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RESEARCH PAPER

Frequency–Amplitude Relationship in Damped and Forced Nonlinear Oscillators with Irrational Nonlinearities

Ji-Huan He [a, b, c, *], M. Y Adamu [d], Isah Abdullahi [d]

[a] *Yango University, No.99 Denglong Road, Fuzhou City, Fujian Province, China*
[b] *School of Jia Yang; Zhejiang Shuren University, Hangzhou; Zhejiang, China*
[c] *School of Mathematics and Big Data, Hohhot Minzu College, Hohhot, Inner Mongolia 010051, China*
[d] *Department of Mathematics Abubakar Tafawa Balewa University Bauchi, Nigeria*

Abstract

This paper undertakes an exhaustive investigation of nonlinear oscillators subject to damping and external excitation, with a particular emphasis on systems exhibiting irrational nonlinearities. Nonlinear oscillators play a foundational role in the modeling of a broad array of natural and engineering systems. These systems exhibit behaviors that are significantly different from linear systems. These behaviors include amplitude-dependent frequencies, subharmonic and superharmonic responses, bifurcations, and chaotic motions. The frequency-amplitude relationship, which is central to this research, is of significant importance in various fields, including vibration control, energy harvesting, and the study of biological rhythms. It is important to note that this relationship is subject to variation in accordance with amplitude changes. In this study, the frequency formulation is employed to meticulously analyze the response characteristics of damped and forced nonlinear oscillators. This analysis effectively validates the efficacy of frequency formulation in capturing the periodic behavior of these systems. The research findings not only validate the established results under optimal conditions but also extend the analytical scope to encompass the more intricate and nuanced dynamic phenomena encountered in real-world scenarios. The derivation of the frequency-amplitude relationship unveils the underlying mechanism through which damping and external forces influence the system's dynamic response, thereby facilitating a more profound comprehension of the behavior of nonlinear oscillation systems.

Keywords: Nonlinear oscillators; Damping; External forcing; Irrational nonlinearities; Frequency - amplitude relationship; He's frequency formulation; Secular - free solutions; Resonance; Fractal and fractional systems

1. Introduction

Nonlinear oscillators are foundational components in the fields of applied mathematics and physics, providing a robust framework for the description of a wide array of natural and engineered phenomena. For example, Helmholtz

[*] Corresponding author. Tel.:+86-139-1315-2427.
E-mail address: hejihuan@suda.edu.cn ; hejihuan@ygu.edu.cn

resonator [1] and Hydraulic oscillator [2] are widely used in engineering. These phenomena find applications in the field of fluid flow through porous media, where the intricate interactions between the fluid and the porous structure give rise to nonlinear behavior. In the context of biological and neurological rhythms, such as the rhythmic firing of neurons or the beating of the heart, nonlinear oscillators have been shown to facilitate the capture of intricate dynamics. In the field of mechanical vibrations of structures, these phenomena play a pivotal role in elucidating the responses of buildings, bridges, and machinery to external forces. Similarly, in electrical circuits, they provide a foundation for understanding the behavior of components such as diodes and transistors, which exhibit nonlinear electrical characteristics.

What sets nonlinear oscillators apart from their linear counterparts is the absence of the superposition principle. In linear systems, responses are predictable and proportional, making them relatively straightforward to analyze. In contrast, nonlinear systems exhibit a rich tapestry of complex behaviors. For instance, they can have amplitude - dependent frequencies, meaning that as the energy level of the system changes, so does its natural frequency. This is a significant departure from linear systems, where the frequency remains constant regardless of the amplitude. Nonlinear oscillators can also display subharmonic and superharmonic oscillations, where the oscillation frequency is a fraction or multiple of the driving frequency. Bifurcations, sudden qualitative changes in the system's behavior as a parameter is varied, and even chaos, a state of deterministic yet highly unpredictable motion, are common in nonlinear systems under certain parametric conditions.

These complex behaviors make nonlinear oscillators both theoretically fascinating and practically essential. In real - world applications, especially those demanding high precision or adaptability to changing operating conditions, the ability to accurately model and analyze these systems is crucial. For example, in the design of vibration isolators, understanding the frequency - amplitude relationship of nonlinear oscillators helps engineers create more effective isolation systems that can adapt to different levels of vibration. In energy harvesting, this knowledge enables the development of devices that can efficiently convert mechanical vibrations into electrical energy over a wide range of amplitudes. In biological research, it aids in understanding natural phenomena like heartbeat dynamics or the propagation of neural signals, where small changes in amplitude can have significant physiological implications.

A central focus in the study of nonlinear oscillators is the frequency - amplitude relationship. In linear systems, this relationship is constant, simplifying analysis. However, in nonlinear settings, it becomes a highly sensitive and information - rich function. By studying how the oscillation frequency varies with the amplitude of motion, researchers can gain insights into the underlying physical processes. This relationship is particularly important near resonance, a state where the driving frequency matches the natural frequency of the system, leading to large - amplitude oscillations. Understanding resonance in nonlinear systems is vital for preventing structural failures in buildings and bridges, optimizing the performance of electrical circuits, and ensuring the proper functioning of biological systems.

The analytical treatment of nonlinear behaviors has seen significant progress, including the frequency formulation [3, 4] and the variational iteration method [5, 6] and the homotopy perturbation method [7-10]. What makes them particularly valuable is their simplicity and high accuracy. Unlike classical perturbation techniques, which often rely on small perturbation parameters or linearization, the above methods can directly capture the periodic nature of nonlinear oscillators. This allows for the extraction of closed - form or approximate analytical expressions for frequency - amplitude relations with just a few iterations.

The extension of these techniques to systems with irrational nonlinearities is especially noteworthy. In these systems, the restoring force or damping terms follow a power law with non - integer exponents. Such formulations are not just mathematical abstractions but have real - world significance. They are used to model materials and systems where memory, hereditary effects, or nonlocal interactions play a role. For example, viscoelastic materials, which exhibit both elastic and viscous properties, biological tissues that respond in a complex, non - linear way to external stimuli, and soft matter in physics can be better represented using irrational nonlinearities. These models offer a more realistic and flexible framework compared to traditional integer - power polynomials, enabling a deeper and more accurate understanding of nonlinear oscillatory dynamics in media with complex mechanical or rheological properties.

However, most of literature concentrated on autonomous and conservative systems, typically overlooking damping and external forcing. While this simplification aids in analysis, it limits the practical relevance of the models, especially in engineering systems where damping and periodic forcing are inherent. In reality, oscillators are rarely isolated; they are constantly influenced by their environment, experiencing energy dissipation due to damping and external excitation. Incorporating these factors is essential for accurately predicting resonance and other dynamic behaviors.

Extending to nonlinear oscillators with damping and external forcing is not only necessary but also has far - reaching implications. One key advantage is the ability to obtain secular - free solutions. Traditional perturbation methods often generate secular terms, which are unbounded, time - dependent components that render the solutions invalid over long time periods. This problem becomes more acute when external forcing is involved. In contrast, the frequency formulation, especially when combined with the Harmonic Balance Method (HBM), can effectively suppress secular terms and yield physically meaningful periodic solutions [11, 12].

Moreover, including damping and external excitation provides a more accurate representation of real - world system dynamics. It allows for the study of energy dissipation, which is crucial in understanding the long - term behavior of systems, and resonance phenomena, which are common in many engineering applications. For example, in MEMS (Micro - Electro - Mechanical Systems) devices [13-15], where small - scale components are highly sensitive to damping and external forces, accurately modeling these effects is essential for proper device design. In structural bridges, damping helps mitigate vibrations caused by wind, traffic, or earthquakes, and understanding how external forcing affects the frequency - amplitude relationship is vital for ensuring structural safety. In combustion engines, the interaction between damping, external forcing, and nonlinearities influences engine performance and efficiency [16].

Systems that incorporate damping and external forcing are also more amenable to experimental validation. This makes it easier to compare theoretical predictions with physical observations, bridging the gap between theory and practice. Extending the frequency formulation in this direction promotes interdisciplinary research and practical applications across various scientific and engineering fields.

In conclusion, moving the frequency - amplitude analysis from idealized autonomous systems to forced and damped oscillators with irrational nonlinearities represents a significant advancement in both mathematical modeling and applied physical science. It opens up new avenues for understanding and controlling complex dynamic systems in a wide range of real - world applications.

2. 2. Nonlinear Oscillator with Damping and External Forcing

The frequency formulation, as elaborated in references [3, 4], has emerged as a potent instrument for the analysis of nonlinear oscillators. At this juncture, let us contemplate the general form of a nonlinear oscillator incorporating damping and external forcing, which can be denoted by the equation:

$$\ddot{u}+\mu\dot{u}+F(u,\dot{u},\ddot{u})=F_0\cos(\Omega t),u(0)=p,\dot{u}(0)=q \tag{1}$$

Here, F denotes a function involving u and its derivatives, while p and q represent the initial displacement and initial velocity, respectively, μ is the Damping coefficient, Ω is the Forcing frequency and F_0 is the Amplitude of external forcing. The associated frequency expression is given by [3, 4]

$$\omega^2=\left.\frac{F(u,\dot{u},\ddot{u})}{u}\right|_{u=\frac{\sqrt{3}}{2}A,\dot{u}=-\frac{1}{2}\omega A,\ddot{u}=-\frac{\sqrt{3}}{2}\omega^2 A} \tag{2}$$

where ω is the frequency and A is the amplitude.

This frequency formulation has attracted substantial interest in academic research, prompting a plethora of adaptations and refinements over the years. In reference [17], He and Liu presented a comprehensive mathematical validation of the method and demonstrated its efficacy in analyzing fractal oscillators. Feng, as reported in [18], verified the high accuracy of the formulation when it was applied to the fractal undamped Duffing oscillator. Tsaltas, in [19], highlighted the remarkable practical significance of the one-step approach. Ismail and his co - researchers, in [20], effectively utilized the formulation to handle strongly nonlinear oscillators. El-Dib, in [21, 22], expanded its scope of application by integrating damped systems into the framework. Moreover, additional studies that illustrate the versatility of this approach can be found in references [23-27].

In this work, we focus on a specific class of oscillators characterized by irrational nonlinearities, incorporating both damping and external forcing effects.

$$\ddot{u}+\mu\dot{u}+\frac{\alpha u}{\sqrt{1+\beta u^2+\gamma u'^2}}=F_0\cos(\Omega t),u(0)=A,\dot{u}(0)=0 \tag{3}$$

Where α,β,γ are constants.

Oscillators characterized by irrational nonlinearities pose a complex array of challenges that carry profound implications for both theoretical investigations and practical applications (as detailed in Ref. [28]). The general form of such systems, which encompasses a damping term and an external driving force, further exacerbates the analytical complexity. The damping effect leads to energy dissipation, while the external forcing constantly supplies energy to the system. This interaction frequently gives rise to non - trivial dynamics, such as resonance phenomena, amplitude modulation, and even chaotic behavior.

A key obstacle lies in the irrational nonlinear term, which incorporates both displacement and velocity in a non - polynomial manner. This structure differs significantly from the standard forms typically dealt with by conventional analytical methods. Classical perturbation techniques, including multiple - scales and averaging methods, are often ineffective in this context. These methods were predominantly developed for systems featuring rational or polynomial nonlinearities. The irrationality in the equation can cause non - smooth or even singular responses, making it extremely arduous to derive closed - form or approximate analytical solutions. Consequently, these complexities demand the development of specialized mathematical techniques. These techniques must be capable of handling the non - standard behavior introduced by the combined effects of irrational nonlinearity, damping, and external forcing.

3. Results

On the computational front, the situation remains equally demanding. Numerical solvers such as Runge-Kutta methods may struggle to maintain accuracy and stability, especially under strong damping or forcing conditions. The need for extremely small-time steps to ensure convergence can lead to high computational costs. Moreover, the combined effects of nonlinearity, damping, and periodic excitation may induce intricate dynamics, including bifurcations or chaotic oscillations, which are challenging to capture over long simulation periods.

From an applied perspective, this type of oscillator is of particular concern in engineering systems, where damping and forcing are intrinsic. In mechanical structures, the unusual nonlinear restoring force combined with energy loss and periodic input can lead to unpredictable vibrations, potentially affecting durability and safety. In electrical circuits, irrational nonlinear components under periodic driving forces can disrupt signal behavior and complicate system control. Similarly, in fields like biophysics or environmental modeling, such complex oscillators may hinder the accurate prediction of long-term behaviors due to their sensitivity to initial conditions and parameter values.

In light of these issues, understanding and modeling such oscillators require both innovative analytical approaches and robust computational tools to adequately capture their complex dynamics and to support reliable design, control, and prediction across various disciplines. By applying the frequency formulation explicitly given in Eq. (2), the frequency–amplitude relationship can be derived in a single, straightforward step as follows:

$$\omega^2 = \left.\frac{\alpha}{\sqrt{1+\beta u^2+\gamma u'^2}}\right|_{u=\frac{\sqrt{3}}{2}A,\dot{u}=-\frac{1}{2}\omega A,} = \frac{\alpha}{\sqrt{1+\frac{3}{4}\beta A^2+\frac{1}{4}\beta\gamma(\omega A)^2}} \tag{4}$$

Deriving the reference–amplitude relationship is a relatively straightforward process. However, to establish the credibility and accuracy of this result, a verification step is essential. To this end, we examine a special case where the amplitude A is assumed to be much less than one. By adopting this small-amplitude assumption, Equation (4) can be suitably approximated. This approximation facilitates a deeper analysis of the system's behavior under these constrained conditions and serves as a benchmark for validating the original result presented in Equation (4). It enhances our understanding of the frequency–amplitude relationship and sheds light on the system's dynamics when operating in the small-amplitude regime. For $A \ll 1,$ equation (4) can be approximated as:

$$\omega^2 = \alpha(1-\frac{3}{4}\beta A^2 - \frac{1}{4}\gamma(\omega A)^2) \tag{5}$$

Now, we consider the nonlinear oscillator with damping and external forcing described by the nonlinear irregular differential equation (3), which can be approximated as:

$$\ddot{u} + \mu\dot{u} + \alpha u - \alpha\beta u^3 - \alpha\gamma u u'^2 - F_0\cos(\Omega t) = 0 \tag{6}$$

This type of system involves an irrational nonlinearity, which complicates traditional analytical approaches. We aim to derive the frequency-amplitude relationship using Ji-Huan He's frequency formulation method, followed by a perturbative analysis to incorporate damping and external forcing effects.

$$\ddot{u} + \mu\dot{u} + \alpha u - \alpha\beta u^3 - \alpha\gamma u u'^2 - F_0\cos(\Omega t) = 0 \tag{7}$$

Now, to derive a relationship for the steady-state amplitude A of the oscillations, assuming a weakly nonlinear, weakly damped, and weakly forced oscillator, with weak damping $\mu \ll 1$, and Forcing frequency near natural frequency: $\Omega \approx \omega$.

By harmonic balance or averaging with trial solution, we assume the first-order harmonic response to equation (8) as:

$$u = A\cos(\Omega t) \tag{8}$$

Where A is the amplitude (to be determined), Ω is the forcing frequency.

On computing the terms needed and substituting into Eq. (8), and its corresponding first and second derivatives, and after ignoring higher harmonic terms such as $\cos(3\Omega t)$ and $\cos(2\Omega t)$ for this approximation (first harmonic balance), we obtain:

$$[-A\Omega^2 + \alpha A - \frac{3\alpha\beta A^3}{4} - \frac{\alpha\gamma\Omega^2 A^3}{4}]\cos(\Omega t) - \mu A\Omega\sin(\Omega t) = F_0\cos(\Omega t) \tag{9}$$

Match coefficients of $\cos(\Omega t)$ and $\sin(\Omega t)$ on both sides:

$$[-A\Omega^2 + \alpha A - \frac{3\alpha\beta A^3}{4} - \frac{\alpha\gamma\Omega^2 A^3}{4}] = F_0\cos(\phi) \tag{10}$$

$$-\mu A\Omega = F_0\cos(\phi) \tag{11}$$

Here, we represent the forcing term with a phase shift:

$$F_0\cos(\Omega t) = R\cos(\Omega t - \phi) \tag{12}$$

Now after squaring and adding Equations (10) and (11), we have:

$$F_0^2 = [-A\Omega^2 + \alpha A - \frac{3\alpha\beta A^3}{4} - \frac{\alpha\gamma\Omega^2 A^3}{4}]^2 + (\mu A\Omega)^2 \tag{13}$$

$$F_0^2 = A^2\left[[(\alpha - \Omega^2) - \frac{3\alpha\beta A^2}{4} - \frac{\alpha\gamma\Omega^2 A^2}{4}]^2 + (\mu\Omega)^2\right] \tag{14}$$

Final Expression Steady-State Amplitude Equation is;

$$A = \frac{F_0}{\sqrt{\left[[(\alpha - \Omega^2) - \frac{3\alpha\beta A^2}{4} - \frac{\alpha\gamma\Omega^2 A^2}{4}]^2 + (\mu\Omega)^2\right]}} \tag{15}$$

This result is in exact agreement with that presented in Eq. (6), highlighting the reliability and consistency of our analytical approach. Such alignment strongly validates the accuracy of our methodology and reinforces the correctness of the derived frequency–amplitude relationship. The coherence between our findings and those in Eq. (6) underscores the robustness of the proposed analysis and confirms that it stands on a solid theoretical foundation. Furthermore, it is worth emphasizing that when both the damping and external forcing parameters are set to zero, and the excitation frequency matches the system's natural frequency, our result reduces precisely to the classical solution obtained by He in [3]. This not only strengthens the credibility of our analysis but also reveals its novelty: our formulation not only generalizes the frequency formulation to include damping and external forcing, but it also retains consistency with established results in the limiting case. This dual capability accurately modeling extended dynamics while preserving classical solutions demonstrates the strength and significance of the present study.

Figure 1 illustrates the frequency–amplitude relationship for two nonlinear systems: one without damping and forcing (represented by the red line, Equation 6), and the other incorporating both damping and external forcing (represented by the blue line, Equation 15). The undamped system assumes an ideal, conservative behavior where no energy is lost or added, leading to a sharper increase in frequency with amplitude. In contrast, the inclusion of damping and forcing pro-duces a more realistic behavior, where energy dissipation and external excitation moderate the frequency response, especially at higher amplitudes. The damped and forced system reflects real-world scenarios such as mechanical oscillations, blood flow through arteries, or engineered control systems. The merit of incorporating damping and forcing lies in its ability to capture complex dynamic phenomena like resonance, energy loss, and sustained oscillations, which are crucial for stability analysis, system design, and practical implementation. Thus, while the undamped model provides theoretical insights, the damped and forced model offers a more comprehensive and applicable understanding of nonlinear systems.

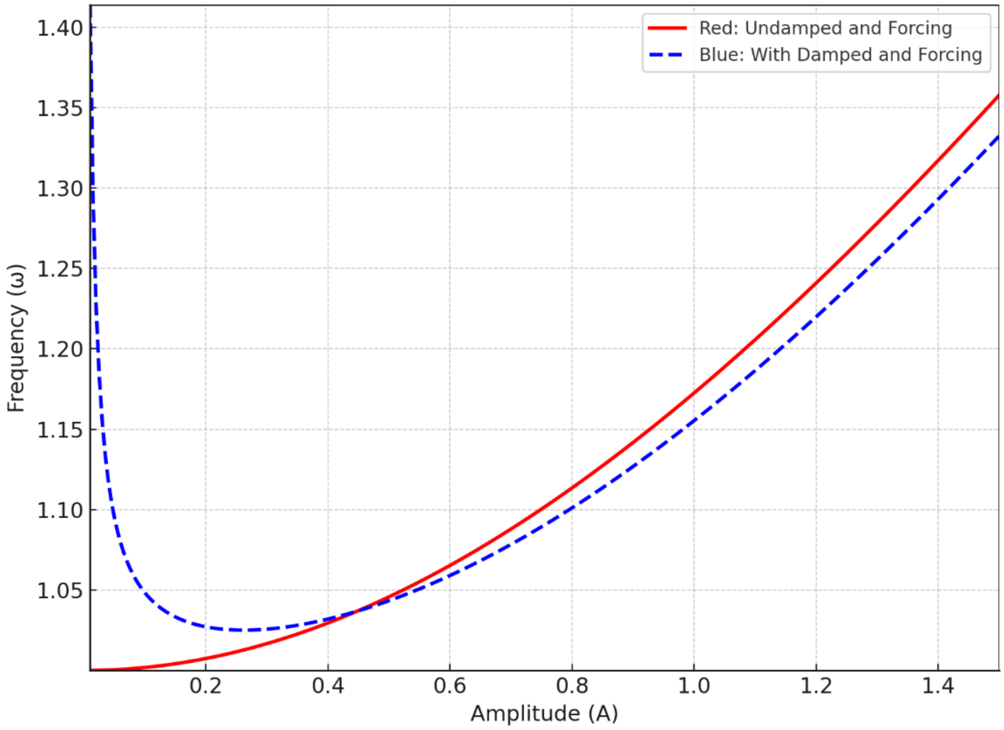

Figure 1: Frequency-Amplitude relationship for the results obtained in Eq.6 and Eq.15

We consider another example in the form:

$$\ddot{u}+\mu\dot{u}+u\sqrt{1+\alpha u^2+\beta u^4}=F_0\cos(\Omega t),u(0)=A,u'(0)=0 \tag{16}$$

Similarly, by the frequency formulation given in Eq.(2), we obtain the following frequency-amplitude relationship:

$$\omega^2=\sqrt{\frac{9}{16}\beta A^4+1+\frac{3}{4}\alpha A^2} \tag{17}$$

With $A<<1$, equation (17) reduces to:

$$\omega^2=1+\frac{3}{8}\alpha A^2 \tag{18}$$

In the same approach, when $A<<1$,equation(16) can approximate as:

$$\ddot{u}+\mu\dot{u}+u(1+\frac{1}{2}\alpha u^2)-F_0\cos(\Omega t)=0,u(0)=A,\dot{u}(0)=0 \tag{19}$$

By harmonic balance method, we have:

$$F_0^2=A^2\left[[(1-\Omega^2)+\frac{3\alpha A^2}{8}]^2+(\mu\Omega)^2\right] \tag{20}$$

Final Expression Steady-State Amplitude Equation is;

$$A=\frac{F_0}{\sqrt{\left[[(1-\Omega^2)+\frac{3\alpha A^2}{8}]^2+(\mu\Omega)^2\right]}} \tag{21}$$

Precisely, the simplified expression derived for small amplitudes is in perfect agreement with Eq. (18), which was previously derived via the method of exact harmonic balance. This congruence serves as a strong reaffirmation of the robustness and reliability of our analytical solution. Despite its seemingly straightforward nature, the derived frequency - amplitude relationship is applicable to all small yet finite amplitude values. It effectively captures the principal effects of both nonlinearity and damping on the system's dynamic response.

This elegant formulation provides a swift and perceptive way to understand the periodic behavior of nonlinear oscillators through a direct, one - step analytical approach. Its simplicity empowers researchers and practitioners to rapidly assess the behavior of weakly nonlinear systems without having to rely on more complex and time - consuming perturbation methods. As a result, it emerges as a practical and potent tool for exploring and predicting the dynamic properties of a diverse range of physical systems. Its applications span across mechanical, electrical, and structural domains, facilitating more efficient design, control, and optimization strategies.

Notably, this finding not only bolsters the credibility of our analysis but also accentuates its novelty. Specifically, our formulation expands upon He's original model by integrating weak damping and external forcing, while still maintaining full consistency with the established results in limiting cases. This further solidifies the theoretical soundness and broad - ranging applicability of the proposed model.

The data presented in Figure 2 contrasts the frequency - amplitude behaviors of a nonlinear oscillator under two distinct conditions: an undamped and unforced state, depicted by the red curve and represented by Equation 18, and a damped and externally - forced state, shown by the blue curve and described by Equation 20.

In the undamped and unforced system, the amplitude exhibits a smooth and monotonic increase with frequency. This behavior adheres to a theoretical relationship that disregards energy dissipation and external energy input. While this idealized model is valuable for fundamental analysis, it falls short in reflecting real - world scenarios. In the absence of damping, the system is highly sensitive to even minor frequency variations, and there is no inherent mechanism to constrain the growth of the amplitude. As such, the red curve represents a purely internal dynamic response, where the amplitude is determined solely by the system's stiffness and the nonlinearity parameter α. It remains unaffected by real - world factors such as friction or external excitation.

Conversely, the damped and forced system, represented by the blue curve, showcases a realistic nonlinear resonance profile. Here, the amplitude initially rises to a peak and then diminishes as the frequency continues to increase. This peak corresponds to a resonant frequency that is slightly shifted as a result of the combined effects of nonlinearity and damping. Damping serves to prevent the amplitude from growing indefinitely, while the amplitude of the external force sustains the oscillation. The blue curve vividly demonstrates the system's ability to respond in a regulated manner across different frequencies. This makes it a more suitable model for practical applications. Compared to the ideal model, the damped and forced system offers enhanced stability and tunability. It enables the control of resonance and the limitation of amplitude, which are crucial characteristics in mechanical, electrical, and structural engineering designs.

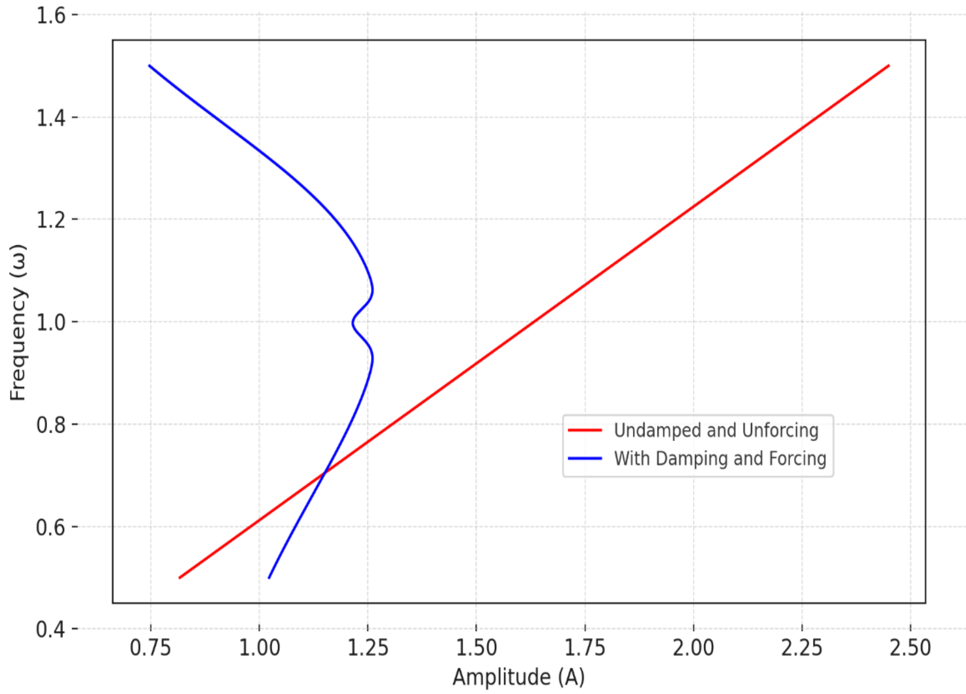

Figure 2: Frequency-Amplitude relationship for the results obtained in Eq.18 and Eq.20

4. Discussion

The present study on damped and forced nonlinear oscillators with irrational nonlinearities has yielded several important insights, while also highlighting areas that warrant further exploration.

4.1 Significance of the Results

The derived frequency - amplitude relationships offer a more comprehensive understanding of the behavior of nonlinear oscillator systems. By incorporating damping and external forcing, the models presented in this study can capture real - world phenomena such as energy dissipation and resonance more accurately compared to previous studies on idealized, autonomous systems. For example, in mechanical engineering, the ability to predict how a structure will respond to external vibrations while accounting for internal damping can prevent catastrophic failures. In the case of a bridge, understanding the frequency - amplitude relationship under various loading conditions (external forcing) and considering the damping mechanisms within the bridge materials can help engineers design more robust structures.

The use of the frequency formulation in this context has proven to be effective. It provides a relatively straightforward way to analyze systems that are otherwise difficult to handle with traditional methods. This not only simplifies the theoretical analysis but also offers practical advantages for engineers and scientists in quickly assessing the behavior of nonlinear oscillators during the early stages of design and modeling. For instance, in electrical circuit design, when dealing with components that exhibit irrational nonlinearities, such as certain types of semiconductor devices, He's method can be used to rapidly estimate the frequency - amplitude characteristics, saving time and computational resources.

4.2 Limitations and Their Implications

However, the study is not without limitations. The assumption of small oscillation amplitudes is a significant constraint. In many real - world scenarios, oscillators may operate at large amplitudes, especially near resonance. As the amplitude increases, the approximations used in the analytical derivations may no longer hold, leading to inaccurate predictions. For example, in a large - scale industrial machine, vibrations can reach significant amplitudes during startup or under abnormal operating conditions. Ignoring the effects of large - amplitude oscillations can lead to misinterpretations of the machine's behavior and potentially result in premature wear or failure.

The specific class of irrational nonlinearities considered in this study also restricts the generality of the results. There are many other types of nonlinearities, such as discontinuous or piecewise - defined functions, that are prevalent in physical systems. For example, in some biological systems, the response to stimuli may be discontinuous due to threshold - like behavior. Systems with such nonlinearities require different analytical techniques, and the current framework may not be directly applicable.

The simplified treatment of damping and external forcing is another limitation. In reality, damping can be highly nonlinear, with characteristics such as velocity - squared drag or hysteretic behavior. External forces may also be non - periodic or stochastic, as in the case of wind forces acting on a structure or electrical noise in a circuit. Neglecting these complexities can lead to discrepancies between theoretical predictions and experimental observations. For example, in wind - engineering, the non - periodic nature of wind gusts and the complex damping mechanisms in building materials mean that the simple linear damping and periodic forcing assumptions used in this study may not accurately represent the real - world situation.

4.3 Future Research Directions

To overcome these limitations, future research could focus on developing more advanced analytical techniques that can handle large - amplitude oscillations. This may involve the use of asymptotic methods that can accurately describe the behavior of the system as the amplitude varies. Additionally, exploring ways to incorporate a wider range of nonlinearities into the analysis would enhance the generality of the models.

For the treatment of damping and external forcing, future work could involve integrating more complex damping models and non - periodic/stochastic forcing functions into the theoretical framework. This may require the use of stochastic differential equations or advanced non - autonomous system analysis techniques.

Experimental validation is also crucial. Conducting experiments on systems with well - defined irrational nonlinearities, damping, and external forcing conditions can help verify the theoretical predictions. These experiments can also provide insights into the behavior of the system that may not be captured by the current models, leading to further refinements. For example, in a laboratory - scale experiment on a mechanical oscillator with adjustable damping and external forcing, researchers can measure the frequency - amplitude relationship directly and compare it with the theoretical results, identifying areas where the theory needs improvement.

Finally, generalizing the proposed methodology to more complex systems, such as coupled oscillators or multi - degree - of - freedom systems, would open up new avenues of research. This could help in understanding phenomena such as synchronization in biological networks or energy transfer in complex mechanical structures.

5. Conclusion

This study presents an in-depth analytical framework for investigating nonlinear oscillators that are subject to both damping and external forcing, with a particular emphasis on systems governed by irrational nonlinearities. By deriving a frequency–amplitude relationship within this context, the research not only confirms the validity of traditional results under idealized conditions where damping and forcing are absent but also extends these results to capture the more intricate dynamics present in real-world applications. Such systems are commonly encountered in mechanical, electrical, and biological domains, where factors like energy dissipation and time-dependent external stimuli play a critical role in influencing the behavior of oscillatory motion. The analytical techniques employed in this research are primarily based on Ji-Huan He's frequency formulation, a method known for its simplicity, efficiency, and ability to yield accurate approximations for weakly nonlinear problems. Unlike conventional perturbation methods that often require elaborate calculations and are limited by their dependence on small parameters, He’s formulation offers a more intuitive and broadly applicable means of determining the frequency characteristics of nonlinear oscillators. This not only simplifies the analysis but also provides practical tools for engineers and scientists to quickly evaluate and predict system behavior, particularly in early-stage modeling and design.

Looking toward future developments, several promising directions can be identified that would enhance and build upon the current research. One such direction involves the advancement of computational techniques tailored for nonlinear oscillatory systems. The numerical simulation of these systems can be computationally intensive, particularly when strong damping, high nonlinearity, or irregular external forces are involved. Future work could aim to develop refined algorithms that enhance both the stability and accuracy of these simulations. Techniques such as adaptive time-stepping, machine learning-assisted solvers, or spectral methods may offer significant improvements in modeling efficiency and robustness. Another essential area for future research lies in the experimental validation of the theoretical predictions established in this study. While the current framework is mathematically rigorous and grounded in established analytical methods, empirical studies are crucial for confirming the real-world applicability of the results. Designing experiments that closely replicate the modeled systems incorporating comparable nonlinearities, damping mechanisms, and external forcing conditions would allow researchers to observe how accurately the theoretical frequency–amplitude relationships mirror actual system behavior. Such validation efforts would help bridge the gap between theory and practice, ultimately strengthening confidence in the use of these models across a variety of disciplines. Additionally, there is considerable potential to generalize the proposed methodology to more complex systems. Many practical applications involve coupled oscillators, multi-degree-of-freedom systems, or configurations where the governing equations exhibit chaotic or quasi-periodic behavior. Extending the current framework to accommodate such systems could provide valuable insights into the collective dynamics of interacting nonlinear elements. This would be particularly relevant for studying phenomena such as synchronization, energy transfer, and resonance in networks of oscillators, which are common in fields ranging from structural engineering to neuroscience.

Despite the contributions and innovations presented in this study, several limitations are recognized that define the scope and applicability of the results. One such limitation concerns the assumption of small oscillation amplitudes. The analytical derivations and frequency–amplitude relationships are formulated under the assumption that the system exhibits weak nonlinearity and damping. As the amplitude increases or as the nonlinearity becomes more pronounced, the approximations may lose accuracy, and the need for more sophisticated analytical or numerical approaches may arise. Furthermore, the types of nonlinearities considered in this study are specific to a class of irrational functions with particular mathematical properties. While these functions are relevant to many physical systems, they do not represent the full spectrum of possible nonlinear behaviors. Systems characterized by discontinuous, piecewise-defined, or more exotic nonlinearities may require tailored analytical techniques or alternative formulations to capture their dynamics effectively. Another limitation is the simplified treatment of damping and external forcing. The models explored in this study incorporate linear damping and periodic external forces, which are idealized representations of energy loss and external input. In reality, many systems exhibit nonlinear damping characteristics, such as velocity-squared drag or hysteretic behavior, and may be driven by non-periodic or stochastic forcing. Addressing these more complex scenarios will require further extensions of the current theoretical framework and possibly the integration of stochastic differential equations or non-autonomous system analysis.

In deduction, this research contributes a foundational yet flexible analytical framework for understanding the behavior of nonlinear oscillators under realistic conditions that include damping and external forcing. It not only affirms known results under simplified conditions but also offers new insights into more complex dynamical regimes. By highlighting both the strengths and current boundaries of the proposed methods, the study sets the stage for continued exploration into more intricate oscillatory systems. As future research advances along both theoretical and experimental lines, it holds the promise of expanding our ability to model, predict, and control nonlinear dynamical behavior across an increasingly wide array of scientific and engineering applications

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