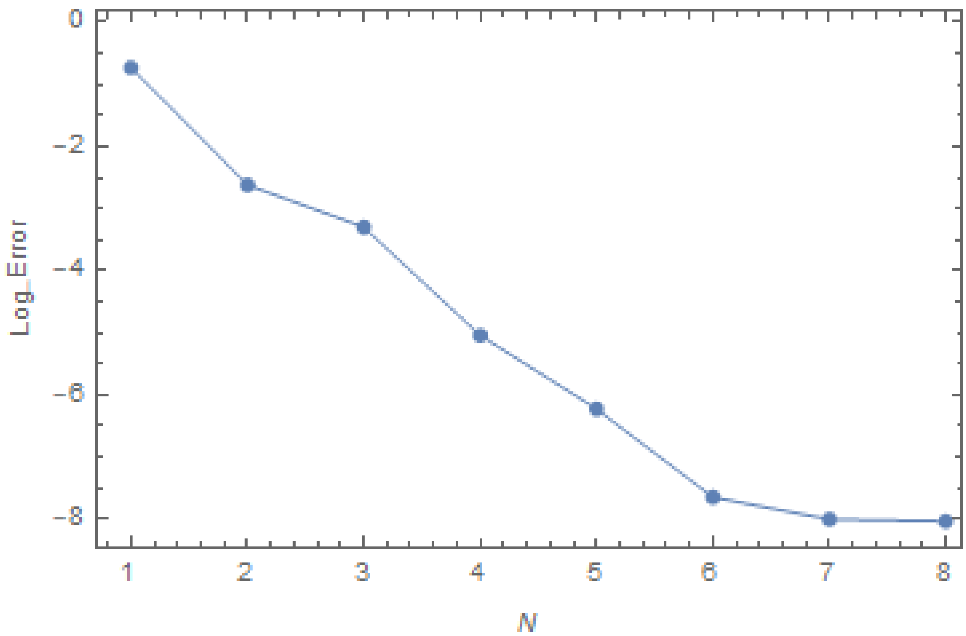

Figure 2: Log error graph for Example (3).

An algorithm that shows accuracy and efficiency for some types of nonlinear two-dimensional integral equations is considered to be more general. It should also be effective for linear two-dimensional integral equations such as the next example.

Example 4. *Consider the following linear two-dimensional integral equation [34]:*

$$u(x,y) = x^2e^y - \frac{1}{3}x^3y^2 + \int_0^x \int_0^1 y^2e^{-t}u(s,t)dt\, ds, \tag{59}$$

where $x, y \in [0,1]$, *with exact solution* $u(x,y) = x^2e^y$. *Shifting the domain to the region* $\{(x,y)|x,y \in [-1,1]\}$ *to get:*

$$u(x,y) = \frac{(x+1)^2e^{\frac{y+1}{2}}}{4} - \frac{(x+1)^3(y+1)^2}{96} + \frac{(y+1)^2}{16\sqrt{e}}\int_{-1}^x \int_{-1}^1 e^{-\frac{t}{2}}u(s,t)dt\, ds, \tag{60}$$

with exact solution $u(x,y) = \frac{(x+1)^2e^{\frac{y+1}{2}}}{4}$.

Using the expansion (44), the residue function takes the form:

$$R_{N,M}(x,y) = \frac{(x+1)^2e^{\frac{y+1}{2}}}{4} - \frac{(x+1)^3(y+1)^2}{96} + \sum_{i=0}^N \sum_{j=0}^M c_{i,j}\left(\frac{(y+1)^2}{16\sqrt{e}}\int_{-1}^x DL_i(s)ds \int_{-1}^1 e^{-\frac{t}{2}}DL_j(t)dt - DL_i(x)DL_j(y)\right) \tag{61}$$

By expanding the exponential function $e^{-\frac{t}{2}}$ *as follows:*

$$e^{-\frac{t}{2}} \approx \sum_{k=0}^L \frac{\left(-\frac{t}{2}\right)^k}{k!}. \tag{62}$$

The residue function becomes:

$$R_{N,M}(x,y) = \frac{(x+1)^2 e^{\frac{y+1}{2}}}{4} - \frac{(x+1)^3(y+1)^2}{96}$$
$$+\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\left(\sum_{k=0}^{L}\frac{(-1)^k(y+1)^2}{2^{k+4}k!\sqrt{e}}\int_{-1}^{x} DL_i(s)ds\int_{-1}^{1} t^k DL_j(t)dt - DL_i(x)DL_j(y)\right). \quad (63)$$

Calculating the integrals in Eq. (63) with the aid of Lemma (2) for $m = 0$ *and Theorem (1) for* $q = 0$.

Table 3 presents the $|E|$ *at* $L = 10$, *for different values of* N *and* M.

Table 3: $|E|$ for Example (4) at $N = M = 4$ and $N = M = 8$.

$x = y$	$N = M = 4$			$N = M = 8$		
	[35]	[34]	Presented Method	[35]	[34]	Presented Method
0.0	$2.27E-08$	$1.04E-17$	$0.00E-00$	$6.40E-10$	$7.39E-17$	$0.00E-00$
0.1	$4.28E-05$	$3.74E-08$	$2.47E-07$	$7.06E-05$	$1.50E-13$	$3.86E-13$
0.2	$2.00E-04$	$7.58E-07$	$4.34E-07$	$3.42E-04$	$1.18E-12$	$1.35E-12$
0.3	$4.72E-04$	$2.16E-06$	$3.40E-06$	$8.37E-04$	$1.36E-12$	$1.44E-12$
0.4	$7.69E-04$	$6.25E-07$	$5.08E-06$	$1.47E-03$	$5.44E-12$	$8.03E-12$
0.5	$8.79E-04$	$6.47E-06$	$2.66E-10$	$2.03E-03$	$8.49E-12$	$4.73E-15$
0.6	$4.22E-04$	$1.45E-05$	$1.18E-05$	$2.13E-03$	$1.14E-11$	$1.84E-11$
0.7	$1.18E-03$	$1.06E-05$	$1.98E-05$	$1.18E-03$	$2.58E-11$	$8.21E-12$
0.8	$2.27E-08$	$1.62E-05$	$7.68E-06$	$1.66E-03$	$1.15E-12$	$2.29E-11$
0.9	$2.27E-08$	$4.16E-05$	$2.29E-05$	$7.49E-03$	$3.73E-11$	$3.38E-11$
1.0	$2.27E-08$	$5.62E-05$	$8.70E-09$	$9.77E-03$	$7.14E-11$	$1.54E-13$

Figure 3 shows the log error, which confirms the stability of the presented method.

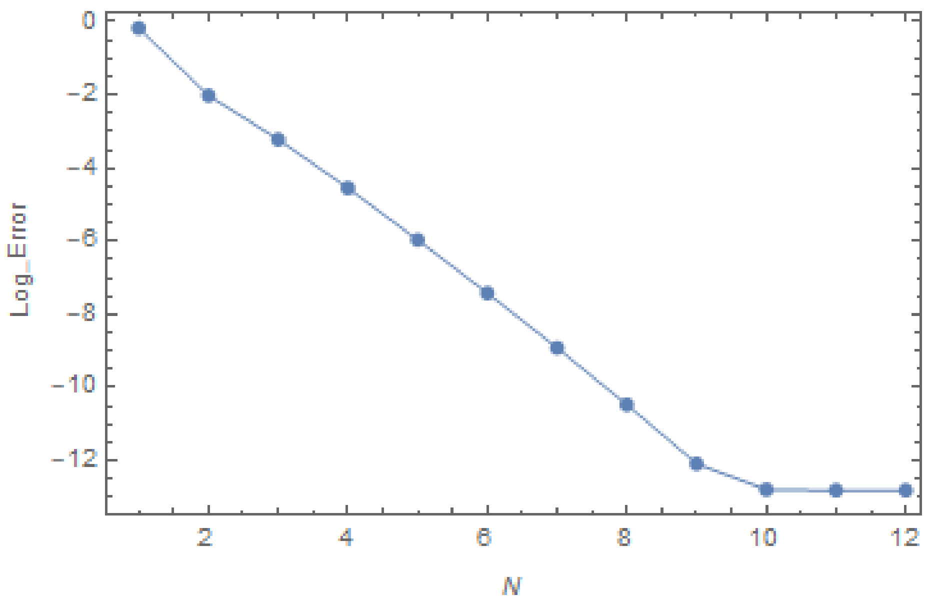

Figure 3: Log error graph for Example (4).

A one-dimensional integral equation can be considered as a special case of a two-dimensional one. Thus, the algorithm that solves two-dimensional integral equations is more general, and it is also valid for one-dimensional ones.

Example 5. *Consider the following nonlinear one-dimensional integral equation:*

$$u(x) = \frac{3}{2} - \frac{1}{2}e^{-2x} - \int_0^x \left(u(t) + \left(u(t)\right)^2\right) dt, \tag{64}$$

where $x \in [0,1]$, *with exact solution* $u(x) = e^{-x}$. *Shifting the domain to* $[-1,1]$ *to get:*

$$u(x) = \frac{3}{2} - \frac{1}{2}e^{-(x+1)} - \frac{1}{2}\int_{-1}^x \left(u(t) + \left(u(t)\right)^2\right) dt, \tag{65}$$

with exact solution $u(x) = e^{-\frac{x+1}{2}}$.

By setting $M = 0$ *in Eq. (44), considering the function* u *is constant with respect to* y, *the expansion takes the form:*

$$u(x) \approx u_{N,0}(x) = \sum_{i=0}^N c_{i,0}\, DL_i(x), \tag{66}$$

Using the expansion (66), the residue function takes the form:

$$R_{N,0}(x) = -\frac{3}{2} + \frac{1}{2}e^{-(x+1)} + \sum_{i=0}^N c_{i,0}\left(DL_i(x) + \frac{1}{2}\int_{-1}^x DL_i(t)dt + \frac{1}{2}\sum_{j=0}^N c_{j,0}\int_{-1}^x DL_i(t)DL_j(t)dt\right). \tag{67}$$

Table 4 presents the MAE for different values of N.

Table 4: MAE for Example (5).

N	[39]	Presented Method
1	$6.37E-02$	$5.85E-02$
3	$5.35E-04$	$3.26E-04$
5	$1.53E-06$	$6.44E-07$
7	$2.17E-09$	$7.03E-10$
9	$7.50E-12$	$4.79E-13$

Figure 4 shows the log error, which confirms the accuracy of the presented method.

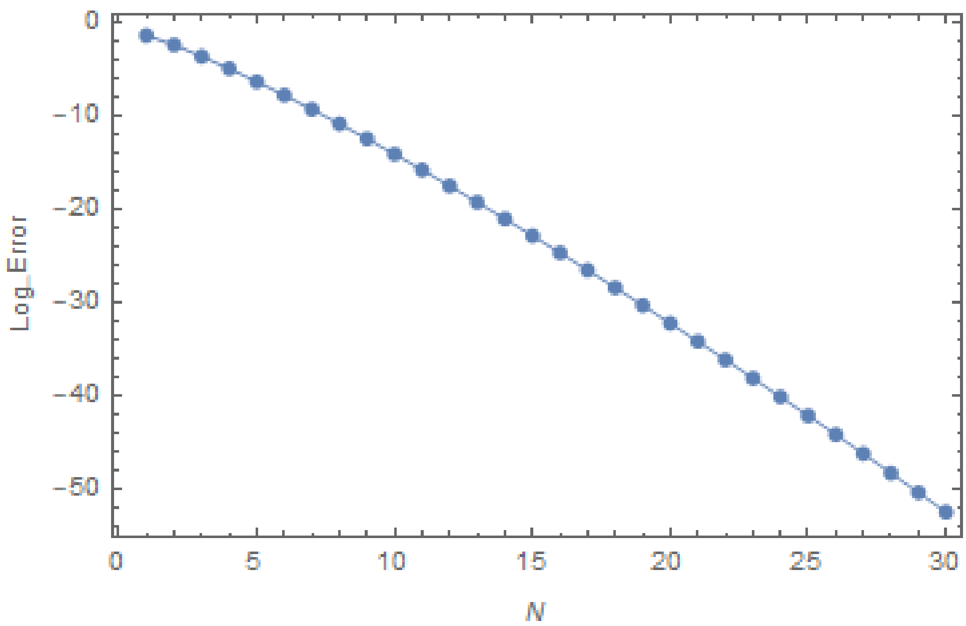

Figure 4: Log error graph for Example (5).

6. Concluding Remarks

The presented method proved its accuracy, efficiency, and stability of the approximate solutions in solving some types of two-dimensional and one-dimensional integral equations. This is affirmed through some test problems, supported by tables and graphs to display the results. In addition, some important integration formulas are created and proved. They played a significant role in calculating the integral of some nonlinear terms.

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