Journal of Computational Applied Mechanics 2025, 56(1): 264-275

DOI: 10.22059/jcamech.2024.386177.1304

RESEARCH PAPER

Time Dependent Harmonic Oscillator via OM-HPM

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Abstract

In this study, we present a semi-analytical technique known as the Optimal and Modified Homotopy Perturbation Method (OM-HPM) for solving nonlinear oscillators with time-dependent mass. The work extends existing approaches, including the standard Homotopy Perturbation Method (HPM), by introducing an auxiliary linear operator that minimizes residual error and enhances the method's efficiency for both singular and non-singular nonlinear ordinary differential equations. The model of a harmonic oscillator with exponentially decaying mass is investigated using this method, and its equation of motion is derived using the Lagrangian formulation. The OM-HPM technique is applied to solve the resulting second-order nonlinear differential equation, and solutions are presented in series form. The method significantly reduces computational cost through the use of Newton-Cotes quadrature. Analytical illustrations demonstrate that the effectiveness of OM-HPM in solving complex nonlinear oscillatory systems.

Keywords: Harmonic Oscillator; Analytical solution; Homotopy methods; time-dependent mass; nonlinear oscillators.

1. Introduction

Simple harmonic oscillating systems are typically represented as a mass connected to a spring, commonly called simple harmonic oscillators. The equation of motion of these systems can be derived through either Newtonian mechanics or the Lagrangian approach, and they can be solved exactly in certain cases. However, such ideal systems do not exist in the macroscopic realm due to dissipative forces inherent in nature. While these forces may be ignored in some instances, they often result in damped oscillations. Linear oscillators are characterized by oscillating at a single frequency, exhibiting sinusoidal and periodic motion. For further insights into both simple and damped oscillators, it is recommended that interested individuals consult classical mechanics literature [1-3].

Nonlinear oscillators result in complex motion, with two primary characteristics: as the amplitude increases, the significance of nonlinearity also increases, and in certain situations, the frequency may vary with amplitude. Numerous examples of such nonlinear oscillators can be observed in the real world, and it is important to recognize that coupled nonlinear oscillators are a topic of interest across various scientific disciplines, including biology and physics. The literature reflects considerable research efforts dedicated to the

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study of these systems [4-6]. A notable instance is the van der Pol oscillator, which features nonlinear damping and was introduced in the 1920s by Balthasar van der Pol (1889 - 1959). This oscillator serves as a prime example of nonlinear damping, where energy is dissipated at high amplitudes and generated at low amplitudes. Where various methods have been applied in dealing with this oscillator either analytical methods using Homotopy Analysis Method (HAM) [7, 8], the Homotopy Perturbation Method (HPM) [9-11], or numerical methods such as perturbation algorithms that combine the Multiple Scales and Modified Lindstedt–Poincaré Techniques [12], a domain decomposition method (ADM) is discussed in [13, 14] and other numerical methods. Nonlinear oscillations have been critically significant in various fields, including engineering, physics, applied mathematics, and several real-world applications for many years. The literature presents a wide array of analytical methods for addressing nonlinear systems, such as the iteration perturbation method [15], the homotopy perturbation method (HMP) [16, 17], the variational method [18], and many others [19]. Researchers interested in this subject may refer to reference [20]. Generally, obtaining an analytical solution for nonlinear oscillators poses considerable challenges, prompting researchers to employ various numerical methods [16, 17, 21]. In [17], the authors examine a nonlinear oscillator characterized by a coordinate-dependent mass, proposing a model with a negative coefficient for the linear term (see Eq. 3 in [17]) and applying the homotopy perturbation method to approximate the period of their equation, while in this paper we will nonlinear oscillator with time-dependent mass

In the present literature, we have introduced an advanced semi-analytical technique called the Optimal and Modified Homotopy Perturbation Method (OM-HPM). This method enhances the traditional Homotopy Perturbation Method (HPM) by redefining the linear operator as an auxiliary linear operator and subsequently optimizing it by minimizing residual error. Furthermore, it can be directly utilized for both singular and non-singular highly nonlinear ordinary differential equations without the need for decomposition, special transformations, or Pade approximation. Consequently, this study aims to investigate the dynamics of a harmonic oscillator employing this advanced optimal analytical technique [10, 11].

2. The Model: Time Dependent Harmonic Oscillator

2.1. Equation of Motion of the Model via Lagrangian

Consider a harmonic oscillator with time dependent mass (TDM) $m(t)$ and also a time dependent angular frequency $\omega^2(t)$. Such a harmonic oscillator has the following Hamiltonian (H):

$$H = \frac{p^2}{2m(t)} + \frac{m(t)\omega^2(t)}{2}x^2. \tag{1}$$

The Lagrangian (L) is related to H through the relation:

$$H = \sum_i p_i \dot{q}_i - L. \tag{2}$$

In our case (one dimension), we have only $p_i = p_1 = p$ and $\dot{q}_i = \dot{q}_1 = \dot{x}$. So (2) reads:

$$H = p\dot{x} - L. \tag{3}$$

Now, using $\frac{dx}{dt} = \frac{\partial H}{\partial p}$ then we have

$$\dot{x} = \frac{p}{m(t)}. \tag{4}$$

Upon using (3) and (4) and simplifying, we got:

$$x\ . \tag{5}$$

Now, to obtain the equation of motion (EOM) or known as Euler- Lagrange equation (ELE) we use (5) into the formula $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$. As a result, we got

$$\frac{\partial L}{\partial x} = m(t)\omega^2(t)x \; . \tag{6}$$

$$\frac{\partial L}{\partial \dot{x}} = m(t)\dot{x} \; . \tag{7}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m(t)\ddot{x} + \frac{dm(t)}{dt}\dot{x} \; . \tag{8}$$

The ELE (i.e., EOM) reads:

$$m(t)\ddot{x} + \frac{dm(t)}{dt}\dot{x} + m(t)\omega^2(t)x = 0 . \tag{9}$$

2.2 Exponential decaying TDM

Here we consider the exponential decaying TDM, where $m(t)$ is given as:

$$m(t) = \frac{m_o}{2}(1+e^{-\varepsilon t}) . \tag{10}$$

Using (10) in (9) we got the EOM in the following form:

$$\ddot{x} - \frac{\varepsilon e^{-\varepsilon t}}{(1+e^{-\varepsilon t})}\dot{x} + \omega^2(t)x = 0 . \tag{11}$$

In general, the angular frequency $(\omega(t))$ of the harmonic oscillator is related to the mass (m), and the stiffness of the oscillator (k) as $\omega_o^2 = \frac{k}{m}$. So, in our case:

$$\omega^2 = \frac{k}{m(t)} = \frac{k}{\frac{m_o}{2}(1+e^{-\varepsilon t})} = \frac{2\omega_o^2}{(1+e^{-\varepsilon t})} . \tag{12}$$

As a result, relation (11) can be written as:

$$\ddot{x} - \frac{\varepsilon}{(1+e^{+\varepsilon t})}\dot{x} + \frac{2\omega_o^2}{(1+e^{-\varepsilon t})}x = 0 . \tag{13}$$

For the case, $\varepsilon \to 0$ relation (9) reduces to the well-known EOM of the one-dimensional harmonic oscillator. On the other hand, the mass $m(t)$ reduced to $m(t) = m_o = constant$ and so, $\omega^2(t) = \omega_o^2 = \frac{k}{m}$ which is the expected angular frequency. The solution of this trivial case is presented in nearly all classical mechanics texts.

Now, we aim to find the solution of Eq. (13) with the boundary condition

$$x(0) = a \text{ and } \dot{x}(0) = 0 . \tag{14}$$

3. Methodology

Let the nonlinear differential equation of the form

$$\mathcal{N}\left[x(t)\right] = 0, \; t \in \Gamma \tag{15}$$

With the boundary condition

$$\mathcal{B}\left(t, \frac{\partial x}{\partial t}\right) = 0 \tag{16}$$

Where $\mathcal{N}$ is nonlinear operator of order n, $\mathcal{B}$ boundary operator with the domain Γ and $x(t)$ and $g(t)$ are unknown and known analytic functions, respectively. By the definition of homotopy in topology

$$\phi(t, p) : \Omega \times [0,1] \to R \tag{17}$$

we construct the zero-th order homotopy equation of the OM-HPM [20-22] as

$$(1-p)\ \mathcal{L}\left[\phi(t, p) - x_0(t)\right] + p\ \mathcal{N}\left[\phi(t, p)\right] = 0 \tag{18}$$

Where $p \in [0,1]$ is an embedding parameter in topology, $x_0(t)$ is an initial approximation, which is the solution of $\mathcal{L}[x] = 0$ with the boundary (16), $\phi(t, p)$ is the solution of the homotopy equation (18) of the form

$$\phi(t, p) = x_0(t) + \sum_{k=1}^{+\infty} x_k(t)\, p^k \tag{19}$$

And $\mathcal{L}$ is the auxiliary linear operator of the form

$$\mathcal{L}\left[x(t)\right] = \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}x}{dt^{n-2}} + \ldots\ldots + a_1\frac{dx}{dt} + (-1)^n a_0 x(t)$$

$$a_{n-k} = \frac{(-1)^k}{k!}\beta_k\left(s_1, -1!s_2, 2!s_3, \ldots\ldots, (-1)^{k-1}(k-1)!s_k\right), \tag{20}$$

Where β_k is the Bell's polynomials [22], $s_k = \sum_{i=1}^{n} \lambda_i^{\,k}$ and $a_0 = \prod_{i=1}^{n} \lambda_i$, λ_i's are the auxiliary roots of the equation $\mathcal{L}\left[x(t)\right] = 0$.

If we set $p = 0$ at the homotopy solution Eq (18), we have

$$\phi(t,0) = x_0(t) \tag{21}$$

and for $p = 1$ we have

$$\phi(t,1) = x(t) \tag{22}$$

Therefore, when p continuously deforms 0 to 1, the initial solution continuously deforms to the final solution. That is, the Taylor's series solution (19) of the governing equation will be convergent when $p = 1$. Therefore, the series based analytical solution takes the form

$$x(t) = x_0(t) + \sum_{n=1}^{+\infty} x_n(t) \tag{23}$$

It is to note that when all the auxiliary root λ_i, $i = 1, 2, .., n$ are not simultaneously equal to zero, then the series solution (23) contains those unknowns say λ_i, $i = 1, 2, .., n$. Then these unknowns will be compute by minimizing the square residuals [8, 23]

$$\Delta(\lambda_i) = \int_{\Omega} \mathcal{N}\left[\ \sum_{k=0}^{m} x_k(t)\ \right]^2 dt,\ \ i = 1, 2, \ldots, n. \tag{24}$$

Where x_k is the k-th order OM-HPM approximation. It found that the integration involving in Eq. (24) consumes more CPU time to calculate the residuals [8]. To decrease the computational cost, we discretized the integration define in (25) based on Newton-Quotes quadrature (Simson's 1/3) rule (ref. [10, 11, 24]) be defined as

$$\Delta(\lambda_1) \approx \frac{h}{3}\sum_{j=1}^{\frac{k}{2}}\left\{\mathcal{N}\left[\sum_{i=0}^{m} x_i\left(t_{2j-2}\right)\right]^2 + 4\mathcal{N}\left[\sum_{i=0}^{m} x_i\left(t_{2j-1}\right)\right]^2 + \mathcal{N}\left[\sum_{i=0}^{m} x_i\left(t_{2j}\right)\right]^2\right\} \tag{25}$$

4. Analytical Illustrations

We construct the homotopy as

$$(1-p)L\left[\phi(t;p)-x_0(t)\right]+pN\left[\phi(t;p)\right]=0, \tag{26}$$

where,

$$\phi(t;p)=x_0(t)+\sum_{k=1}^{+\infty} x_k(t)\,p^k \tag{27}$$

be the series solution of the homotopy equation (26), $x_0(t)$ be the initial approximation, L be the linear operator as defined in Eq.(20) and p be an embedding parameter.

Since EOM (13) is a second order differential equation, therefore, from eq.(20) we have the general form of linear operator for the second order differential equation as

$$L[x]=\ddot{x}-(\lambda_1+\lambda_2)\dot{x}+\lambda_1\lambda_2 x \tag{28}$$

There is numerous possibility to choose λ_1 and λ_2 , Here we assume that one root is equal to zero (say, $\lambda_2=0$) and other is unknown., then the linear operator (28) becomes

$$L[x]=\ddot{x}-\lambda_1\dot{x} \tag{29}$$

Therefore, from (26), (27) and (29) we have

$$\begin{aligned}&(1-p)\left\{\left(\ddot{\phi}(t;p)-\lambda_1\dot{\phi}(t;p)\right)-\left(\ddot{x}_0-\lambda_1\dot{x}_0\right)\right\}+\\&p\left\{\ddot{\phi}(t;p)-\frac{\epsilon}{1+e^{\epsilon t}}\dot{\phi}(t;p)+\frac{2\omega_0^2}{1+e^{-\epsilon t}}\phi(t;p)\right\}=0\end{aligned} \tag{30}$$

Therefore, substituting the series (26) into the homotopy equation (30) we have

$$(1-p)\left\{\left(\left(\ddot{x}_0+p\ddot{x}_1+p^2\ddot{x}_2+\ldots\right)-\lambda_1(\dot{x}_0+p\dot{x}_1+p^2\dot{x}_2+\ldots\right)-\left(\ddot{x}_0-\lambda_1\dot{x}_0\right)\right\}$$

$$+p\left\{\left(\ddot{x}_0+p\ddot{x}_1+p^2\ddot{x}_2+\ldots\right)-\frac{\epsilon}{1+e^{\epsilon t}}\left(\dot{x}_0+p\dot{x}_1+p^2\dot{x}_2+\ldots\right)+\frac{2\omega_0^2}{1+e^{-\epsilon t}}\left(x_0+px_1+p^2x_2+\ldots\right)\right\}=0$$

$$(1-p)\left\{\left(\left(p\ddot{x}_1+p^2\ddot{x}_2+\ldots\right)-\lambda_1(p\dot{x}_1+p^2\dot{x}_2+\ldots\right)\right\}$$

$$+p\left\{\left(\ddot{x}_0+p\ddot{x}_1+p^2\ddot{x}_2+\ldots\right)-\frac{\epsilon}{1+e^{\epsilon t}}\left(\dot{x}_0+p\dot{x}_1+p^2\dot{x}_2+\ldots\right)+\frac{2\omega_0^2}{1+e^{-\epsilon t}}\left(x_0+px_1+p^2x_2+\ldots\right)\right\}=0$$

On comparing the coefficient of p both side, we get

$$p^1:\ddot{x}_1(t)-\lambda_1\dot{x}_1(t)+\ddot{x}_0(t)-\frac{\epsilon}{1+e^{\epsilon t}}\dot{x}_0(t)+\frac{2\omega_0^2}{1+e^{-\epsilon t}}x_0(t)=0$$

$$p^2:(\ddot{x}_2(t)-\lambda_1\dot{x}_2(t))-\left(\ddot{x}_1(t)-\lambda_1\dot{x}_1(t)\right)+\ddot{x}_1(t)-\frac{\epsilon}{1+e^{\epsilon t}}\dot{x}_1(t)+\frac{2\omega_0^2}{1+e^{-\epsilon t}}x_2(t)=0$$

And so on.

Now, assuming the initial approximation $x_0(t)=A\cos\omega t$ we solve the linear equation, the coefficient of p with the boundary $x_m(0)=0$ and $\dot{x}_m(0)=0,\ m\geq 1$. Then we have the first order correction as

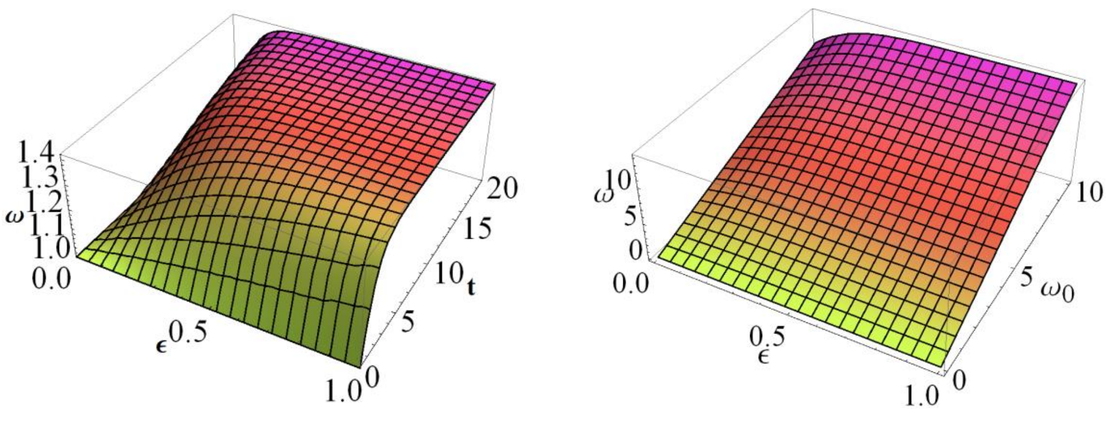
$$x_1(t)=\frac{A\mathrm{e}^{t\lambda_1}\omega^2}{\omega^2+\lambda_1^2}-\frac{A\omega^2\mathrm{Cos}[t\omega]}{\omega^2+\lambda_1^2}+\frac{A\epsilon\omega\mathrm{Sin}[t\omega]}{\left(1+\mathrm{e}^{t\epsilon}\right)\left(\omega^2+\lambda_1^2\right)}+\frac{A\epsilon\omega^2}{\left(1+\mathrm{e}^{t\epsilon}\right)\lambda_1\left(\omega^2+\lambda_1^2\right)}$$
$$-\frac{A\mathrm{e}^{t\lambda_1}\epsilon\omega^2}{\left(1+\mathrm{e}^{t\epsilon}\right)\lambda_1\left(\omega^2+\lambda_1^2\right)}+\frac{A\epsilon\lambda_1}{\left(1+\mathrm{e}^{t\epsilon}\right)\left(\omega^2+\lambda_1^2\right)}-\frac{A\epsilon\mathrm{Cos}[t\omega]\lambda_1}{\left(1+\mathrm{e}^{t\epsilon}\right)\left(\omega^2+\lambda_1^2\right)}-\frac{A\omega\mathrm{Sin}[t\omega]\lambda_1}{\omega^2+\lambda_1^2}$$
$$-\frac{2A\mathrm{e}^{t\lambda_1}\omega_0^2}{\left(1+\mathrm{e}^{-t\epsilon}\right)\left(\omega^2+\lambda_1^2\right)}+\frac{2A\mathrm{Cos}[t\omega]\omega_0^2}{\left(1+\mathrm{e}^{-t\epsilon}\right)\left(\omega^2+\lambda_1^2\right)}+\frac{2A\mathrm{Sin}[t\omega]\lambda_1\omega_0^2}{\left(1+\mathrm{e}^{-t\epsilon}\right)\omega\left(\omega^2+\lambda_1^2\right)}$$

Now, the condition for which it becomes periodic is that the coefficient of cosine and sine should be zero. That is, we eliminate the secular term which may occur in the next iteration. Therefore, we have secular term as

$$\begin{cases} -\dfrac{A\left(\left(1+\mathrm{e}^{t\epsilon}\right)\omega^2+\epsilon\lambda_1-2\mathrm{e}^{t\epsilon}\omega_0^2\right)}{\left(1+\mathrm{e}^{t\epsilon}\right)\left(\omega^2+\lambda_1^2\right)}=0 \\ \dfrac{A\left(\epsilon\omega^2-\lambda_1\left(\left(1+\mathrm{e}^{t\epsilon}\right)\omega^2-2\mathrm{e}^{t\epsilon}\omega_0^2\right)\right)}{\left(1+\mathrm{e}^{t\epsilon}\right)\omega\left(\omega^2+\lambda_1^2\right)}=0 \end{cases} \tag{31}$$

Solving the above equations and utilizing the boundary condition (14) we have,

$$\omega=\sqrt{\frac{2\omega_0^2}{1+\mathrm{e}^{-t\epsilon}}} \quad \text{and} \quad A=\frac{10a+0.1\mathrm{e}^t+a\mathrm{e}^t}{10+\mathrm{e}^t} \tag{32}$$



(a) **(b)**

Fig.1: Frequency for (a) $\omega_0=1$ and unknown ϵ, t and (b) $t=10$ and unknown ϵ, ω_0

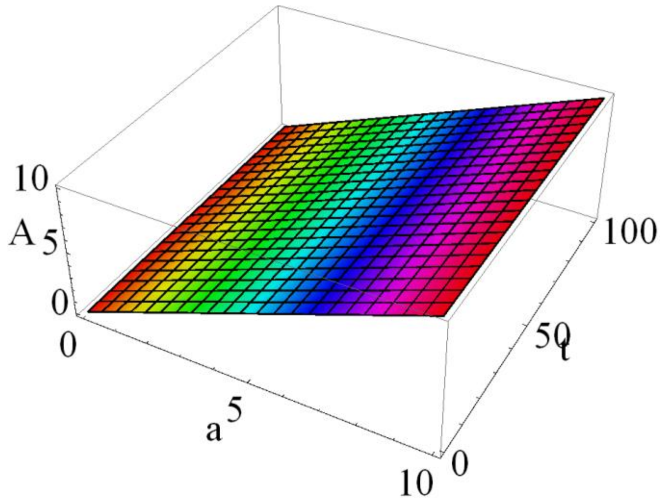
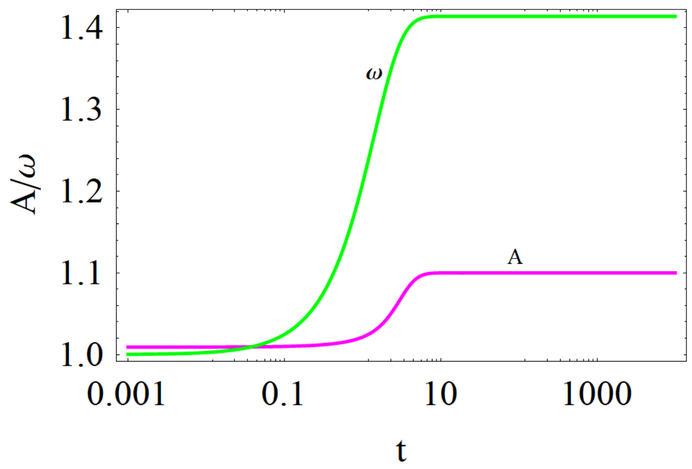


Fig2: Amplitude for unknown a and t

From Eq. (32) it is to note that the amplitude A and frequency ω is time dependent. Now to illuminate we the time dependencies we present Fig. 3 for the parameter values $a=1, \epsilon=1, \omega_0=1$. From the Fig.3 one can see that though it is initially dependent on time t but after certain time $(t \geq 3)$. it became asymptotic or independent on time t



F.3: Graphical presentation of Amplitude (A) and frequency (ω) which is noted in Eq. (32).

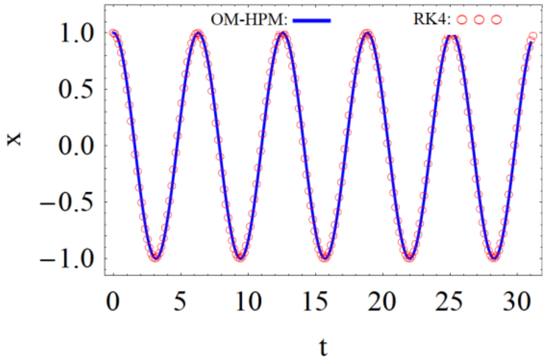
Therefore, the first order OM-HPM approximated solution is

$$x(t) \approx x_0(t) + x_1(t) \tag{33}$$

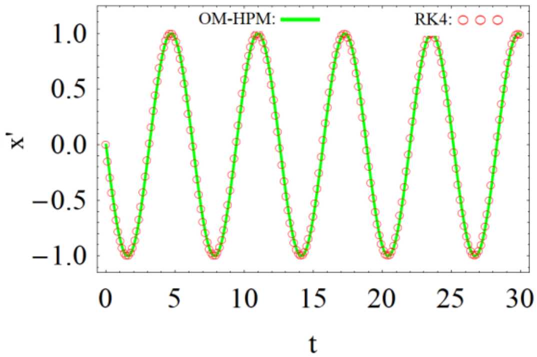
To illustrate the dynamics of considered EOM (13) analytically we considered few cases as of following:

A. For $a=1$, $\epsilon=0$ and $\omega_0=1$

Using the residual Eq.(25) we have the optimal λ_1 as 0, substituting this values we have symbolic approximate solution. To visualize the dynamical behavior for this case we have present the time series solution in Fig.4(a) while the velocity presented in Fig.4(b). From the time series it is evident that motion is independent on time. As we vanished ϵ from the governing EOM (Eq.(13)), so it is natural that the solution will independent on time. However, the time series Fig. 4(a) indicates that the amplitude of oscillation remains constant over time, hence the motion is simple harmonic with no damping effect. To confirm that argument we present the phase portrait of the motion against the displacement and velocity in Fig.5. We have seen that the phase diagram shows a circular trajectory, which is characteristic of an undamped simple harmonic oscillator. In addition, the oscillator exhibits periodic motion with constant energy. Both numerical methods (OM-HPM and RK4) show excellent agreement, indicating that the solution is stable and reliable.



(a)



(b)

Fig.4: Comparison of computed solution with those of numerical solution (a) $x \sim t$ (b) $x' \sim t$

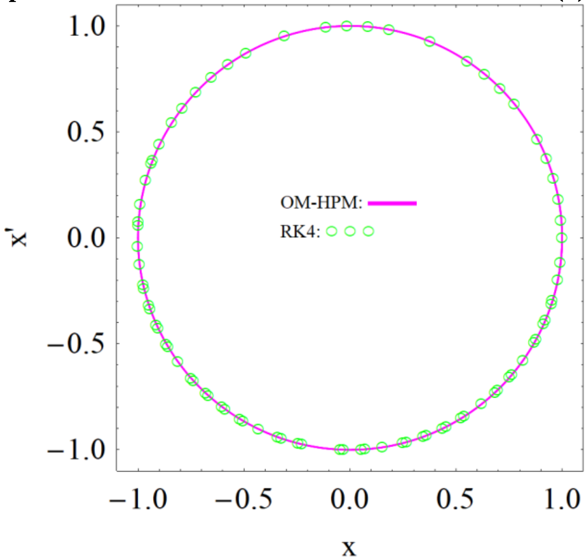
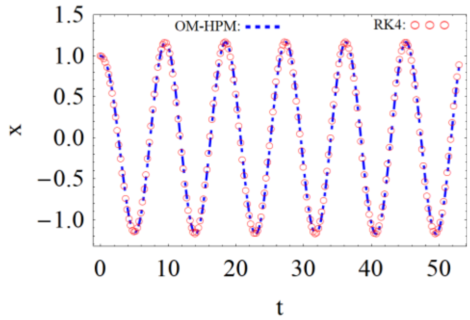


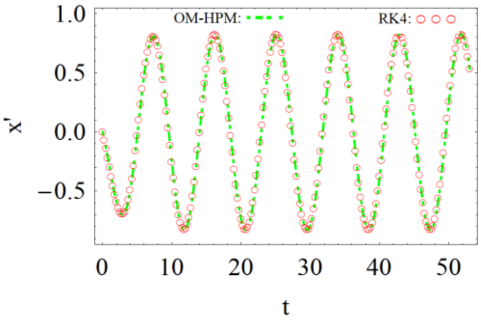
Fig.5: Comparison of phase portrait $x \sim x'$

B. For $a = 1$, $\epsilon = 0.5$ and $\omega_0 = 0.5$

For this case we have optimal value of λ_1 as -3.5317758 where the error is 0.0032394. Therefore, substituting this values we have solution for this case and to visualize the dynamical nature we present time series solution in Fig.6(a) and velocity in Fig.6(b). From the time series Fig.6(a) and the velocity of the motion Fig.6(b) we seen that the motion is time dependent. However, the dependency is exponentially decay. That is, when time is increase the dependency decrease exponentially and after a certain time it becomes independent. Which also noticed from the analytical form of the amplitude and frequency (ref. Eq.(32) and Fig.3). From the time series Fig.6(a) and the phase diagram Fig.7 suggests that this is a **damped oscillator** with the system gradually approaching equilibrium as the oscillations decay over time. The good match between the two methods OM-HPM (blue line) and RK4 (red circles) indicates their reliability in capturing the behavior of the system.



(a)



(b)

Fig.6: (a) $x \sim t$ and (b) $x' \sim t$

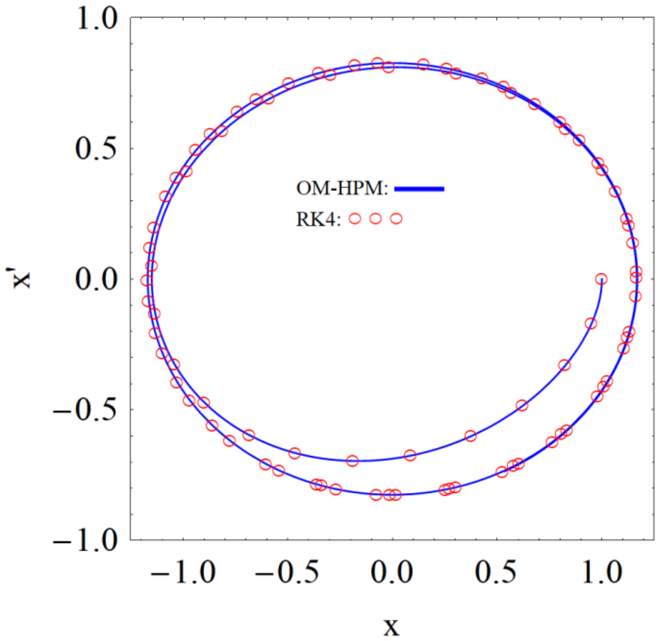
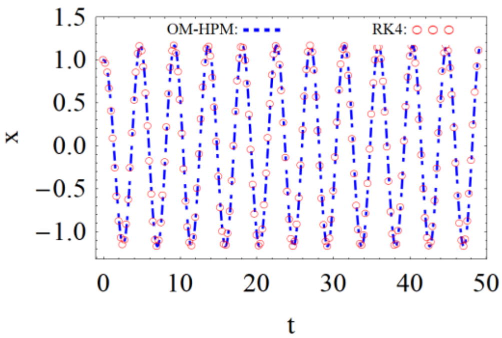
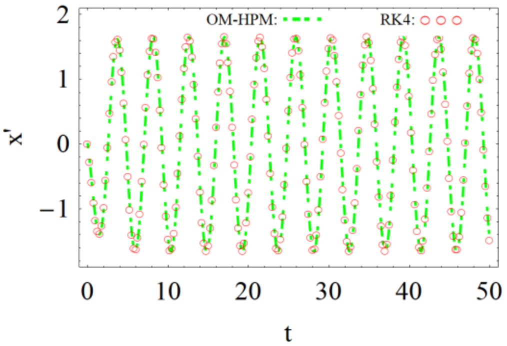


Fig.7: Comparison of computed phase portrait $x \sim x'$

C. For $a=1,\ \epsilon=1$ and $\omega_0=1$

Similarly for this case we have optimal value of $\lambda_1 = -6.5463174$ and corresponding error is 0.0226123. Therefore, substituting this values finally we have approximated solution and presented in Fig.7 and Fig.8. From Fig.7 the time evolution of the oscillator's position $x(t)$ is shown using OM-HPM (blue dashed line) and RK4 (red circles). The oscillator undergoes periodic oscillations with a consistent amplitude of approximately ± 1.5, indicating that the motion is stable and regular over time. The oscillations persist without significant decay in amplitude within the given time frame. This suggests that any damping effect in the system may be minimal or that the plotted time interval is not long enough to show the full decay. Both methods (OM-HPM and RK4) produce nearly identical results. On the other hand, Fig.8 provides the phase portrait of the oscillator, which displays the relationship between position $x(t)$ and velocity $x'(t)$. The trajectory traces an elliptical path that spirals inward, indicating that the oscillator is damped. Over time, the amplitude of the oscillations decreases, leading the system toward a stable equilibrium point. The inward spiral suggests that the system is losing energy, most likely due to damping and will eventually settle into an equilibrium state. From Table-1 we have the maximum relative percentage error is 3%, therefore, the Figs 1-9 and relative error defining in Table-1 demonstrate that the analytical OM-HPM solutions are align closely align with numerical solution by RK4 method.





(a) **(b)**

Fig. 8 (a) $x \sim t$ and (b) $x' \sim t$

Table 1: Computation of maximum relative percentage error (max % error) $\left|\dfrac{x_m - x_{num}}{x_{num}}\right| \times 100$ for different values of a, ϵ and ω_0. Where x_m is the m^{th} order OM-HPM solution and x_{num} is the numerical solution.

a	ϵ	ω_0	Max % error
1	0	1	1.3
1	0.5	0.5	3.3
1	1	1	2.9
10	1	2	3.0

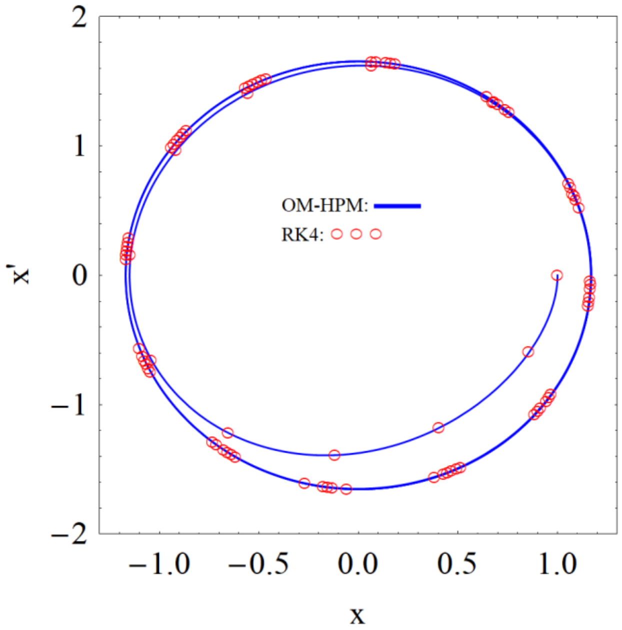


Fig. 9: $x \sim x'$

5. Conclusion and future scope

This paper introduces an effective and computationally efficient approach to solving the time-dependent harmonic oscillator with exponential mass decay model using the Optimal and Modified Homotopy Perturbation Method (OM-HPM). The method provides an accurate and efficient semi-analytical solution to the system's equation of motion, which was derived using the Lagrangian formalism. By optimizing the linear operator to minimize the residual error, OM-HPM generates highly precise results without requiring special transformations or numerical decomposition methods typically used for nonlinear problems. Compared to the widely used Runge-Kutta fourth-order (RK4) method, OM-HPM demonstrates improved accuracy with reduced computational complexity, making it an effective alternative for analyzing such systems. The solutions derived for oscillators with time-dependent mass suggest that the OM-HPM can be a valuable tool for addressing a broader class of nonlinear problems across various scientific and engineering domains.

Acknowledgements

The authors Tapas Roy gratefully acknowledge to Govt. of West Bengal, INDIA for supports scholarship for Ph.D through the Swami Vivekananda Merit Cum Means Scholarship Scheme.

The authors Taqwa Al- khader and Jihad Asad gratefully acknowledge financial support by Palestine Technical University- Kadoorie.

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